J. Math. Tokushima Univ.
Vol. 28 (1994), 29–37

Harmonic Maps of Nonorientable Surfaces into Complex Grassmann Manifolds

By

Toru ISHIHARA
*Department of Mathematical Sciences,
Faculty of Integrated Arts and Sciences,
The University of Tokushima
Tokushima 770, JAPAN*
(Received September 14, 1994)

Abstract

We study harmonic maps of nonorientable surfaces into complex Grassmannians. J. C. Wood coded a factorable harmonic maps of a Riemann surfaces into a complex Grassmannian by a sequence of holomorphic maps of the surface into Grassmannians. We investigate codes of harmonic maps of nonorientable surfaces. Especially the degrees of codes are studies.

1991 *Mathematics Subject Classification.* Primary 53E20.

§1. Introduction

The construction of all harmonic maps from the two-sphere to a complex Grassmannian was discussed by many authors(see, for example, [3,5,9,10, 11, 12]). Especially, J. C. Wood [12] gave the explicit construction of all harmonic maps $S^2 \to G_k(C^n)$. In the present paper, we shall invesitigate harmonic maps of nonorientable surfaces into a complex Grassmann manifold $G_k(C^n)$. We deal with a nonorientable surface $\underline{M}$ which is a quotient of a Riemann surface M by the equivalence relation $z \sim w$ if and only if $w = I(z)$, where I is an anti-holomorphic involution of M with no fixed point. Let $\pi : M \to N$ be the natural projection. A necessary and sufficient condition for a map ϕ of M into a manifold N to be factored as $\phi = \underline{\phi} \cdot \pi$, where $\underline{\phi}$ is a map of $\underline{M}$ into N, is that, $\phi(I(p)) = \phi(p)$ for each $p \in M$. Let g be a Riemannian metric compatible with the conformal structure of M. Then there exists a Riemannian structure $\underline{g}$ on $\underline{M}$ such that π is locally isometric. The assignment $\phi \to \underline{\phi}$ is a bijective correspondence between harmonic maps $\phi : M \to N$ with $\phi \cdot I = \phi$ and hrmonic maps $\underline{\phi} : \underline{M} \to N$. We study harmonic maps $\phi : M \to G_k(C^n)$ with $\phi \cdot I = \phi$.

If $M = S^2$, we identify S^2 with $C \cup \{\infty\}$. The antipodal map is an involution given by $I(z) = -1/z$. The quotient space is the projective plane P^2. If a harmonic map $\phi : S^2 \to S^{2m}$ satisfies $\phi \cdot I = \pm\phi$, there corresponds the map $\underline{\phi} : P^2 \to P^{2m}$. N. Egiri [6] studied these maps.

Let $\phi : M \to G_k(C^n)$ be a harmonic map and $\tilde{\phi}$ be obtained by some reduction/extensin. In §2, we show $\tilde{\phi} \cdot I$ is obtained from $\phi \cdot I$ by the naturally corresponding reduction/extension. J. C. Wood [12] coded a factorable harmonic map $\phi : M \to G_k(C^n)$ by a sequence of holomorphic maps from M into complex Grassmannians. In §3, we shall get the necessary and sufficient conditions about codes under which harmonic maps ϕ satisfy $\phi \cdot I = \phi$. In the final section, we investigate the degrees of the codes of harmonic maps ϕ with $\phi \cdot I = \phi$.

§2. Preliminaries

For the definition and basic properties of harmonic maps into a complex Grassmannian, see [4, 11, 12]. For any integers n, k with $0 \leq k \leq n$, let $G_k(C^n)$ be the Grassmannian of complex k-dimensional subspaces of C^n with its standard Kahler structure. Let M be a Riemann surface with antiholomorphic involution I. We identify a smooth map $\phi : M \to G_k(C^n)$ with the smooth complex subbundle $\underline{\phi}$ of *rankk* of the trivial bundle $\underline{C}^n$. Denote by $\underline{\phi \cdot I}$ the subbundle with fibre $\underline{\phi \cdot I}_x = \underline{\phi}_{I(x)}$. Let $\underline{\phi}^\perp$ be the orthogonal complement of $\underline{\phi}$. It is evident that $(\underline{\phi \cdot I})^\perp = \underline{\phi}^\perp \cdot I$. Let ∂ denote the flat connection on $\underline{C}^n$ and $<,>$ the standard Hermitian metric. A subbundle $\underline{\phi}$ inherits a metric $<,>_\phi$ and connection ∇_ϕ by $< v, w >_\phi = < v, w >, v, w \in \underline{\phi}_x, x \in M$ and $(\nabla_\phi)_w V = \pi_\phi \cdot \partial_w V, w \in TM, V \in C^\infty(\underline{\phi})$. We give $\underline{\phi}$ its Koszul-Malgrange holomorphic structure. It is known that $\underline{\phi}$ is a holomorohic (resp. antiholomorhic) subbundle of $\underline{C}^n$ if and only if ϕ is a holomorphic (resp. antiholomorphic) map. Moreover, it is called harmonic if ϕ is a harmonic map.

