On the Betti Series of Local Rings

By

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Let \( R \) be a commutative Noetherian ring and let \( \mathfrak{a} \) be an ideal in \( R \). In \([6]\), Tate has shown that it is always possible to construct a free resolution of \( R/\mathfrak{a} \) which, at the same time, is a skew commutative differential graded algebra over \( R \), and he successfully applied his "\( R \)-algebra resolutions" to the study of the homology theory of Noetherian rings. In the case when \( R \) is a local ring with maximal ideal \( \mathfrak{m} \), it would be more desirable, however, to construct a "minimal" \( R \)-algebra resolution if it is possible.

In § 1, we prove, first of all, that such a resolution always exists. In fact, an \( R \)-algebra resolution \( X \) of the residue field \( K \), which is constructed in theorem 1 in \([6]\), is actually minimal.

For any integer \( p \geq 0 \), the \( p \)-th Betti number \( B_p \) of \( R \) is defined to be the dimension of the vector space \( \text{Tor}^R_p(K, K) \) over \( K \). The power series \( \mathcal{A}(R) = \sum B_p Z^p \) is called the Betti series of \( R \). Based on the existence of a minimal \( R \)-algebra resolution, we can express \( \mathcal{A}(R) \) as a quotient of two power series:

\[
\mathcal{A}(R) = \left( \frac{1+Z}{1-Z} \right)^{\varepsilon_1} \cdot \left( \frac{1+Z^2}{1-Z^2} \right)^{\varepsilon_2} \cdot \left( \frac{1+Z^3}{1-Z^3} \right)^{\varepsilon_3} \cdot \left( \frac{1+Z^4}{1-Z^4} \right)^{\varepsilon_4} \cdot \left( \frac{1+Z^5}{1-Z^5} \right)^{\varepsilon_5} \cdots,
\]

where \( n \) is the embedding dimension of \( R(=\dim_K \mathfrak{m}/\mathfrak{m}^2) \) and \( \varepsilon_i \)'s are non negative integers. In the case when \( K \) is of characteristic 0, as was pointed out by Scheja \([3]\), this formula is also obtained by applying the Hopf-Borel structure theorem to the Hopf algebra \( \text{Tor}^R(K, K) \) \([1]\), but the formula is true in general as we mentioned above.

In the following sections 2 and 3, we will give alternating proofs of the theorems, due to Scheja \([3]\), by the systematic use of the \( R \)-algebra method which would simplify the original arguments in some points. By making use of the formula (\(*\)), we investigate in § 2 the relationship between \( \mathcal{A}(R) \) and \( \mathcal{A}(\bar{R}) \), where \( \bar{R} \) is the residue ring of \( R \) by a non zero divisor of \( R \). In § 3, we represent \( \mathcal{A}(R) \) as a rational function in the case \( n \leq 2 \), and show that the multiplicative property of the Koszul complex of \( R \) gives us an information about the classification of the possible types of \( \mathcal{A}(R) \).

Throughout, the terminology and notations are the same as those of \([6]\). We shall use freely the \( R \)-algebra techniques, all of which can be found in
[1] and [6]. By a local ring \((R, m)\) we mean \(R\) is a commutative Noetherian local ring and \(m\) its maximal ideal.

1. Let \((R, m)\) be a local ring of embedding dimension \(n\) and let \(K\) be the residue field \(R/m\). First, we recall that the limit of the following ascending sequence of \(R\)-algebras \(X^{(k)} (k=0, 1, 2, \ldots)\) gives us an \(R\)-algebra resolution \(X\) of \(K\) \([6]\).

We take \(X^{(0)}=R\) and fix a minimal system of generators \(t_1, \ldots, t_n\) of \(m\). Viewing \(t_i\)'s as 0-cycles, we adjoin variables \(T_1, \ldots, T_n\) of degree 1 to \(R\) which kill \(t_1, \ldots, t_n\) and put

\[X^{(1)}=R<T_1, \ldots, T_n>; \ dT_i=t_i.\]

Then, \(H_0(X^{(1)})=K\). Denote by \(\varepsilon_i\) the dimension of \(H_1(X^{(1)})\) over \(K\) and choose 1-cycles \(s_1, \ldots, s_{\varepsilon_i} \in Z_i(X^{(1)})\) such that whose homology classes generate \(H_i(X^{(1)})\), and adjoin variables \(S_i (1 \leq i \leq \varepsilon_i)\) of degree 2 to \(X^{(1)}\) which kill the cycles \(s_i\). Then we get the next \(R\)-algebra

\[X^{(2)}=X^{(1)}<S_1, \ldots, S_{\varepsilon_i}>; \ dS_i=s_i.\]

Continuing in this way we get a sequence of \(R\)-algebras \(X^{(k)} (k=0, 1, 2, \ldots)\).

