The Conductor of Some Special Points in $P^2$

By

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Abstract

We describe a way of calculating the conductor of some "special" points in $P^2$, which are constructed by complete intersection finite sets of points. As examples, we calculate the conductor of pure configurations in $P^2$. Furthermore, we give a necessary and sufficient condition for a pure configuration to have the Cayley-Bacharach property.

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Introduction

Let $A$ be the homogeneous coordinate ring of a set of $s$ points $X = \{P_1, \cdots, P_s\}$ in $\mathbb{P}^n = \mathbb{P}^n_k$, where $k$ is an algebraically closed field, and let $\overline{A}$ be the integral closure of $A$ in its total quotient ring $Q = Q(A)$, i.e., $\overline{A} \cong \prod_{i=1}^s k[t_i]$, where $k[t_i]$ is isomorphic to the homogeneous coordinate ring of $P_i$. We denote by $C_X$ the conductor of $A$ in $\overline{A}$, namely

$$C_X = \{ \alpha \in \overline{A} \mid \alpha \overline{A} \subset A \}.$$

F. Orecchia [7, Theorem 4.3] showed that

$$C_X = \prod_{i=1}^s e_i \overline{A}^i k[t_i],$$

where $e_i$ is the least degree of any hypersurface which passes through all of $X$ except for $P_i$. Accordingly, we call $e_i$ the degree of conductor of $P_i$ in $X$ and write $d_X(P_i) = e_i$. Also, we refer to $C_X$ as the conductor of $X$. 
In this note, we describe a way of calculating the conductor of some “special” points in $\mathbb{P}^2$, which are constructed by complete intersection finite sets of points (see Theorem 3.1). Theorem 3.1 is viewed as an extension of Cayley-Bacharach Theorem in $\mathbb{P}^2$ (see Remark 2.2 (2)). As examples, we calculate the conductor of pure configurations in $\mathbb{P}^2$ (see Theorem 4.1). Furthermore we give a necessary and sufficient condition for a pure configuration to have the Cayley-Bacharach property (see Corollary 4.7).

1. Preliminaries

Throughout this note, let $k$ be an algebraically closed field. Let $R = k[x_0, x_1, \cdots, x_n]$ be a homogeneous coordinate ring of $\mathbb{P}^n = \mathbb{P}_k^n$ and let $I$ be a homogeneous ideal of $R$. The ring $A = R/I = \bigoplus_{i \geq 0} A_i$ is a graded $k$-algebra of finite type. Hence the dimension of $A_i$ as a $k$-vector space is finite. The Hilbert function of $A$ is defined by $H(A, i) = \dim_k A_i$ for all $i = 0, 1, \cdots$, and the Hilbert series of $A$ is defined by $F(A, \lambda) = \sum_{i \geq 0} H(A, i) \lambda^i \in \mathbb{Z}[[\lambda]]$. We put $d = \dim A$. Then it is well-known that we can write $F(A, \lambda)$ in the form

$$F(A, \lambda) = \frac{h_0 + h_1 \lambda + \cdots + h_s \lambda^s}{(1 - \lambda)^d}$$

for certain integers $h_0, h_1, \cdots, h_s$ satisfying $\sum h_i \neq 0$ and $h_s \neq 0$. We put $s(A) = s$ and $e(A) = \sum_{i=0}^{s(A)} h_i$.

Assume that $A$ is Cohen-Macaulay, and let

$$0 \longrightarrow \bigoplus_{i=1}^{l_g} R(-l_{g,i}) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{l_1} R(-l_{1,i}) \longrightarrow R \longrightarrow A \longrightarrow 0$$

be a minimal free resolution of $A$, where $g = n + 1 - d$. The socle type of $A$ is defined by

$$S(A, \lambda) = \sum_{i \geq 0} (\dim_k [\text{Tor}_g^R(A, k)(g)]_i) \lambda^i, \text{ i.e., } S(A, \lambda) = \sum_{i=1}^{l_g} \lambda^{s(A) - g}.$$ 

It is well-known that $l_{g,i} - g \leq s(A)$. We say that $A$ is level if $l_{g,i} - g = s(A)$ for all $i = 1, \cdots, t_g$. The Cohen-Macaulay type of $A$ is defined by

$$r(A) = \dim_k \text{Tor}_g^R(A, k), \text{ i.e., } r(A) = S(A, 1).$$

Next, let $A$ be the homogeneous coordinate ring of a finite set $X$ of points in $\mathbb{P}^n$, i.e., $A = R/I(X)$, where $I(X)$ is the homogeneous ideal of $X$ generated by \{ $f \in R \mid f$ is homogeneous and $f(P) = 0$ for all $P \in X$ \}. We note that $A$ is an 1-dimensional reduced ring. The Hilbert function, the Hilbert series, the socle type and the Cohen-Macaulay type of $X$ are defined by $H(X, i) = H(A, i)$ for all $i \geq 0$, $F(X, \lambda) = F(A, \lambda)$, $S(X, \lambda) = S(A, \lambda)$ and $r(X) = r(A)$, respectively. We denote by $| X |$ the number of points in $X$ and $\mu(X)$ the
minimal number of generators of $I(X)$. Furthermore we put $e(X) = e(A)$ and $s(X) = s(A)$.