Let $\underline{\phi}$ and $\underline{\psi}$ be mutually orthogonal subbundle of $\underline{C}^n$. Denote by $\underline{\phi} \oplus \underline{\psi}$ the subbundle with fibre $\underline{\phi}_x + \underline{\psi}_x$ at $x \in M$. Let $\tilde{A}'_{\phi,\psi} : T^{1,0}M \otimes \underline{\phi} \to \underline{\psi}$ be the global ∂'-second fundamental form of $\underline{\phi}$ in $\underline{\phi} \oplus \underline{\psi}$. Then, $(\tilde{A}'_{\phi,\psi})_w v = \pi\psi \cdot \partial_w V$, where $w \in T^{1,0}M, v \in \underline{\phi}_x$ and V is a smooth extension of v. The global ∂''-second fundamental form $\tilde{A}''_{\phi,\psi} : T^{0,1}M \otimes \underline{\phi} \to \underline{\psi}$ is defined similarly. Choose a local holomorphic vector field Z on M, for example $Z = \partial/\partial z$ for some local complex coordinate z and denote the representatives $(\tilde{A}'_{\phi,\psi})_Z$ and $(\tilde{A}''_{\phi,\psi})_{\bar{Z}}$ by $A'_{\phi,\psi}$ and $A''\phi, \psi$ respectively, which are again called ∂'- and ∂''- second fundamental forms. Particularly, the second fundamental forms of $\underline{\phi}$ in $\underline{C}^n$, $\tilde{A}'_\phi = \tilde{A}'_{\phi,\phi^\perp}$ and $\tilde{A}''_\phi = \tilde{A}''_{\phi,\phi^\perp}$ are immportant. These are called the fundamental collineations of $\underline{\phi}$ in [5, 11]. We also put $A'_\phi = (\tilde{A}'_\phi)_Z$ and $A''_\phi = (\tilde{A}''_\phi)_Z$. Note that $\phi : M \to G_k(C^n)$ is holomorphic (resp. antiholomorphic) if and only if $A''_\phi = 0$ (resp. $A'_\phi = 0$. Moreover, ϕ is harmonic if and only if $A'_\phi : \underline{\phi} \to \underline{\phi}$ is holomorphic or $A''_\phi : \underline{\phi} \to \underline{\phi}^\perp$ is antiholomorphic (see [4,12]).

Suppose that $\phi : M \to G_k(C^n)$ is harmonic. Then the ∂'-Gauss (resp. ∂''-Gauss) bundle $G'(\phi)$(resp. $G''(\phi)$ is the holomorphic subbundle $\underline{ImA'_\phi}$ of $\underline{\phi}$ (resp. antiholomorphic subbundle $\underline{ImA''_\phi}$ of $\underline{\phi}$. These bundes are harmonic. We define $G^{(i)}(\phi), (i \in Z)$ by $G^{(0)}(\phi) = \underline{\phi}$, $G^{(i)} = G'(G^{(i-1)})(\phi), (i \geq 1)$, $G^{(-i)} =$

$G''(G^{(-i+1)}), (i \geq 1)$.

If $\underline{\phi}$, $\underline{\psi}$ are subbundle of $\underline{C}^n$ with $\underline{\psi} \subset \underline{\phi}$. Then $\underline{\psi}^\perp \cap \underline{\phi}$ is denoted by $\underline{\phi} \ominus \underline{\phi}$. Let $\phi : M \to G_k(C^n)$ be harmonic. Assume that $\underline{\alpha} \subset \underline{\phi}$ and $\underline{\beta} \subset \underline{\phi}$ satisfy the ∂'-replacement (resp. ∂''-replacement) conditions, that is, α is holomorphic(resp. antiholomorphic) subbundle of $\underline{\phi}$ and β is holomorphic(resp. antiholomorphic) subbundle of $\underline{\phi}^\perp$ with $A'_\phi(\underline{\alpha}) \subset \underline{\beta}$ and $A'_{\phi^\perp} \subset \underline{\alpha}$ (resp. $A''_\phi(\underline{\alpha}) \subset \underline{\beta}$ and $A''_{\phi^\perp} \subset \underline{\alpha}$. Then $\tilde{\underline{\phi}} = (\underline{\phi} \ominus \underline{\alpha}) \oplus \underline{\beta}$ is harmonic (see Proposition 2.1 in [12]). The transformation $\phi \to \tilde{\phi}$ is called the ∂'-repracement (resp. ∂"-replacement) of the holomorphic (resp. antiholomorphic) subbundle $\underline{\alpha}$ by $\underline{\beta}$.