We remark that \(X^{(k)} (k=0, 1, 2, \ldots)\) enjoy the following properties:

1. \(X^{(k+1)} \supset X^{(k)}\) and \(X^{(k+1)}=X^{(k)}\) if \(k=k+1\).
2. \(H_0(X^{(k)})=K\) and \(H_0(X^{(k)})=0\) for \(1 \leq k < k\).

3. \(X^{(k+1)}_{k+1}\) is a direct sum of \(X^{(k)}_{k+1}\) and \(\varepsilon_k\)-copies of \(R\), where \(\varepsilon_k\) is a number of variables adjoined to \(X^{(k)}\) which is equal to the dimension of the vector space \(H_k(X^{(k)})\) over \(K\). \(X^{(1)}\) is nothing but the Koszul complex of \(R\) and will be denoted by \(E\). We remark further that \(\varepsilon_i\) is independent of the choice of the cycles in \(Z_i(X^{(1)})\) so that \(\varepsilon_i\) and, consequently, \(X\) are the homological invariants of \(R\). We call \(\varepsilon_i\) the \(i\)-th deflection of \(R\) since \(\varepsilon_i\)'s give us an information about the degree of irregularity of \(R\).

A projective resolution \(P\) of \(K\) is called minimal if it satisfies the condition, \(dP \subseteq mP\), where \(d\) is the differential operator defined on the complex \(P\). Now, we shall prove that the \(R\)-algebra resolution \(X\) of \(K\), constructed above, has this additional property. For this we need the following lemma which provides us with the basis of an inductive argument.

**Lemma 1.** Let \(X\) be an \(R\)-algebra and assume \(X\) satisfies the following two

1) It is well known that \(\varepsilon_i=0\) if and only if \(R\) is regular, and \(\varepsilon_2=0\) if and only if \(R\) is a complete intersection \([1],[3],[7]\).
conditions:

(1) \( Z_\lambda(X) \subseteq mX_\lambda (\lambda \geq 1) \).

(2) If \( x \in X_\lambda \) and \( dx \in m^2X_{\lambda-1} \), then \( x \in mX_\lambda (\lambda \geq 1) \).

Now, let \( t \in Z_{\rho -1}(X) \) be a cycle of degree \( \rho -1 (\rho > 0) \) and let \( Y = X < T> ; \) \( dT = t \). Then, (1) and (2) also hold in \( Y \).

**Proof.** We treat the cases of odd and even \( \rho \) separately.

\( \rho \) odd: In this case, \( Y = X + XT \). Let \( y_\lambda = x_\lambda + x_{\lambda - \rho} T \) be an element of \( Z_\lambda(Y) \). Since \( dy_\lambda = 0 \), we have \( dx_\lambda + (-1)^{\lambda - \rho} x_{\lambda - \rho} t = 0 \) and \( dx_{\lambda - \rho} = 0 \). From (1), we have \( x_{\lambda - \rho} \in mX_{\lambda - \rho} \). Hence, \( dx_\lambda \in mX_\lambda \) \( (mX_\lambda \cap m^2X_{\lambda-1} \subseteq mX_{\lambda-1} \). Whence \( x_\lambda \in mX_\lambda \) by virtue of (2). Consequently, \( y_\lambda \in mY_\lambda \). Next, we assume \( dy_\lambda \in m^2Y_{\lambda-1} \). Then, \( dx_\lambda + (-1)^{\lambda - \rho} x_{\lambda - \rho} t \in m^2X_{\lambda-1} \) and \( dx_{\lambda - \rho} \in m^2X_{\lambda - \rho - 1} \). Hence, by the similar argument as above, we can easily verify that \( y_\lambda \in mY_\lambda \).