**Remark 1.1.** Let $A$ be the homogeneous coordinate ring of a finite set $X$ of points in $\mathbb{P}^n$, and let $y \in A_1$ be a non zero-divisor. Put $B = A/yA$ and $\text{Soc}(B) = \{ f \in B \mid fg = 0 \text{ for all } g \in \oplus_{i \geq 1} B_i \} = \oplus_{i \geq 0} \text{Soc}(B)_i$. It is well-known that $\text{Soc}(B) \cong \text{Tor}_n^A(A, k)(n)$ as graded $k$-vector spaces. Furthermore, we note that $\text{Soc}(B) \supset B_{s(A)}$ and $\text{Soc}(B)_i = (0)$ for all $i > s(A)$, and it is easy to check that $X$ is level (i.e., $A$ is level) if and only if $\text{Soc}(B) = B_{s(A)}$.

Finally, we shall recall some basic facts about Hilbert functions of points in $\mathbb{P}^n$.

**Proposition 1.2 (cf. [3]).** Let $X$ be a finite set of points in $\mathbb{P}^n$. Then
1. $e(X) = \mid X \mid$.
2. $H(X, i) \leq H(X, i + 1)$ for all $i \geq 0$.
3. $H(X, i) = H(X, i + 1) \Rightarrow H(X, i + 2) = H(X, i)$.
4. $s(X) = \min \{ i \mid H(X, i) = \mid X \mid \}$.
5. $s(Y) = \min \{ i \mid H(X, i) = \mid X \mid \}$.
6. If $Y \subset X$ then $s(Y) \leq s(X)$.

2. The Cayley-Bacharach property

A. V. Geramita, P. Maroscia and L. Roberts gave a simple combinatorial characterization of those sequences $S = \{ b_i \}_{i \geq 0}$ which are the Hilbert function of some set of points in $\mathbb{P}^b$, namely, $S = \{ b_i \}_{i \geq 0}$ is the Hilbert function of some set of points in $\mathbb{P}^b$ if and only if $S$ is a zero-dimensional differentiable O-sequence (cf. [3, Theorem 4.1] for the details).

**Definition (cf. [3]).** Let $S$ be a zero-dimensional differentiable O-sequence. We say that $\zeta$ is a permissible value for $S$ if the sequence $S' = \{ b'_i \}_{i \geq 0}$, where

$$b'_i = \begin{cases} b_i & 0 \leq i < \zeta, \\ b_i - 1 & \zeta \leq i, \end{cases}$$

is a zero-dimensional differentiable O-sequence.

**Remark 2.1.** Let $X$ be a finite set of points in $\mathbb{P}^n$ and let $P \in X$. We can check that the degree of conductor of $P$ in $X$, $d_X(P)$ is necessarily a permissible value for $\{ H(X, i) \}_{i \geq 0}$,
i.e.,
\[ H(X \setminus \{ P \}, i) = \begin{cases} H(X, i) & 0 \leq i < d_X(P), \\ H(X, i) - 1 & d_X(P) \leq i. \end{cases} \]
Also we have \( d_X(P) \leq s(X) \) for all \( P \in X \) (cf. [3, Lemma 3.3]). Furthermore, there exists a point \( P \in X \) such that \( d_X(P) = s(X) \) (cf. [3, Theorem 3.4]).

\textbf{Definition.} Let \( X \) be a finite set of points in \( \mathbb{P}^n \). We say that \( X \) has the \textit{Cayley-Bacharach property} (CBP for short) if \( d_X(P) = s(X) \) for all \( P \in X \).

\textbf{Remark 2.2.} (1) We can easily calculate the Hilbert function of a finite set \( X \) of points in \( \mathbb{P}^1 \), that is
\[ H(X, i) = \begin{cases} i + 1 & 0 \leq i |X|, \\ |X| & |X| \leq i. \end{cases} \]
Hence \( \{H(X, i)\}_{i \geq 0} \) has the unique permissible value \( \zeta = |X| - 1 \). Thus by Remark 2.1, we obtain \( d_X(P) = |X| - 1 \) for all \( P \in X \). Therefore, all finite sets of points in \( \mathbb{P}^1 \) has CBP.

(2) In general, if \( X \) is a finite set of points in \( \mathbb{P}^n \) such that the coordinate ring of \( X \) is Gorenstein, then \( X \) has CBP (cf. [1, Theorem 5]). Hence, all complete intersection finite sets of points in \( \mathbb{P}^n \) has CBP (Cayley-Bacharach Theorem). Therefore if \( X \subseteq \mathbb{P}^n \) is a complete intersection of type \( (a_1, \ldots, a_n) \), i.e., \( X \) is a set of \( a_1 \cdots a_n \) points which is the intersection of hypersurfaces of degree \( a_i \) \((1 \leq i \leq n)\), then \( d_X(P) = s(X) = a_1 + \cdots + a_n - n \) for all \( P \in X \).