Proposition 2.1. *Let $\phi : M \to G_k(C^n)$ be harmonic. Put $\psi = \phi \cdot I$. Then ψ is also harmonic. If $\underline{\alpha}(\underline{\alpha} \subset \underline{\phi})$ and $\underline{\beta}(\underline{\beta} \subset \underline{\phi}^\perp)$ satisfy the ∂'-replacement(resp. ∂''-replacement) conditions, then $\underline{\alpha \cdot I}(\underline{\alpha \cdot I} \subset \underline{\phi})$ and $\underline{\beta \cdot I}(\underline{\beta \cdot I} \subset \underline{\phi}^\perp)$ satisfy the ∂''-replacement(resp. ∂'- replacement) conditions, and $\tilde{\underline{\phi}} = (\underline{\phi} \ominus \underline{\alpha}) \oplus \underline{\beta}$ and $\tilde{\underline{\psi}} = (\underline{\psi} \ominus \underline{\alpha \cdot I}) \oplus \underline{\beta \cdot I}$ satisfy $\tilde{\underline{\psi}} = \underline{\phi \cdot I}$.*

Proof. Assume that $\underline{\alpha}$ and $\underline{\beta}$ satisfy the ∂'-replacement conditions. For a local holomorphic vector field Z, we can put $dI(Z) = a\bar{Z}$ and $dI(\bar{Z}) = \bar{a}Z$. For $v \in \underline{\alpha \cdot I_x}$,

$$A''_\psi(v) = \pi_{\psi^\perp(x)} \cdot \partial_{\bar{Z}} V = a\pi_{\phi^\perp(I(x))}\partial_Z(V \cdot I) = aA'_\phi(v),$$

where $V \in C^\infty(\underline{\alpha \cdot I})$ is a smooth extension of v which is also regarded as an element of $\underline{\alpha}_{I(x)}$. As $A'_\phi(v) \in \underline{\beta}_{I(x)} = \underline{\beta \cdot I_x}$, ws we show that $A''_\psi(\underline{\alpha \cdot I}) \subset \underline{\beta \cdot I}$. Similarly we have $A''_{\psi^\perp}(\underline{\beta \cdot I}) \subset \underline{\alpha \cdot I}$. It is evident that $\tilde{\underline{\psi}} = \underline{\phi \cdot I}$. We can show tha dual statement similarly.

Let $\underline{\alpha}$ is a holomorphic subbundle of $ker\underline{A}'_{\phi^\perp}$. Then with $\underline{\beta} = 0$, it satisties the replacement conditions. The resulting harmonic bundles $\tilde{\underline{\phi}} = \underline{\phi} \oplus \underline{\alpha}$ and $\tilde{\underline{\phi}}^\perp = \underline{\phi}^\perp \ominus \underline{\alpha}$ are said to be the extension of ϕ by the holomorphic subbundle $\underline{\phi}^\perp$ by the holomorphic subbundle $\underline{\alpha}$. Similarly for a holomorphic subbundle $\underline{\alpha}$ of $ker\underline{A}'_\phi$, we have harmonic bundles $\tilde{\underline{\phi}} = \underline{\phi} \ominus \underline{\alpha}$ and $\tilde{\underline{\phi}}^\perp = \underline{\phi}^\perp \oplus \underline{\alpha}$, the reduction/ extension by a holomorphic subbundle $\underline{\alpha}$. There a dual notion of reduction/extension by an antiholomoarphic subbundle. J. C. Wood[12] codes these eight types of reduction/extension as follows;