\( \rho \) even: In this case, \( Y = X + XT + XT^{(2)} + \cdots \). Let \( y_\lambda = x_\lambda + x_{\lambda - \rho} T + x_{\lambda - 2\rho} T^{(2)} + \cdots + x_{\lambda - n\rho} T^{(n)} \) be an element of \( Z_\lambda(Y) \). Then, we see at once that \( dx_{\lambda - n\rho} = 0 \), \( dx_{\lambda - (n-1)\rho} + (-1)^{n-1} x_{\lambda - n\rho} t = 0 \), \( \cdots \), \( dx_\lambda + (-1)^{\lambda - \rho} x_{\lambda - \rho} t = 0 \). Therefore, we can prove, step by step, each \( x_{\lambda - ip} (i = n, n-1, \ldots, 2, 1) \) is contained in \( mX_{\lambda - ip} \), which shows that \( y_\lambda \in mY_\lambda \). As for the proof of (2), we will leave it to the reader.

Observe that \( X^{(0)}(=R) \) trivially satisfies the condition (1) and (2). Therefore, by the successive applications of lemma 1 to each step of the adjoining variables in the construction of \( X \), we get our following important theorem.

**Theorem 1.** A minimal \( R \)-algebra resolution of \( K \) always exists.

Another consequence of the particular construction of the minimal resolution \( X \) of \( K \) is stated in

**Theorem 2.** The Betti series \( \mathcal{A}(R) \) of \( R \) is given by the following formula:

\[
\mathcal{A}(R) = \frac{(1 + Z)^n}{(1 + Z^3)^{\varepsilon_1}} \cdot \frac{(1 + Z^3)^{\varepsilon_1}}{(1 + Z^6)^{\varepsilon_2}} \cdot \frac{(1 + Z^6)^{\varepsilon_2}}{(1 + Z^9)^{\varepsilon_3}} \cdots,
\]

where \( n \) is the embedding dimension of \( R \) and \( \varepsilon_i \) the \( i \)-th deflection of \( R \). In particular,

\[
B_1 = n, \quad B_2 = \binom{n}{2} + \varepsilon_1, \quad B_3 = \binom{n}{3} + \left( \binom{n}{2} + \varepsilon_2 \right) \varepsilon_1 + \varepsilon_2,
\]

\[
B_4 = \binom{n}{4} + \left( \binom{n}{2} \varepsilon_1 + \varepsilon_2 \right) - \left( \binom{n}{2} \varepsilon_1 + \binom{n}{2} \varepsilon_2 \right) + \left( \binom{n}{2} + \varepsilon_3 \right) \varepsilon_2 + \varepsilon_3,
\]
\[ B_{\delta} = \left( \frac{n}{5} \right) + \left( \frac{n}{3} \right) \varepsilon_1 + \left( \frac{n}{1} \right) \frac{\varepsilon_1}{2} + \left( \frac{n}{1} \right) \varepsilon_1 + \frac{\varepsilon_1 \varepsilon_2}{2} + \frac{\varepsilon_1}{1} \frac{\varepsilon_3 + \varepsilon_4}{1}, \cdots \]

**Proof.** Since \( X \) is minimal, we have \( \text{Tor}^R(K, K) \neq H(X \otimes K) \neq X \otimes K \). Therefore the \( p \)-th Betti number \( B_p \) and the number of generators of the free module \( X_p \) are exactly the same. Hence, counting the number of generators of \( X_{\delta} \), we obtain the desired result.

We point out here that, in view of theorem 2, we see our \( \varepsilon_i \) coincides with that of Scheja \([8]\) and of Uehara \([7]\) for \( i \leq 3 \).

2. In this section, as an application of theorem 2, we shall see how the Betti series of \( R \) is affected if we pass to its residue ring by a non zero divisor. We begin with the following lemma which has the general character.

**Lemma 2.** Let \( R \) and \( \overline{R} \) be Noetherian rings and let \( X \) and \( \overline{X} \) be \( R \)- and \( \overline{R} \)-algebras such that there exists an \( R \)-homomorphism \( \varphi \) from \( X \) to \( \overline{X} \) which induces an isomorphism \( \varphi_* \) of \( H(X) \) onto \( H(\overline{X}) \). Suppose \( \iota \) and \( \overline{\iota} \) are \((\rho - 1)\)-cycles in \( X \) and \( \overline{X} \) such that \( \varphi(\iota) = \overline{\iota} \) and let \( Y = X < \iota > ; d \overline{Y} = \iota \) and \( \overline{Y} = \overline{X} < \overline{\iota} > ; d \overline{\overline{Y}} = \iota \). Then, \( \varphi \) can be extended to an \( R \)-homomorphism (again denoted by \( \varphi \)) from \( Y \) to \( \overline{Y} \) and it induces an isomorphism of \( H(Y) \) onto \( H(\overline{Y}) \).

