On the Mean Values of Integral Functions and Their Derivatives Defined by Dirichlet Series

By

A. K. Agarwal and G. P. Dikshit*

(Received September 30, 1968)

1. Consider the Dirichlet Series \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \), where \( \lambda_{n+1} > \lambda_n \), \( \lambda_1 > 0 \), \( \lim_{n \to \infty} \lambda_n = \infty \), \( s = \sigma + it \) and

\[
\lim_{n \to \infty} \sup_{\sigma} \frac{\log n}{\log \lambda_n} = E < \infty.
\]

(1.1)

Let \( \sigma_c \) and \( \sigma_a \) be the abscissa of convergence and abscissa of absolute convergence, respectively, of \( f(s) \). Let \( \sigma_c = \infty \) then \( \sigma_a \) will also be infinite, since according to a known result ([1], p. 4) a Dirichlet Series which satisfies (1.1) has its abscissa of convergence equal to its abscissa of absolute convergence, and so, \( f(s) \) is an integral function.

The Mean Value of \( f(s) \) is

\[
I_2(\sigma) = I_2(\sigma, f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt,
\]

(1.2)

and extending this definition to \( f^{(p)}(s) \), the \( p^{\text{th}} \)-derivative of \( f(s) \),

\[
I_2(\sigma, f^{(p)}) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f^{(p)}(\sigma + it)|^2 dt.
\]

(1.3)

Let \( \mu(\sigma) = \max\{|a_n| e^{\lambda_n s}\} \); \( M(\sigma) = \text{l.u.b.} |f(\sigma + it)| \) be respectively the maximum term and the maximum modulus of an integral function.

It is known ([2], p. 67) that

\[
\log \mu(\sigma) = \log(1) + \int_{\sigma}^{\infty} \lambda_{\nu(t)} dt,
\]

where \( \nu(\sigma) \) is the rank of the maximum term.

Further, we know ([3], p. 265, Theorem 5) that

\[
\log M(\sigma) \sim \log \mu(\sigma),
\]

(1.5)

* This research has been supported by the Junior Fellowship award of the Council of Scientific and Industrial Research, New Delhi (INDIA).
provided (1.1) holds and \( f(s) \) is of finite order.

It is also known* ([4], p. 523) that for functions of finite non-zero linear order \( \rho \) and lower order \( \lambda \),

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \log \log \frac{I_2(\sigma)}{\sigma} = \frac{\rho}{\lambda}
\]

and

\[
(1.7) \quad \log \{ I_2(\sigma) \}^{1/2} \sim \log M(\sigma).
\]

Throughout this paper we shall assume that the function \( f(s) \) is of finite non-zero linear order and satisfies (1.1). In this paper we have obtained a few properties of \( I_2(\sigma) \) and its derivative, also of \( I_2(\sigma, f^{(p)}) \).

2. Theorem 1. Let \( f(s) \) be an integral function of linear order \( \rho \) and lower order \( \lambda \), then

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \left\{ \log \frac{I_2(\sigma)}{I_2(\sigma)} \sigma \right\} = \frac{\rho}{\lambda}
\]

Proof. We know ([4], p. 521) that \( \log I_2(\sigma) \) is an increasing convex function of \( \sigma \). Therefore, \( \log I_2(\sigma) \) is differentiable almost everywhere with an increasing derivative; the set of points where the left hand derivative is less than the right-hand derivative is of measure zero. This enables us to express \( \log I_2(\sigma) \) in the following form:

\[
\log I_2(\sigma) = \log I_2(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{I_2(x)}{I_2(x)} \, dx,
\]

for an arbitrary \( \sigma_0 \).