(1) $\tilde{\underline{\phi}} = \underline{\phi}^\perp \ominus \underline{\alpha}$, (2)$\underline{\phi} = \underline{\phi} \oplus \underline{\alpha}$ for a holomorphic subbundle$\underline{\alpha} \subset \underline{kerA'_{\phi^\perp}}$,
(3) $\tilde{\underline{\phi}} = \underline{\phi}^\perp \ominus \underline{\alpha}$, (4)$\underline{\phi} = \underline{\phi} \oplus \underline{\alpha}$ for a antiholomorphic subbundle$\underline{\alpha} \subset \underline{kerA''_{\phi^\perp}}$,
(5) $\tilde{\underline{\phi}} = \underline{\phi} \ominus \underline{\alpha}$, (6)$\underline{\phi} = \underline{\phi}^\perp \oplus \underline{\alpha}$ for a holomorphic subbundle$\underline{\alpha} \subset \underline{kerA'_\phi}$,
(7) $\tilde{\underline{\phi}} = \underline{\phi} \ominus \underline{\alpha}$, (8)$\underline{\phi} = \underline{\phi}^\perp \oplus \underline{\alpha}$ for a holomorphic subbundle$\underline{\alpha} \subset \underline{kerA''_\phi}$.

Moreover, J. C. Wood gave the one-to-one correspondence between holomorphic maps $f : M \to G_s(C^t)$ and reductions/extensions $\phi \to \tilde{\phi}$ of type ζ

and said that $\tilde{\phi}$ is obtained by the teduction/extension coded by (f,ζ). If $\underline{\alpha}$ is a holomorphic subbundle of $\underline{kerA'}_{\phi}$(resp. $\underline{kerA'}_{\phi^\perp}$, $\underline{\alpha \cdot I}$ is an antiholomorphic subbundle of $\underline{kerA''_{\phi \cdot I}}$ (resp. $\underline{kerA''_{\phi \cdot I}}$). The dual fact is also true. Hence we have

Lemma 2.2. *Let $\phi : M \to G_k(C^n)$ be harmonic. Put $\psi = \phi \cdot I$. Let $\underline{\tilde{\phi}}$(resp. $\underline{\tilde{\psi}}$ be obtained by reduction/extension coded by (f,ζ) (resp. $(f \cdot I, \zeta^*)$, where $1^* = 3, 2^* = 4, 3^* = 1, 4^* = 2, 5^* = 7, 6^* = 8, 7^* = 5, 8^* = 6$. Then $\underline{\tilde{\psi}} = \underline{\tilde{\phi} \cdot I}$.*

§3. Factorizations

If a hrmonic map $\phi : M \to G_k(C^n)$ is factorable by reduction and extensions (see Definition 3.1 in [12]), J. C. Wood found a sequence $((f_1,\zeta_1),\cdots,(f_r,\zeta_r))$ of holomorphic maps $f_i : M \to G_{s_i}(C^{t_i})$ and integers $\zeta i \in \{1,\cdots,8\}$ such that aubbundles $\phi_0,\cdots,\phi_r = \phi$ are given iteratively as follows: $\phi_0 = 0$, for $i \geq 1$, ϕ_i is obtained from ϕ_{i-1} or $(\phi_{i-1}^\perp)$ by performing the reduction/extension coded by (f_i,ζ_i)(see Theorem 5.1 in[12]). We shall say that ϕ is coded by the sequence $((f_1,\zeta_1),\cdots,(f_r,\zeta_r))$. From Lemma 2.2, we obtain

Proposition 3.1. *Let $\phi : M \to G_k(C^n)$ be harmonic which is factorable by reduction and extensions. Assume that ϕ is coded by a sequence $((f_1,\zeta_1),\cdots,(f_r,\zeta_r))$. Then ϕ satisfies $\phi \cdot I = \phi$ if and only if ϕ is also coded by the sequence $((f_1 \cdot I,\zeta_1^*),\cdots,(f_r \cdot I,\zeta_r^*))$.*