**Proof.** We again treat the cases of odd and even \( \rho \) separately.

\( \rho \) odd: In this case, \( Y = X + XT \). Consider the exact sequence

\[ 0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{j} X \longrightarrow 0, \]

where \( i \) is injective and \( j \) is defined by \( j(x_1 + x_2 T) = x_2 \). Then, \( i \) and \( j \) commute with \( d \) and \( \varphi \). Hence, we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & X_\lambda \xrightarrow{i} Y_\lambda \xrightarrow{j} X_{\lambda - \rho} \longrightarrow 0 \\
\downarrow \varphi & & \downarrow \varphi \\
0 & \longrightarrow & \overline{X}_\lambda \xrightarrow{i} \overline{Y}_\lambda \xrightarrow{j} \overline{X}_{\lambda - \rho} \longrightarrow 0
\end{array}
\]

with exact rows. From this we get a commutative diagram

\[
\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_{\lambda - \rho + 1}(X) & \longrightarrow & H_\lambda(X) & \xrightarrow{d} & H_{\lambda - \rho}(X) & \longrightarrow & H_{\lambda - 1}(X) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots \\
\cdots & \longrightarrow & H_{\lambda - \rho + 1}(\overline{X}) & \longrightarrow & H_\lambda(\overline{X}) & \longrightarrow & H_{\lambda - \rho}(\overline{X}) & \longrightarrow & H_{\lambda - 1}(\overline{X}) & \longrightarrow & \cdots
\end{array}
\]

where both rows are exact. Therefore \( H_\lambda(Y) \cong H_\lambda(\overline{Y}) \) \((\lambda = 1, 2, 3, \cdots)\) by the "five lemma" \([2]\).
\( \rho \) even: In this case, \( Y = X + XT + XT^{(2)} + \cdots \), and we have an exact sequence

\[
0 \to X \xrightarrow{i} Y \xrightarrow{j} Y \to 0,
\]

where \( i \) is injective and \( j \) is defined by \( j(x_0 + x_1 T + x_2 T^{(2)} + \cdots) = x_1 + x_2 T + x_3 T^{(2)} + \cdots \). As in the first case, \( i \) and \( j \) commute with \( d \) and \( \varphi \), and

\[
0 \to X_\lambda \xrightarrow{i} Y_\lambda \xrightarrow{j} Y_{\lambda - \rho} \to 0
\]

is a commutative diagram with exact rows. This yields the following commutative homology diagram

\[
\cdots \to H_1(Y) \xrightarrow{d} H_0(X) \xrightarrow{i_*} H_0(Y) \xrightarrow{j_*} H_0(Y) \to H_{\rho-1}(X) \xrightarrow{i_*} H_{\rho-1}(Y) \to 0
\]

where both rows are exact. Since \( H_0(Y) \approx H_0(\bar{Y}) \approx K \), we can easily see that \( H_{\rho-1}(Y) \approx H_{\rho-1}(\bar{Y}) \). On the other hand, from the construction of \( Y \) and \( \bar{Y} \), we have \( H_i(Y) \approx H_i(\bar{Y}) \) for \( i < \rho - 1 \). Thus, applying the five lemma, we have \( H_\lambda(Y) \approx H_\lambda(\bar{Y}) \) and similarly \( H_\lambda(Y) \approx H_\lambda(\bar{Y}) \) for all \( \lambda \).

Let again \((R, m)\) be a local ring and let \( t_1, \ldots, t_n \) be a minimal system of generators of \( m \). Suppose \( t_n \) is a non zero divisor in \( R \) and not in \( m^2 \). Let \( \bar{R} = R/t_n R \) and let \( \bar{t}_i \) be the residue class of \( t_i \) \((i = 1, 2, 3, \ldots)\). We consider two \( R \)- and \( \bar{R} \)-algebras

\[
E = R < T_1, \ldots, T_n >; \quad dT_i = t_i \text{ and } F = \bar{R} < \bar{T}_1, \ldots, \bar{T}_{n-1} >; \quad dT_i = \bar{t}_i.
\]

If \( x \) is a homogeneous element of degree \( \rho \) in \( E \), then \( x \) can be written as \( x = x_1 + x_2 T_n \), where \( x_1 \) and \( x_2 \) are homogeneous elements in \( E' = R < T_1, \ldots, T_{n-1} > \) of degree \( \rho \) and \( \rho - 1 \) respectively. Then, the canonical map \( \varphi: E \to F \) defined by \( \varphi(x) = \bar{x}_1 \) induces a homomorphism \( \varphi_*: H(E) \to H(F) \).