The following result tells us the relation between the degree of conductor and the socle type of finite set of points in \( \mathbb{P}^n \).

\textbf{Proposition 2.3.} Let \( X \) be a finite set of points in \( \mathbb{P}^n \), and let \( S(X, \lambda) = \sum_{i=0}^{n} a_i \lambda^i \) be the socle type of \( X \). Then we have
\[ d_X(P) \in \{ i \mid a_i \neq 0 \} \]
for all \( P \in X \).

\textbf{Proof.} We may assume that \( x_0 \) is not a zero-divisor on \( A = R/I(X) \). Put \( B = R/(I(X), x_0) = \bigoplus_{i=0}^{n} B_i \) and \( Y = X \setminus \{ P \} \). By Remark 2.1, we obtain
\[ H(Y, i) = \begin{cases} H(X, i) & 0 \leq i < d_X(P), \\ H(X, i) - 1 & d_X(P) \leq i. \end{cases} \]
Hence we have
\[ \Delta H(Y, i) = \begin{cases} \Delta H(X, i) - 1 & i = d_X(P), \\ \Delta H(X, i) & \text{otherwise}, \end{cases} \]
where \( \Delta H(X, i) \) is the difference function of \( X \) which is defined by
\[ \Delta H(X, i) = H(X, i) - H(X, i - 1) \quad (\text{here } H(X, -1) = 0) \]
and is equal to \( H(B, i) \). Therefore it holds
\[ \dim_k J_i = \begin{cases} 1 & i = d_X(P), \\ 0 & \text{otherwise}, \end{cases} \]
where \( J = \bigoplus J_i \) is the image of \( I(Y) \) in \( B \). Thus, there is an element \( \xi \in J_{d_X(P)} \) such that \( \xi \neq 0 \), and we have \( B_i \xi = (0) \). Hence \( \xi \in \text{Soc}(B) \). This implies our assertion.

The following is clear from Proposition 2.3, so we omit the proof.

**Corollary 2.4.** If \( X \) is level, then \( X \) has CBP.

**Remark 2.5.** In general, the converse of Corollary 2.4 is not true. For example, we consider the following set \( X \) of 7-points in \( \mathbb{P}^2 \)

\[
\begin{array}{ccc}
0 & & 0 \\
0 & & 0 \\
0 & 0 & 0
\end{array}
\]

It is easy to check that \( X \) has CBP and \( S(X, \lambda) = \lambda^2 + \lambda^3 \).

3. An extension of Cayley-Bacharach Theorem in \( \mathbb{P}^2 \)

Let \( g, g' \in R = k[x_0, x_1, x_2] \). We write \( g \mid g' \) if \( g' \in gR \) and \( \deg g < \deg g' \).

The main theorem of this note is the following.

**Theorem 3.1.** Let \( Y_1, \ldots, Y_t \) be finite sets of points in \( \mathbb{P}^2 \) which are complete intersection, i.e., there exist forms \( g_i, h_i \in R = k[x_0, x_1, x_2] \) such that \( I(Y_i) = (g_i, h_i)(1 \leq i \leq t) \). Put \( X = \bigcup_{i=1}^t Y_i \) and \( s(Y_i) = \deg g_i + \deg h_i - 2 \) for all \( i = 1, \ldots, t \). Assume that \( g_{i+1} \mid g_i \) and \( h_{i+1} \mid h_i \) for all \( i = 1, \ldots, t-1 \), and \( \gcd\{g_1, h_1\} = 1 \). Then we have
\[
\Delta_X(P) = \max_{1 \leq i \leq t} \{s(Y_i) \mid P \in Y_i\}.
\]
for all $P \in X$.

We need some lemmas to prove Theorem 3.1.

**Lemma 3.2.** Let $X$ be a finite set of points in $P^n$, let $g \in R = k[x_0, x_1, \ldots, x_n]$ be a homogeneous polynomial and put $Z = \{P \in X \mid g(P) \neq 0\}$ and $X \setminus Z = \{P \in X \mid P \notin Z\}$. Then we have the following.

1. $d_X(P) \leq d_Z(P) + \deg g$ for all $P \in Z$.
2. If $I(X \setminus Z) = (I(X), g)$, then $d_X(P) = d_Z(P) + \deg g$ for all $P \in Z$.

**Proof.** Let $P \in Z$ and let $f$ be a homogeneous polynomial such that $\deg f = d_Z(P)$, $f(Q) = 0$ for all points $Q \in Z \setminus \{P\}$ and $f(P) \neq 0$. Since $g(Q) = 0$ for all $Q \in X \setminus Z$, $f g(Q) = 0$ for all $Q$ of $X$ except for $P$. Furthermore since $f(P) \neq 0$ and $g(P) \neq 0$, we have $f g(P) \neq 0$. Hence $\deg f g \geq d_X(P)$. This implies the assertion of (1).