We have,

\[
\log I_2(\sigma) \leq \log I_2(\sigma_0) + \frac{I_2(\sigma)}{I_2(\sigma)} (\sigma - \sigma_0)
\]

or

\[
\lim_{\sigma \to \infty} \sup_{\sigma} \left\{ \log \log \frac{I_2(\sigma)}{\sigma} \right\} \leq \lim_{\sigma \to \infty} \sup_{\sigma} \left\{ \frac{1}{\sigma} \left( \log \left( \frac{I_2(\sigma)}{I_2(\sigma)} \right) \right) \right\}.
\]

Again, for an arbitrary fixed \( k > 0 \)

---

* Results (1.6) and (1.7) has been proved under the condition \( \lim_{n \to \infty} \frac{\log n}{\lambda_n} = D = 0 \), though the results also hold for \( E < \infty \).
\[ \log I_2(\sigma + k) = \log I_2(\sigma) + \int_0^{\sigma + k} \frac{I_2(x)}{I_2(x)} \, dx \geq k \frac{I_2(\sigma)}{I_2(\sigma)}, \]

and therefore,

\[
\lim_{\sigma \to \infty} \sup \left\{ \frac{\log \log I_2(\sigma + k)}{\sigma} \right\} \geq \lim_{\sigma \to \infty} \inf \left\{ \frac{1}{\sigma} \log \left( \frac{I_2(\sigma)}{I_2(\sigma)} \right) \right\}.
\]

Thus

\[
\lim_{\sigma \to \infty} \sup \left\{ \frac{\log \log I_2(\sigma)}{\sigma} \right\} = \lim_{\sigma \to \infty} \inf \left\{ \frac{1}{\sigma} \log \left( \frac{I_2(\sigma)}{I_2(\sigma)} \right) \right\}.
\]

Further, from (1. 7) we have

\[
\lim_{\sigma \to \infty} \sup \left\{ \frac{\log \log I_2(\sigma)}{\sigma} \right\} = \lim_{\sigma \to \infty} \sup \left\{ \frac{\log \log M(\sigma)}{\sigma} \right\}.
\]

Hence

\[
\lim_{\sigma \to \infty} \sup \left\{ \frac{\log I_2(\sigma)/I_2(\sigma)}{\sigma} \right\} = \lim_{\sigma \to \infty} \inf \left\{ \frac{\log \log M(\sigma)}{\sigma} \right\} = \rho.
\]

**Theorem 2.** Let \( f(s) \) be an integral function of linear order \( \rho \) and lower order \( \lambda \), then

\[
(2.2) \quad \lim_{\sigma \to \infty} \sup \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_\nu(\sigma)} \right\} \leq 2(1 - \lambda/\rho)
\]

and

\[
(2.3) \quad \lim_{\sigma \to \infty} \sup \left\{ \frac{\log I_2(\sigma)}{\lambda_\nu(\sigma) \log \lambda_\nu(\sigma)} \right\} \leq 2(1/\lambda - 1/\rho).
\]

**Proof.** It is known* ([5], p. 84) that, for \( 0 < \rho < \infty \)

\[
(2.4) \quad \lim_{\sigma \to \infty} \sup \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_\nu(\sigma)} \right\} \leq 1 - \lambda/\rho
\]

and

\[
(2.5) \quad \lim_{\sigma \to \infty} \sup \left\{ \frac{\log \mu(\sigma)}{\lambda_\nu(\sigma) \log \lambda_\nu(\sigma)} \right\} \leq 1/\lambda - 1/\rho.
\]

Using (1.5) and (1.7), we have

* Results (2.4) and (2.5) has been proved under the condition \( \lim_{n \to \infty} \log \frac{n}{\lambda_n} = D = 0 \), though the result also hold for \( E < \infty \).
\[
\limsup_{\sigma \to \infty} \left[ \frac{\log I_2(\sigma)}{\sigma \lambda_2(\sigma)} \right]^{1/2} = \limsup_{\sigma \to \infty} \left\{ \frac{\log M(\sigma)}{\sigma \lambda_2(\sigma)} \right\} \\
= \limsup_{\sigma \to \infty} \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_2(\sigma)} \right\} \\
\leq 1 - \lambda/\rho.
\]