**Lemma 3.** In the situation just described, \( \varphi_* \) is an isomorphism and \( \bar{\varepsilon}_1 = \bar{\varepsilon}_1 \) where \( \bar{\varepsilon}_1 \) is the first deflection of \( \bar{R} \).

**Proof.** First we show that \( \varphi \) induces an \( R \)-homomorphism of \( Z_\rho(E) \) onto \( Z_\rho(F) \). Take \( \bar{x}_1 \in Z_\rho(F) \). Since \( d\bar{x}_1 = 0 \), we can write \( dx_1 = y_1 t_n + y_2 T_n \), with \( y_1 \)}
and $y_2$ in $E'$. From the relation

$$0 = d^2 x_1 = (d y_1) t_n + (d y_2) T_n + (-1)^{n-1} y_2 t_n,$$

we get $d y_1 = (-1)^{n-1} y_2$ and $d y_2 = 0$ since $t_n$ is a non zero divisor. Now, by a direct calculation, we easily see that the element $x = x_1 + (-1)^n y_1 T_n$ belongs to $Z_n(E)$ such that $\varphi(x) = \bar{x}_1$. By the similar argument we can show $\varphi^{-1}(B_\rho(F')) = B_{\rho}(E)$. Therefore $\varphi_*$ is an isomorphism of $H(E)$ onto $H(F)$ as we asserted.

**Theorem 3.** Let $(R, m)$ be a local ring and let $x$ be an element of $m$, which is not a zero divisor in $R$. Put $\bar{R} = R/xR$ and denote by $\mathcal{B}(\bar{R})$ and $\bar{e}_i$ the Betti series and the $i$-th deflection of $\bar{R}$ respectively. Then:

(i) If $x \notin m^2$, we have $\bar{e}_i = e_i$ ($i=1, 2, \ldots$) and $\mathcal{B}(\bar{R}) = \mathcal{B}(R)(1 + Z)^{[3]}$.

(ii) If $x \in m^2$, we have $\bar{e}_i = e_{i+1}, \bar{e}_i = e_i$ ($i=2, 3, \ldots$) and $\mathcal{B}(\bar{R}) = \mathcal{B}(R)(1 - Z^2)$.

**Proof.** (i) If $x \notin m^2$, we can take $x$ as a member of a minimal generating system of $m$. Observe that $\dim \bar{R}^g = \dim R - 1$. Hence, by lemma 2 and 3, combining with the formula of Betti series in theorem 2, we have our assertion.

(ii) We remark first that $E = E/xE$ is the Koszul complex of $\bar{R}$ since $x \in m^2$. Write $x = \sum a_i t_i$. Then, $s = \sum a_i T_i$ is in $E_i$ and satisfies $ds = x$. The residue class $s (\mod xE)$ is a 1-cycle in $E$, whose homology class we denote by $\sigma$. The canonical map $j: E \to \bar{E}$ induces an isomorphism $j_*$ of $H(E)$ into $H(\bar{E})$ and $H(\bar{E}) = (j_* H(E)) < \sigma > [6, \text{theorem 3}']$. Hence, $\bar{e}_i = e_i + 1$.

We adjoin a variable $S$ of degree 2 to $E$ which kills $s$ and obtain

$$E' = E < S >; \quad dS = S.$$

Since $\sigma$ is a skew non zero divisor in $j_*(H(E))$, we have

$$H(E') \cong H(E)/\sigma H(E) \cong H(E)$$

by theorem 2 in [6]. Now, we can conclude our proof by applying lemma 2 to the $R$-algebra $E$ and the $\bar{R}$-algebra $E'$.

**Corollary.** (i) If $R$ is a regular local ring of dimension $n$, then $\mathcal{B}(R) = (1 + Z)^n [3], [4]$.

(ii) If $R$ is a complete intersection of embedding dimension $n$, then $\mathcal{B}(R) = \frac{(1 + Z)^n}{(1 - Z^2)^{n-d}}$, where $d = \dim R$ [6].

3. As we stated in the introduction, the results concerning the struc-

\[2\) We denote by $\dim R$ the dimension of $R$ in the sense of Krull.
ture of Betti series in this section were originally obtained by Scheja [3]. He used the Syzygy theory of modules and his method was rather ideal theoretic. We shall present here a simplified proof which is based on the theory of R-algebras.