Next, let $P \in Z$ and let $f$ be a homogeneous polynomial such that $\deg f = d_X(P)$, $f(Q) = 0$ for all points $Q$ of $X$ except for $P$ and $f(P) \neq 0$. Since $f(Q) = 0$ for all $Q \in X \setminus Z$, we have $f \in I(X \setminus Z) = (I(X), g)$. Hence, $f = qg + r$ for some $q \in R$ and $r \in I(X)$. If $q(Q) = 0$ for all points $Q$ of $Z$ except for $P$ and $q(P) \neq 0$, then $\deg q \geq d_Z(P)$. By noting $d_X(P) = \deg f = \deg q + \deg g$, we have $d_X(P) \geq d_Z(P) + \deg g$. So, it is enough to show that $q(Q) = 0$ for all points $Q$ of $Z$ except for $P$ and $q(P) \neq 0$. Let $Q \in Z$ and $Q \neq P$. Since $f(Q) = 0$ and $r(Q) = 0$, we have $qg(Q) = 0$. Furthermore since $Q \in Z$, $g(Q) \neq 0$. Hence $q(Q) = 0$. Also, since $f(P) \neq 0$ and $r(P) = 0$, $qg(P) \neq 0$. Furthermore since $P \in Z$, $g(P) \neq 0$. Hence $q(P) \neq 0$. This completes the proof of (2).

**Lemma 3.3.** With the same notations as in Theorem 3.1, we have the following.

1. The set $\{g_1, g_2 h_1, g_3 h_2, \ldots, g_t h_{t-1}, h_t\}$ is a minimal generating set for the ideal $I(X)$ of $X$.
2. $Y_i \not\subseteq \bigcup_{j \neq i} Y_j$ for all $i = 1, \ldots, t$.

**Proof.** (1) Obviously, $I(X) = \cap_{i=1}^t (g_i, h_i)$. Put

$$I = (g_1, g_2 h_1, g_3 h_2, \ldots, g_t h_{t-1}, h_t).$$

Let $P \in Y_j$, i.e., $g_j(P) = 0$ and $h_j(P) = 0$. Then we have $g_i h_{i-1}(P) = 0$ for all $i \leq j$ in view of $g_{i+1} \mid g_i$. On the other hand, we have, for all $i > j$, $g_i h_{i-1}(P) = 0$ by noting $h_i \mid h_{i+1}$. This implies that $I(X) \supset I$. Next, we show that $I(X) \subset I$. We prove this by induction on $t$. Our assertion is true for $t = 1$. Let $t > 1$. By the assumption of induction,
we have
\[ \bigcap_{i=1}^{t-1} I(Y_i) = (g_1, g_2 h_1, \ldots, g_{t-1} h_{t-2}, h_{t-1}). \]

For any element \( f \in I(X) \), we can write \( f \) in the forms
\[ f = g_1 a_1 + g_2 h_1 a_2 + \cdots + g_{t-1} h_{t-2} a_{t-1} + h_{t-1} a_t \]
or
\[ f = gb + htc \]
for certain elements \( a_i, b, c \in R \). Hence we have
\[ g_i(g'_1 a_1 + g'_2 h_1 a_2 + \cdots + g'_{t-1} h_{t-2} a_{t-1} - b) = h_{t-1}(h'_c - a_t), \]
where \( g_i = g'_i g_i \) and \( h_t = h'_t h_{t-1} \). Thus, since \( g.c.d\{g_i, h_{t-1}\} = 1 \), we have \( h'_c - a_t \in g_i R \), i.e., \( a_t = g_i u + h'_c c \) (\( u \in R \)). Hence
\[ f = g_1 a_1 + g_2 h_1 a_2 + \cdots + g_{t-1} h_{t-2} a_{t-1} + g_t h_{t-1} a_t + h_{t-1} u + h_t c \in I. \]