Hence
\[
\limsup_{\sigma \to \infty} \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_2(\sigma)} \right\} \leq 2(1 - \lambda/\rho).
\]

Proceeding as above and using (2.5), we have
\[
\limsup_{\sigma \to \infty} \left\{ \frac{\log I_2(\sigma)}{\lambda_2(\sigma) \log \lambda_2(\sigma)} \right\} \leq 2(1/\lambda - 1/\rho).
\]

**Theorem 3.** Let \( f(s) \) be an integral function of finite linear order \( \rho \) and lower order \( \lambda \), then
\[
(2.6) \quad \limsup_{\sigma \to \infty} \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_2(\sigma)} \right\} \leq 2.
\]

**Proof.** Using (1.5) and (1.7), we have
\[
\limsup_{\sigma \to \infty} \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_2(\sigma)} \right\} = 2 \limsup_{\sigma \to \infty} \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_2(\sigma)} \right\}.
\]

From (1.4), we get
\[
\limsup_{\sigma \to \infty} \left\{ \frac{\log \mu(\sigma)}{\sigma \lambda_2(\sigma)} \right\} \leq 1.
\]

Hence
\[
\limsup_{\sigma \to \infty} \left\{ \frac{\log I_2(\sigma)}{\sigma \lambda_2(\sigma)} \right\} \leq 2.
\]

**Theorem 4.** Let \( f(s) \) be an integral function of finite linear order \( \rho \) and lower order \( \lambda \), then for \( \sigma > \sigma_0 \) and \( \varepsilon > 0 \)
\[
(2.7) \quad I_2(\sigma) > \frac{I_2(\sigma) \log I_2(\sigma)}{(1 + \varepsilon)\sigma},
\]
where \( \varepsilon = \varepsilon(\sigma) \to 0 \) as \( \sigma \to \infty \).

**Proof.** For the left hand derivative of \( \log I_2(\sigma) \), we have
\[
\frac{I_2'(\sigma)}{I_2(\sigma)} > \frac{\log I_2(\sigma) - \log I_2(\sigma_1)}{\sigma - \sigma_1}
\]

\[
> \frac{\log I_2(\sigma)}{(1 + \varepsilon)\sigma}
\]

where \(\sigma_1 < \sigma\) and \(\varepsilon = \varepsilon(\sigma) \to 0\) as \(\sigma \to \infty\). Hence,

\[
I_2'(\sigma) > \frac{I_2(\sigma) \log I_2(\sigma)}{(1 + \varepsilon)\sigma},
\]

where \(\varepsilon = \varepsilon(\sigma) \to 0\) as \(\sigma \to \infty\).

**Corollary.** Let \(f(s)\) be an integral function of finite order \(\rho\) and if \(I_2'(\sigma)\) is the derivative of \(I_2(\sigma)\). Then for \(\sigma_0 < \sigma_1 < \sigma_2\)

(2.8) \[
\frac{I_2'(\sigma_1)}{I_2(\sigma_1)} < \frac{\log I_2(\sigma_2) - \log I_2(\sigma_1)}{\sigma_2 - \sigma_1} < \frac{I_2'(\sigma_2)}{I_2(\sigma_2)}.
\]

**Proof.** From (1.4), we have

(2.9) \[
\log I_2(\sigma_2) = \log I_2(\sigma_1) + \int_{\sigma_1}^{\sigma_2} \frac{I_2'(x)}{I_2(x)} \, dx
\]

\[
< \log I_2(\sigma_1) + \frac{I_2'(\sigma_2)}{I_2(\sigma_2)} (\sigma_2 - \sigma_1)
\]

and

(2.10) \[
\log I_2(\sigma_2) = \log I_2(\sigma_1) + \int_{\sigma_1}^{\sigma_2} \frac{I_2'(x)}{I_2(x)} \, dx
\]

\[
> \log I_2(\sigma_1) + \frac{I_2'(\sigma_1)}{I_2(\sigma_1)} (\sigma_2 - \sigma_1).
\]

Combining (2.9) and (2.10) we get the result.