We shall use the same notations as in §1 and the Koszul complex of R will be denoted by E. For the obvious reason we treat only non regular case.

**Theorem 4.** If \((R, m)\) is a local ring of embedding dimension 1 and if \(H_i(E) \neq 0\), then we have

\[
\varepsilon_1 = 1, \quad \varepsilon_i = 0 \quad (i \geq 2) \quad \text{and} \quad \mathcal{A}(R) = \frac{1 + Z}{1 - Z^2}.
\]

**Proof.** In this case, we have \(H_0(E) = K\), \(H_i(E) = (0 : m) T\) and \(H_i(E) = 0\) for \(i \geq 2\). And, moreover, \(R\) is a principal ideal ring. Hence our hypothesis implies that there is an element \(a \neq 0\) in \(R\) such that \(0 : m = aR\).

Now, we adjoin a variable \(S\) of degree 2 which kills 1-cycle \(aT\), and obtain an \(R\)-algebra \(X = E < S>\); \(dS = aT\). Consider the exact sequence

\[
0 \rightarrow E^i \rightarrow X^j \rightarrow X \rightarrow 0,
\]

where \(i\) is the injective map and \(j\) is defined by \(j(x_0 + x_1S + x_2S^2 + \ldots) = x_1 + x_2S + \ldots\). \(i\) and \(j\) commute with \(d\) and we get the following exact sequence:

\[
\cdots \rightarrow H_0(E) \rightarrow H_0(X) \rightarrow H_1(X) \rightarrow H_2(E) \rightarrow H_2(X) \rightarrow K
\]

\[
\xrightarrow{d_{00}} H_1(E) \rightarrow H_1(X) = 0.
\]

Since \(H_1(E) \cong K\), \(d_{00}\) is an isomorphism. Hence, \(H_2(X) = 0\) by virtue of \(H_2(E) = 0\). In the same way, the relations \(H_3(E) = 0\) and \(H_1(X) = 0\) imply that \(H_3(X) = 0\). Thus, step by step, we have \(H_i(X) = 0\) for \(i = 1, 2, 3, \ldots\), and hence \(\varepsilon_1 = 1\) and \(\varepsilon_i = 0\) for \(i \geq 2\). Consequently, \(\mathcal{A}(R) = \frac{1 + Z}{1 - Z^2}\) in view of theorem 2.

**Theorem 5.** Let \((R, m)\) be a local ring of embedding dimension 2 and suppose that \(R\) is not regular. Then:

(i) If \(H_1(E)^2 = 0\), then \(\varepsilon_1 \geq 1, \varepsilon_2 = \varepsilon_1 - 1, \varepsilon_3 = \binom{\varepsilon_1}{2}\) and \(\mathcal{A}(R) = \frac{(1 + Z)^2}{1 - \varepsilon_1 Z^2 - \varepsilon_2 Z^3}\).

(ii) If \(H_1(E)^2 \neq 0\), then \(\varepsilon_1 = 2, \varepsilon_2 = 0\) and \(\mathcal{A}(R) = \frac{(1 + Z)^2}{(1 - Z^2)^2}\).

**Proof.** First we remark that the vector space \(0 : m\) over \(K\) has dimension \(\varepsilon_1 - 1\). In fact, since \(\text{dim } R < 2\), the Euler-Poincaré characteristic of the Koszul complex \(E\) is equal to zero \([5]\), i.e.,
\[ \dim_K H_2(E) - \dim_K H_1(E) + \dim_K H_0(E) = 0. \]

Since \( H_2(E) \cong 0 : m \), \( \dim_K H_1(E) = \varepsilon_1 \) and \( \dim_K H_0(E) = 1 \), we have our assertion.

From this remark and from the facts \( H_2(X^{(2)}) \cong H_2(E)/H_1(E)^2 \) [1, Proposition 2.5] and \( \varepsilon_2 = \dim_K H_2(X^{(2)}) \), we find \( \varepsilon_2 = \varepsilon_1 - 1 \) if \( H_1(E)^2 = 0 \), and \( \varepsilon_2 \leq \varepsilon_1 - 2 \) if \( H_1(E)^2 \neq 0 \).