Let $\phi : M \to G_k(C^n)$ be harmonic. Let $\tilde{\phi}$ be a holomorphic extension of $G'(\phi)$ of a subbundle of *rank* s. (see Example 2.7 in [12]). It is also a reduction of ϕ of *type* 3(see Lemma 2.11 in [12]). Put $t =$ *rank* $\underline{kerA''_{\phi}}$. For any $s \in \{0,1,\cdots,t\}$, there is a cononical one-to-one correspondence between holomorphic maps $f : M \to G_s(C^t)$ and holomorphic extensions $\tilde{\phi}$ of $G'(\phi)$ by subbundle of rank s (see Lemma 4.13 in [12]). In this case, $\tilde{\phi}$ is said to be the holomorphic extension of $G'(\phi)$ coded by f. ϕ is a harmonic map of finite ∂''-order if $G^{(-r)}(\phi) = 0$ for some positive integer r (see Definition 3.7 in [12]). Let S_0 be the set of all sequences $(f_1,\cdots,f_\ell)$ of holomorphic maps $f_i : M \to G_{s_i}(C_i^t)$, where $0 \leq s_i \leq t_i$, which give harmonic subbundles $\underline{\phi}_i$, $0 \leq i \leq \ell$ of $\underline{C}^n$ iteratively by :$\phi_0 = 0$, , ϕ_i obatained from ϕ_{i-1} as the holomorphic extension of $G'(\phi_{i-1}$ coded by f_i and $\underline{kerA'_{\phi_{i-1}}} = 0, 0 \leq i \leq \ell$. Then the assignment $(f_1,\cdots,f_\ell) \to \phi = \phi_\ell$ is a bijection between S_0 and the set of all harmonioc maps $\phi : M \to G_k(C^n)$ of finite ∂''-order(see Theorem 5.2 in [12]). The hrmonic map is said to be coded by the sequence $(f_1,\cdots,f_\ell)$ of holomorphic maps. Dually, we have an antiholomorphic extension of $G''(\phi)$ of *rank* s' coded by an antiholomorphic map $g : M \to G_{s'}(C^{t'})$, where $t' =$ *rank* $\underline{kerA'_{\phi}}$. Moreover, dually all harmonic maps of finite ∂'-order are coded by unique sequences $(g_1,\cdots,g_{\ell'})$ of antiholomorphic maps. Let $\phi : M \to G_k(C^n)$ is harmonic map of finite ∂-order and ∂''-order. Let ϕ be coded by a sequence $(f_1,\cdots,f_\ell)$ of holomorphic maps and coded by a sequence $(g_1,\cdots,g_{\ell'})$

of antiholomorphic maps. We shall call $(g_1, \cdots, g_{\ell'})$ the polar of $(f_1, \cdots, f_\ell)$. If the polar of $(f_1, \cdots, f_\ell)$ is $(f_1 \cdot I, \cdots, f_\ell \cdot I)$, $(f_1, \cdots, f_\ell)$ is said to be symmetric with respect to I. Let S_1 be the subset of S_0 whose elements are all sequences $(f_1, \cdots, f_\ell)$ symmetric with respect to I. We can see with ease that $\underline{\tilde{\phi}}$ is a holomorphic extension of $G'(\phi)$ coded by f if and only if $\tilde{\phi} \cdot I$ is a holomorphic extension of $G''(\phi \cdot I)$ coded by $f \cdot I$. Assume $\phi : M \to G_k(\overline{C^n})$ satisfies $\phi \cdot I = \phi$. Then if *phi* is a harmonic map of finite ∂'-order if and only if it is of ∂''-order. In this case, ϕ is coded by the sequence $(f_1, \cdots, f_\ell)$ symmetric with respect to I. Conversely, if ϕ is coded by the sequence $(f_1, \cdots, f_\ell)$ symmetric with respect to I, it satisfies $\phi \cdot I = \phi$. Thus we have

Theorem 3.2. *The assignment $(f_1, \cdots, f_\ell) \to \phi$ is the bijection between the subset S_1 and the set of all harmonic maps $\phi : M \to G_k(C^n)$ of finite ∂''-order(or ∂'-order) with $\phi \cdot I = \phi$.*

if $M = S^2$, all harmonic maps $\phi : M \to G_k(C^n)$ are of finite ∂''-order. Particularly, for a full harmonic map $\phi : S^2 \to CP^2$, there is a unique holomnorphic map, called the directrix curve such that $\phi = G^{(i)}(f)$, for some i, $0 \leq i \leq n$. $G^{(n)}(f)$ is the polar of f (see [7]). Thus we get

Corollary 3.3. *Let $\phi : S^2 \to CP^n$ be a full harmonic map. Let f be the directrix curve of ϕ. The polar of f is $G^{(n)}(f)$. The harmonic map ϕ satisfies $\phi \cdot I = \phi$ if and only if n is even, that is, $n = 2m$, $\phi = G^{(m)}(f)$ and $G^{(n)}(f) = f \cdot I$.*

Remarks. (1) The author was imformed by N. Egiri that the above corollary has been obtained by J. Bolton, L. Vrancken and L. M. Woodward in [3]. (2) For a harmonic map $\phi : M \to S^{2m}$, N. Egiri[6] showed that ϕ satisfies $\phi \cdot I = \pm\phi$ if and only if its directrix curve f satisties $f(z) = \pm z^{2m}\overline{f(I(z))}$. (3) J. Bolton, G. Jensen, M. Rigoli and L. Woodward investigated harmonic maps of S^2 into CP^n with induced metrics of constant curvature. They determined such harmonic maps ϕ with $\phi \cdot I = \phi$ explicitly(Theorems 5.2 and 5..4 in [2]).