3. **Theorem 5.** Let \(f(s)\) be an integral function of linear order \(\rho\) and lower order \(\lambda\), then for \(\text{Re}(s) = \sigma\) and \(\lambda \gg \delta > 0\)

(3.1) \[
I_2(\sigma, f) < I_2(\sigma, f^{(1)}) < \cdots < I_2(\sigma, f^{(p)}),
\]

where \(p\) an integer.

**Proof.** It is known ([4], p. 522)
(3.2) \[ I_2(\sigma, f^{(1)}) \geq \frac{1}{2^2} \left( \frac{\log I_2(\sigma, f)}{\sigma} \right)^2 I_2(\sigma, f). \]

Taking limits on both the sides and using (1.6), we get
\[
\lim \inf_{\sigma \to \infty} \left[ \frac{1}{\sigma} \log \left( \frac{I_2(\sigma, f^{(1)})}{I_2(\sigma, f)} \right)^{1/2} \right] \geq \lambda.
\]

Again, for \( \epsilon > 0 \) and \( \sigma \) sufficiently large
\[
\left\{ \frac{I_2(\sigma, f^{(1)})}{I_2(\sigma, f)} \right\} > e^{2\sigma(\lambda - \epsilon)}.
\]

If \( \lambda \geq \delta > 0 \),
\[
I_2(\sigma, f) < I_2(\sigma, f^{(1)}),
\]
and the result follows for subsequent derivatives.

**Theorem 6.** Let \( f(s) \) be an integral function, then for \( \sigma > 0 \) and \( \lambda \geq \delta > 0 \)
\[
(3.3) \quad I_2(\sigma, f^{(p)}) \geq \frac{1}{2^{2^p}} \left\{ \frac{\log I_2(\sigma, f)}{\sigma} \right\}^{2^p} I_2(\sigma, f),
\]
where \( p \) is an integer.

**Proof.** Writing (3.2) for \( p \)-th-derivative, we have
\[
I_2(\sigma, f^{(p)}) \geq \frac{1}{2^s} \left\{ \frac{\log I_2(\sigma, f^{(p-1)})}{\sigma} \right\}^2 I_2(\sigma, f^{(p-1)})
\]
\[
\geq \frac{1}{2^{2^p}} \left\{ \frac{\log I_2(\sigma, f^{(p-1)})}{\sigma} \right\} \left\{ \frac{\log I_2(\sigma, f^{(p-2)})}{\sigma} \right\}^2 I_2(\sigma, f^{(p-2)})
\]
\[
\geq \frac{1}{2^{2^p}} \left\{ \frac{\log I_2(\sigma, f^{(p-1)})}{\sigma} \right\} \left\{ \frac{\log I_2(\sigma, f^{(p-2)})}{\sigma} \right\} \cdots \left\{ \frac{\log I_2(\sigma, f)}{\sigma} \right\} I_2(\sigma, f).
\]
Using (3.1), we get
\[
I_2(\sigma, f^{(p)}) \geq \frac{1}{2^{2^p}} \left\{ \frac{\log I_2(\sigma, f)}{\sigma} \right\}^{2^p} I_2(\sigma, f).
\]

**Corollary.** Let \( f(s) \) be an integral function of linear order \( \rho \) and lower order \( \lambda \), then
\[
(3.4) \quad \lim \sup \inf_{\sigma \to \infty} \left[ \frac{1}{\sigma} \log \left( \frac{I_2(\sigma, f^{(p)})}{I_2(\sigma, f)} \right)^{1/2^{2^p}} \right] \geq \rho,
\]
\[
\lambda.
\]
where $p$ is an integer.

This follows immediately from (3.3).

It is a privilege to thank Dr. S. K. Bose for proposing this problem and giving helpful suggestions in the preparation of this paper.

Department of Mathematics and Astronomy,
Lucknow University,
Lucknow, India.

References