Now we consider the first case, \( H_1(E)^2 = 0 \). Let \( X \) be a minimal \( R \)-algebra resolution of \( K \) constructed in §1:

\[ X: \ldots \rightarrow X_p \rightarrow \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \overset{\varepsilon}{\rightarrow} K \rightarrow 0 \]

where \( X_0 = R \) and \( \varepsilon \) is the augmentation homomorphism. Then, by the construction of \( X \), \( X_3 \) has the following form:

\[ X_3 = \sum_{j=1}^{\varepsilon_1} (RT_1 + RT_2)S_j + \sum_{k=1}^{\varepsilon_2} RU_k, \]

where \( S_j(1 \leq j \leq \varepsilon_1) \) and \( U_k(1 \leq k \leq \varepsilon_2) \) are variables of degree 2 and 3 which kill cycles \( s_j \) and \( u_k \) respectively. We remark that, since \( 0 : m \cong H_2(E) \cong H_2(X^{(2)}) \), we can take \( c_i T_1 T_2 (i = 1, \ldots, \varepsilon_1) \) as \( u_i \), where \( c_1, \ldots, c_{\varepsilon_1} \) is a minimal generating system of the ideal \( 0 : m \).

Let \( M = dX_3 \) and write \( M = M_1 + M_2 \), where \( M_1 \) (resp. \( M_2 \)) is an \( R \)-module generated by \( t_1S_j - T_1s_j \) and \( t_2S_j - T_2s_j (1 \leq j \leq \varepsilon_1) \) (resp. \( c_i T_1 T_2 (1 \leq k \leq \varepsilon_2) \)). We contend first that \( M_1 \) is isomorphic to the direct sum of \( \varepsilon_1 \)-copies of \( m \). To see this, it is enough to prove that the projection \( \phi: M_1 \rightarrow \sum (R T_1 + R T_2)S_j = \sum mS_j \) defined by

\[ \varphi(\sum \lambda_j (t_1S_j - T_1s_j) + \sum \mu_j (t_2S_j - T_2s_j)) = \sum (\lambda_j t_1 + \mu_j t_2) S_j \]

is an isomorphism. Assume \( \alpha = \sum \lambda_j (t_1S_j - T_1s_j) + \sum \mu_j (t_2S_j - T_2s_j) \in \text{Ker } \varphi \). Then, we have \( \lambda_j t_1 + \mu_j t_2 = 0 \) and hence \( \lambda_j T_1 + \mu_j T_2 \in Z_i(E) \) for \( j = 1, \ldots, \varepsilon_1 \). Therefore

\[ \alpha = - (\sum \lambda_j s_j T_1 + \sum \mu_j s_j T_2) = - \sum (\lambda_j T_1 + \mu_j T_2) s_j = 0 \]

in view of \( H_1(E)^2 = 0 \). Hence \( \varphi \) is injective, and whence bijective because clearly it is surjective. By a similar argument, \( M_2 \) is isomorphic to the direct sum of \( \varepsilon_2 \)-copies of \( K \). In this case, we consider the free module \( RZ_1 + \cdots + RZ_{\varepsilon_2} \) and consider the map \( \phi: RZ_1 + \cdots + RZ_{\varepsilon_2} \rightarrow 0 : m \) defined by

\[ \phi(\sum \nu_i Z_i) = \sum \nu_i c_i. \]

Then, \( \phi \) induces, obviously, an isomorphism between \( \bigoplus K \) and \( 0 : m \), since \( c_1, \ldots, c_{\varepsilon_1} \) is a minimal generating system of \( 0 : m \). Finally, we mention that
$M$ is actually the direct sum of $M_1$ and $M_2$ in view of $H_1(E)^2 = 0$. Summarizing, we obtain

$$M \cong (\oplus m) \oplus (\oplus K).$$

Now, since the torsion functor has an additive property, we have

$$\text{Tor}^R_p(M, K) = (\oplus \text{Tor}^R_p(m, K)) \oplus (\oplus \text{Tor}^R_p(K, K))$$

for $p \geq 0$. But, clearly $\text{Tor}^R_p(M, K) = \text{Tor}^R_{p+3}(K, K)$ and $\text{Tor}^R_p(m, K) = \text{Tor}^R_{p+1}(K, K)$. Hence, we obtain the following recurrence relation of Betti numbers:

$$B_{p+3} = \varepsilon_1 B_{p+1} + \varepsilon_2 B_p \quad (p \geq 0)$$

Combining this with the fact $B_0 = 1$, $B_1 = 2$ and $B_2 = 1 + \varepsilon_1$, we obtain

$$\mathfrak{B}(R) = \frac{(1+Z)^2}{1-\varepsilon_1 Z^2 - \varepsilon_2 Z^3}.$$ 

The fact $\varepsilon_3 = \frac{\varepsilon_1}{2}$ follows from theorem 2 by a direct computation.