Next we show that \( \mu(X) = t + 1 \). If
\[ g_t \in (g_2 h_1, \ldots, g_{t-1} h_{t-1}, h_t) \]
or
\[ h_t \in (g_1, g_2 h_1, \ldots, g_{t-1} h_{t-1}), \]
then we have \( h_t | g_t \) or \( g_t | h_t \), a contradiction. Assume that
\[ g_i h_{i-1} \in (g_1, g_2 h_1, \ldots, g_{i-1} h_{i-2}, g_{i+1} h_i, \ldots, g_{t-1} h_{t-1}, h_t) \]
for some \( i \) (\( 1 < i < t \)). Then there exist \( a_i \in R \) such that
\[ g_i h_{i-1} = g_1 a_1 + g_2 h_1 a_2 + \cdots + g_{i-1} h_{i-2} a_{i-1} + g_{i+1} h_i a_i + \cdots + g_{t-1} h_{t-1} a_{t-1} + h_t a_t. \]
Hence we have
\[ g_i h_{i-1} = g_i u + h_i v, \]
where \( g_j = g'_j g_{i-1} \) for all \( j \leq i-2 \), \( h_j = h'_j h_i \) for all \( j \geq i+1 \), \( u = g'_i a_1 + g'_2 h_1 a_2 + \cdots + g'_{i-1} h_{i-2} a_{i-1} \) and \( v = g_{i+1} a_i + g_{i+2} h'_{i+1} a_{i+1} + \cdots + h'_i a_t \). Thus
\[ g_i(h_{i-1} - g'_{i-1} u) = h_i v, \]
where \( g_{i-1} = g'_{i-1} g_i \). Since \( g.c.d\{g_i, h_i\} = 1 \), we have \( h_{i-1} - g'_{i-1} u \in h_i R \), i.e., \( h_{i-1} - g'_{i-1} u = h_i r \) for some \( r \in R \). Hence
\[ h_{i-1}(1 - h'_i r) = g'_{i-1} u, \]
where \( h_i = h_{i-1} h'_i \). Thus since \( g.c.d\{h_{i-1}, g'_{i-1}\} = 1 \), we have
\[ 1 - h'_i r \in g'_{i-1} R. \]
Therefore \( 1 \in (h'_i, g'_{i-1}) R \). But, since \( g'_{i-1} \) and \( h'_i \) are homogeneous polynomials with positive degree, \( 1 \not\in (g'_{i-1}, h'_i) \), a contradiction.

(2) Put \( X' = \bigcup_{j \neq i} Y_j \). We show that \( I(X') \neq I(X) \). From (1), we have \( g_{i+1} h_{i-1} \in I(X') \). Assume that \( I(X') = I(X) \). Hence \( g_{i+1} h_{i-1} \in I(X) \). By noting \( I(X) \subset I(Y_i) \), we can write
\[ g_{i+1} h_{i-1} = g_i u + h_i v, \]
where \( u, v \in R \). Hence \( g_{i+1}(h_{i-1} - g'_i u) = h_i v, \) where \( g_i = g'_i g_{i+1} \).
Thus, since \(g.c.d.\{g_{i+1}, h_i\} = 1\), we have \(h_{i-1} - g'_{i}u \in h_iR\), i.e., \(h_{i-1} - g'_{i}u = h_ir\) \((r \in R)\).

Hence, \(h_{i-1}(1 - h'_{i}r) = g'_{i}u\), where \(h_i = h'_{i}h_{i-1}\). Thus \(1 - h'_{i}r \in g'_{i}R\). Therefore \(1 \in \langle g'_{i}, h'_i \rangle\).

But, since \(g'_{i}\) and \(h'_i\) are homogeneous polynomials with positive degree, \(1 \not\in \langle g'_{i}, h'_i \rangle\), a contradiction.

**Lemma 3.4.** With the same notations as in Theorem 3.1, we put \(Z = \{P \in X \mid g_i(P) \neq 0\}\) and \(W = \{P \in X \mid h_i(P) \neq 0\}\) for some \(i\). Then we have \(I(X\setminus Z) = (I(X), g_i)\) and \(I(X\setminus W) = (I(X), h_i)\).

**Proof.** Obviously, \(I(X\setminus Z) \supseteq (I(X), g_i)\). By noting Lemma 3.3 and \(g_{j+1} | g_j\), we have

\[
I(Y_i \cup \cdots \cup Y_i) = (g_i, g_{i+1}h_i, \ldots, g_ih_{i-1}, h_i) = (I(X), g_i).
\]

Since \(X\setminus Z \supset Y_i \cup \cdots \cup Y_i\), we have \(I(X\setminus Z) \subseteq I(Y_i \cup \cdots \cup Y_i)\). Hence \(I(X\setminus Z) \subseteq (I(X), g_i)\).

Thus \(I(X\setminus Z) = (I(X), g_i)\). The proof of the equality \(I(X\setminus W) = (I(X), h_i)\) is the same as above. We note that \(X\setminus Z = Y_i \cup \cdots \cup Y_i\) and \(X\setminus W = Y_i \cup \cdots \cup Y_i\).

**Lemma 3.5.** With the same notations as in Lemma 3.4, let \(Y_j' (1 \leq j < i)\) be subsets of \(Y_j\) defined by \(g_j', h_j\), where \(g_j = g'_{j}g_i\). Furthermore let \(Y_j' (i < j \leq t)\) be subsets of \(Y_j\) defined by \(g_j, h'_j\), where \(h_j = h'_jh_i\). Then \(Z = \bigcup_{j=1}^{t} Y_j'\) and \(W = \bigcup_{j=i+1}^{t} Y_j'\).

**Proof.** Let \(P \in Z\), i.e., \(g_i(P) \neq 0\). Hence, since \(g_j(P) \neq 0\) for all \(j \geq i\), we have \(P \not\in Y_j\) for all \(j \geq i\). Therefore \(P \in Y_j\) for some \(j\), where \(1 \leq j < i\). Since \(g_j(P) = 0\) and \(g_i(P) \neq 0\), we have \(g'_j(P) = 0\). Thus \(P \in Y_j', i.e., Z \supset \bigcup_{j=1}^{i-1} Y_j'\). Let \(P \in \bigcup_{j=1}^{i-1} Y_j'\), i.e., \(P \in Y_j'\) for some \(j\) \((1 \leq j < i)\). Hence \(g'_j(P) = 0\). We note that \(Y_j\) is the set of \((\deg g_j)(\deg h_j)\) distinct points. Hence, by Bezout's Theorem, the intersection number of \(C_1\) and \(C_2\) at \(P\) is one, where \(C_1\) and \(C_2\) are the curves defined by \(g_j\) and \(h_j\), respectively. Thus, since \(g'_j(P) = 0\) and \(g_j = g'_{j}g_i\), the intersection number of \(C_1'\) and \(C_2\) at \(P\) is zero, where \(C_1'\) is the curve defined by \(g_j\). Therefore \(g_i(P) \neq 0\), i.e., \(Z \supset \bigcup_{j=1}^{i-1} Y_j'\). Hence \(Z = \bigcup_{j=1}^{i-1} Y_j'\).

The proof of \(W = \bigcup_{j=i+1}^{t} Y_j'\) is the same as above.

**Lemma 3.6.** With the same notations as in Theorem 3.1, we have

\[
s(X) = \max\{s(Y_i) \mid 1 \leq i \leq t\}.
\]

**Proof.** We use induction on \(t\). By Remark 2.2 (2), our assertion is true for \(t = 1\). Let
$t > 1$. By Lemma 3.3, we have

$$I(Y_1 \cup \cdots \cup Y_{t-1}) = (g_1, g_2 h_1, \ldots, g_{t-1} h_{t-2}, h_{t-1}).$$

Hence, by $g_i \mid g_t$ for all $1 \leq i \leq t - 1$ and $h_{t-1} \mid h_t$, we have

$$I(Y_1 \cup \cdots \cup Y_{t-1}) = I(Y_t) = (g_t, h_{t-1}).$$

Thus we obtain the following exact sequence

$$0 \longrightarrow R/I(X) \longrightarrow R/I(Y_1 \cup \cdots \cup Y_{t-1}) \oplus R/I(Y_t) \longrightarrow R/(g_t, h_{t-1}) \longrightarrow 0.$$

Hence, in view of Proposition 1.2 (5) and Remark 2.2 (2), we obtain

$$s(X) \leq \max \{s(Y_1 \cup \cdots \cup Y_{t-1}), s(Y_t), s(R/(g_t, h_{t-1}))\}.$$

From the assumption of induction, we have

$$s(Y_1 \cup \cdots \cup Y_{t-1}) = \max \{s(Y_i) \mid 1 \leq i \leq t - 1\}.$$

Furthermore we can check that

$$s(Y_{t-1}) \geq \deg g_t + \deg h_{t-1} - 2 = s(R/(g_t, h_{t-1})).$$

Thus, we have

$$s(X) \leq \max \{s(Y_i) \mid 1 \leq i \leq t\}.$$

On the other hand, from Proposition 1.2 (6), we have $s(Y_i) \leq s(X)$ for all $i = 1, \ldots, t$.

Hence we obtain

$$s(X) = \max \{s(Y_i) \mid 1 \leq i \leq t\}.$$

We now start to prove Theorem 3.1.

**Proof of Theorem 3.1.** We use induction on $t$. By Remark 2.2 (2), our assertion is true for $t = 1$. Let $t > 1$. By Lemma 3.6, there exists an integer $j$ such that $s(X) = s(Y_j)$. Since $d_X(P) \geq d_{Y_j}(P)$ for all $P \in Y_j$, we have $d_X(P) = s(Y_j)$ for all $P \in Y_j$ by Remark 2.1.

Put $Y = \{P \in X \mid P \not\in Y_j\}$, $Z = \{P \in X \mid g_j(P) \neq 0\}$ and $W = \{P \in X \mid h_j(P) \neq 0\}$. Obviously $Y = Z \cup W$. By Lemma 3.4, we have $I(X \setminus Z) = (I(X), g_j)$ and $I(X \setminus W) = (I(X), h_j)$. Hence, by Lemma 3.2, we have, for all $P \in X$,

$$d_X(P) = \begin{cases} 
    d_Z(P) + \deg g_j & \text{if } P \in Z, \\
    d_W(P) + \deg h_j & \text{if } P \in W.
\end{cases}$$
Thus by Lemma 3.5 and by the assumption of induction, we have, for all $P \in X$,
\[
d_X(P) = \begin{cases} 
\max_{1 \leq i < j} \{s(Y_i') \mid P \in Y_i'\} + \deg g_j & \text{if } P \in Z, \\
\max_{i \leq j \leq t} \{s(Y_i') \mid P \in Y_i\} + \deg h_j & \text{if } P \in W.
\end{cases}
\]
Therefore it follows that for all $P \in X$,
\[
d_X(P) = \max_{1 \leq i \leq t} \{s(Y_i) \mid P \in Y_i\}.
\]
This completes the proof.