Next we consider the case when $H_1(E)^2 \neq 0$. In this case, $0 \leq \varepsilon_2 \leq \varepsilon_1 - 2$, as we already mentioned, and hence we shall have $\varepsilon_2 = 0$ if we show $\varepsilon_1 \leq 2$.

Observe that everything is unchanged when we pass to the completion of $R$. Therefore, we can assume $R$ is complete. By the structure theorem of Cohen, there exists a minimal embedding of $R$, that is, there exists a regular local ring $\tilde{R}$ of dimension 2 and an ideal $\tilde{a}$ of $\tilde{R}$ such that $R = \tilde{R}/\tilde{a}$, $\tilde{a} \subset \tilde{m}^2$, where $\tilde{m}$ is the maximal ideal of $\tilde{R}$. Denote by $h: \tilde{R} \to R$ the canonical map and let $\tilde{t}_i$ be an element of $\tilde{R}$ such that $h(\tilde{t}_i) = t_i$. Then, obviously $\tilde{m} = (\tilde{t}_1, \tilde{t}_2) \tilde{R}$. Let $\tilde{a}_1, \ldots, \tilde{a}_r$ be a minimal system of generators of $\tilde{a}$ and write $\tilde{a}_i = \tilde{\lambda}_i \tilde{t}_1 + \tilde{\mu}_i \tilde{t}_2$. Then, $s_i = \lambda_i T_1 + \mu_i T_2$ $(1 \leq i \leq r)$ constitutes a minimal generating system of $Z_1(E)$ modulo $B_1(E)$ [1, p. 196] where $\lambda_i = h(\tilde{\lambda}_i)$ and $\mu_i = h(\tilde{\mu}_i)$. Since $H_1(E)^2 \neq 0$, there exist at least two elements, say $\tilde{a}_1$ and $\tilde{a}_2$, in $\tilde{a}_1, \ldots, \tilde{a}_r$, such that $(\lambda_1 \mu_2 - \lambda_2 \mu_1) T_1 T_2 \neq 0$.

Let $\alpha_1 = (\tilde{a}_1, \tilde{a}_2) \tilde{R}$, $\bar{R} = \tilde{R}/\alpha_1$ and $\bar{m} = \tilde{m}/\alpha_1$. Since $\tilde{a}_1, \tilde{a}_2$ is a minimal system of generators of $\alpha_1$, we have $\dim_K H_1(\bar{E}) = 2$, where $\bar{E}$ is the Koszul complex of $\bar{R}$. Hence, by the remark at the first paragraph of the proof, we have $\dim_K 0: \bar{m} = 1$, since $\bar{R}$ is not regular. Therefore, it follows that $0: \bar{m} = (\lambda_1 \mu_2 - \lambda_2 \mu_1) \bar{R}$, where $\lambda_i$ and $\mu_i$ are the residue classes of $\tilde{\lambda}_i$ and $\tilde{\mu}_i$ in $R$ respectively. Thus $H_2(\bar{E}) = H_1(\bar{E})^2$ and $\bar{R}$ is a complete intersection and of dimension 0 [1, theorem 2.7]. Hence the zero ideal of $\bar{R}$ is irreducible [8, IV theorem 34].

Suppose $\varepsilon_1 > 2$, then $\tilde{a} \supsetneq \alpha_1$. Hence $0: \bar{m} \subset \tilde{a}/\alpha_1$ [8, IV theorem 34] and
therefore \( \lambda_1 \mu_2 - \lambda_2 \mu_1 = 0 \). But, this contradicts the choice of \( a_1 \) and \( a_2 \).

As for the Betti series of \( R \), it is enough to mention that \( \bar{a} = a_1 \) is generated by an \( \bar{R} \)-sequence and hence coroll. of theorem 3 can be applied to \( R \).

**Corollary.** Let \( (R, \mathfrak{m}) \) be a local ring of embedding dimension 2, and assume that \( R \) is not regular. If \( \mathfrak{m} \) contains at least one non-zero divisor, then we have \( \mathcal{B}(R) = \frac{(1 + \mathfrak{m}^2)}{1 - \mathfrak{m}^2} \).

**Proof.** Our hypothesis implies that \( H_2(E) \approx 0; \mathfrak{m} = 0 \). Therefore \( H_1(E)^2 = 0 \). Hence, we have the corollary in view of theorem 5.

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**References**