The following is clear from Theorem 3.1, so we omit the proof.

**Corollary 3.7.** With the same notations of Theorem 3.1, if $s(Y_i) = s(Y_{i+1})$ for all $i = 1, \ldots, t - 1$, then $X$ has CBP.

4. Examples

Let $R = k[x_0, x_1, x_2]$ be a homogeneous coordinate ring of $\mathbb{P}^2$. In this section, we calculate the conductor of pure configurations in $\mathbb{P}^2$.

**Definition** (cf. [5]). A pure configuration in $\mathbb{P}^2$ is a finite set $X$ of points in $\mathbb{P}^2$ which satisfies the following conditions:

There exist distinct elements $c_1, \ldots, c_u \in k$ such that

(i) $X$ is the disjoint union of $X \cap L_1, \ldots, X \cap L_u$, where $L_i$ is the line defined by $x_2 - c_i x_0 = 0$.

(ii) $\varphi(X \cap L_i) \supset \varphi(X \cap L_{i+1})$ for all $i = 1, \ldots, u - 1$, where $\varphi : \mathbb{P}^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{P}^1$ is the map defined by sending the point $(a_0, a_1, a_2)$ to the point $(a_0, a_1)$.

We put $d_i = \dim X \cap L_i$ for all $i = 1, \ldots, u$. By the condition (ii), we have $d_1 \geq \cdots \geq d_u$. The type of $X$ is defined by $\text{type}(X) = (d_1, \ldots, d_u)$ and we write $X = X(d_1, \ldots, d_u)$.

If $d_1 > d_u$, then we put $r_1 = \min\{j \mid d_j > d_{j+1}\}$, and, inductively, if $d_{r_{i+1}} > d_u$, then $r_{i+1} = \min\{j > r_i \mid d_j > d_{j+1}\}$. If $d_1 = d_u$, then we put $r_1 = u$ Furthermore we denote by $t(X)$ the number of distinct natural numbers in $\{d_1, \ldots, d_u\}$. The $r$-type of $X$ is defined by $(r_1, \ldots, r(t(X)))$, where $r(t(X)) = u$.

Let $\{b_1, \ldots, b_d\}$ be the $x_1$-coordinates of points in $X \cap L_1$, and let $L'_j$ be the line defined by $x_1 - b_j x_0 = 0$ for all $j = 1, \ldots, d_1$. Note that $X$ is the disjoint union of $X \cap L'_1, \ldots, X \cap L'_d$. We may assume that $|X \cap L'_j| \geq |X \cap L'_{j+1}|$ for all $j = 1, \ldots, d_1 - 1$. 

The Conductor of Some Special Points in $P^2$

Put
\[ g_i = \prod_{j=1}^{d_i} (x_1 - b_j x_0) \quad \text{and} \quad h_i = \prod_{j=1}^{r_i} (x_2 - c_j x_0) \]
for all $i = 1, \ldots, t$ ($= t(X)$). Furthermore we put
\[ Y_i = \{ P \in X \mid g_i(P) = 0 \text{ and } h_i(P) = 0 \} \]
for all $i = 1, \ldots, t$ ($= t(X)$), and we call $Y_1, \ldots, Y_t$ the CI-subsets of $X$.

The following theorem can be proved by using Theorem 3.1. So we omit the proof.

**Theorem 4.1.** Let $X = X(d_1, \ldots, d_n)$ be a pure configuration in $P^2$, $(r_1, \ldots, r_t)$ be the $r$-type of $X$ where $t = t(X)$ and let $Y_1, \ldots, Y_t$ be the CI-subsets of $X$. Then we have
\[ d_X(P) = \max \{ d_i + r_i - 2 \mid P \in Y_i \} \]
for all $P \in X$.

**Example 4.2.** Let $X$ be the following set of 38 points in $P^2$

```
  o  
  o  
  o  o  
  o  o  
  o  o  
  o  o  o  o  o  o  o  o  o  o  
  o  o  o  o  o  o  o  o  o  
  o  o  o  o  o  o  o  
```

Then we have $X = X(9,8,8,3,3,2,2)$, and the $r$-type of $X$ is $(1,3,6,8)$. Furthermore we obtain

\[
\begin{align*}
i : & \quad 1 \quad 2 \quad 3 \quad 4 \\
r_i : & \quad 1 \quad 3 \quad 6 \quad 8 \\
d_{r_i} : & \quad 9 \quad 8 \quad 3 \quad 2 \\
d_i + r_i - 2 : & \quad 8 \quad 9 \quad 7 \quad 8 .
\end{align*}
\]

Also, the CI-subsets $Y_1, Y_2, Y_3, Y_4$ of $X$ are as follows.
Accordingly we can calculate the degree of conductor of all points in $X$ as follows.

\[ d_X(P) = \begin{cases} 
7 & \text{for 3 points such that } P \in Y_3, P \notin Y_2 \text{ and } P \notin Y_4, \\
8 & \text{for 11 points such that } P \in Y_1 \cup Y_4 \text{ and } P \notin Y_2, \\
9 & \text{for 24 points } P \in Y_2.
\end{cases} \]

Consequently we have

\[ C_X = \prod_{i=1}^{3} t_i^3 k[t_i] \prod_{i=4}^{14} t_i^8 k[t_i] \prod_{i=15}^{38} t_i^9 k[t_i]. \]

The following is clear from Theorem 4.1, so we omit the proof.

**Corollary 4.3.** Let $X$ and $X'$ be pure configurations in $\mathbb{P}^2$. If $\text{type}(X) = \text{type}(X')$ then $C_X = C_{X'}$.

**Example 4.4.** Let $X'$ be the following set of 38 points in $\mathbb{P}^2$

\[
\begin{array}{ccccccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

Then we have $X' = X'\langle 9, 8, 8, 3, 3, 3, 2, 2 \rangle$. Thus by Corollary 4.3, we obtain

\[ C_{X'} = \prod_{i=1}^{3} t_i^3 k[t_i] \prod_{i=4}^{14} t_i^8 k[t_i] \prod_{i=15}^{38} t_i^9 k[t_i]. \]
Next, we recall some results about pure configurations in $\mathbb{P}^2$ from [5].

**Definition.** Let $\tau_1 = \{\tau_{1,i}\}, \cdots, \tau_m = \{\tau_{m,i}\}$ be sequences of non-negative integers. We denote by $H^{(\tau_1, \cdots, \tau_m)}$ the sequence obtained as follows.

Write down the sequences $\tau_1, \cdots, \tau_m$, successively shifted to the right and add:

\[
\begin{array}{cccc}
\tau_m : & \tau_{m,0}, & \cdots \\
\tau_{m-1} : & \tau_{m-1,0}, & \tau_{m-1,1}, & \cdots \\
& \cdots & \cdots & \cdots \\
\tau_2 : & \tau_{2,0}, & \tau_{2,1}, & \cdots & \cdots \\
\tau_1 : & \tau_{1,0}, & \tau_{1,1}, & \tau_{1,2}, & \cdots & \cdots \\
\end{array}
\]

Hence

\[
H^{(\tau_1, \cdots, \tau_m)}(i) = \sum_{j=1}^{m} \tau_{j,i+1-j}, \text{ where } \tau_{j,i} = 0 \text{ for } l < 0.
\]

**Theorem 4.5** (cf. [5, Theorem 3.1]). Let $X = X(d_1, \cdots, d_u)$ be a pure configuration in $\mathbb{P}^2$, and $(r_1, \cdots, r_t)$ be the $r$-type of $X$ where $t = t(X)$. For all $i = 1, \cdots, u$, let $\tau_i$ be the sequence $1, 2, \cdots, d_i$. Then

1. $H(X) = H^{(\tau_1, \cdots, \tau_m)}$.
2. $\mu(X) = t + 1$.
3. $S(X, \lambda) = \sum_{i=1}^{t} \lambda^{d_i+r_i-2}$.
4. $r(X) = t$.
5. $X$ is level if and only if $d_i + r_i = d_{i+1} + r_{i+1}$ for all $i = 1, \cdots, t - 1$.

**Remark 4.6.** Let $X$ be a finite set of points in $\mathbb{P}^2$ and $S(X, \lambda) = \sum_{i=0}^{t(X)} a_i \lambda^i$ the socle type of $X$. In general, for an integer $i$ such that $a_i \neq 0$, it does not necessarily exist a point $P \in X$ such that $d_X(P) = i$. For example, see Remark 2.5. But it follows from Lemma 3.3 (2) and Theorem 4.1 that if $X$ is a pure configuration, then for each $i$ ($1 \leq i \leq t$), there exists a point $P \in X$ such that $d_X(P) = d_i + r_i - 2$.

**Corollary 4.7.** Let $X = X(d_1, \cdots, d_u)$ be a pure configuration in $\mathbb{P}^2$ and $(r_1, \cdots, r_t)$ be the $r$-type of $X$. Then $X$ has CBP if and only if $d_i + r_i = d_{i+1} + r_{i+1}$ for all $i = 1, \cdots, t - 1$. 
PROOF. The assertion follows from Theorem 4.1 and Remark 4.6.

Example 4.8. Let $X$ be the following set of points in $\mathbb{P}^2$

\[
\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

Then we have $X = X(5, 5, 4, 2, 2)$, and the r-type of $X$ is $(2, 3, 5)$. Therefore, we obtain

\[
i : \quad 1 \quad 2 \quad 3 \\
r_i : \quad 2 \quad 3 \quad 5 \\
d_{ri} : \quad 5 \quad 4 \quad 2 \\
d_{ri} + r_i : \quad 7 \quad 7 \quad 7.
\]

Thus by Corollary 4.7, $X$ has CBP in this case.

References