We shall construct some examples of harmonic maps ϕ of S^2 to CP^{2m} with $\phi \cdot I = \phi$. Let $\Xi : S^2 \to CP^{2m}$ be a full holomorphic curve. Consider S^2 covered by isothermal coordinates given by stereographic projections. Then $I(z) = -1/\bar{z}$. We can represent Ξ by a polynomial $\xi : C \to C^{2m+1}$, given by $\xi(z) = \sum a_i z^i$. Particularly, we deal with the polynomials considered by J. Barbosa [1]: $\xi_{km}(z) = a_0 + \sum_{i=1}^{2m-1} a_{k-m+i} z^{k-m+i} + a_{2k} z^{2k}$, $k \geq m$. Let denote by ξ^j_{km} be the j-th derivative of ξ_{km}. Put $\xi^*(z) = a_0 \bar{z}^{2k} + \sum_{i=1}^{2m-1} a_{k-m+i}(-1)^{k-m+i} \bar{z}^{-k+m-i} + a_{2k}$. Let Φ_{km} be the harmonic map such that its directrix curve is ξ and $\Phi_{km} = G^{(m)}(\xi_{km})$. Then $\Phi_{km} \cdot I = \Phi$ if and only if ξ^* is the polar of ξ_{km}, that is, for $j = 0, k-m+1, k-m+2, \cdots, k+m-1$, $< \xi^j_{km}, \xi^* >= 0$, where $<,>$ is the Hermitian inner product of C^{2m+1}. From $< \xi^{k+m-1}_{km}, \xi^* >= 0$, we get $< a_{k+m-1}, a_{2k} >= 0$. Inductively, we have

$< a_i, a_j >= 0, i \neq j$. Moreover, we get that $k-m$ is even and that for $i = 1, \cdots, 2m-1$, $\| a_{k+m-i} \|^2 = \begin{pmatrix} 2k \\ k-m+i \end{pmatrix} \begin{pmatrix} k-m+i-1 \\ k-m \end{pmatrix} \| a_{2k} \|^2$, $\| a_0 \|^2 = (\sum_{i=1}^{2m-1} (-1)^{i+1} \begin{pmatrix} 2k \\ k-m+i \end{pmatrix} \begin{pmatrix} k-m+i-1 \\ k-m \end{pmatrix} - 1) \| a_{2k} \|^2$.

4. The energy and the degree

For any smooth map of a closed Riemann surface M to $G_k(C^n)$, we define the (1,0) and (0,1) energy integral(see [12])

$$E'(\phi) = \int_M | \partial\phi |^2 \, dM, \quad E''(\phi) = \int_M | \bar{\partial}\phi |^2 \, dM.$$

Then they satisfy

$$E'(\phi) - E''(\phi) = 4\pi deg\phi = -4\pi c_1(\phi). \tag{1}$$

Let $\tilde{\phi}$ be obtained by replacing $\underline{\alpha}(\subset \underline{\phi})$ by $\underline{\beta}(\subset \underline{\phi})$ where α, β satisfy the ∂'-replacement conditions. In [12], J. C. Wood got

$$E'(\tilde{\phi}) - E'(\phi) = -4\pi c_1(\beta), E''(\tilde{\phi}) - E''(\phi) = -4\pi C_1(\alpha). \tag{2}$$

Dually, if α, β satisfy the ∂''-replacement conditions, then

$$E'(\tilde{\phi}) - E'(\phi) = 4\pi c_1(\alpha), E''(\tilde{\phi}) - E''(\phi) = 4\pi C_1(\beta). \tag{3}$$

As $\phi^\perp$ is obtained by ∂'-replaceing of ϕ by $\phi^\perp$, we have $E'(\phi^\perp) = E'(\phi) + 4\pi c_1(\phi), E''(\phi^\perp) = E''(\phi) - 4\pi C_1(\phi)$. Since the antiholomorphic involution I is an isometry, we have $E'(\phi \cdot I) = E'(\phi)$, $E''(\phi \cdot I) = E''(\phi)$. Hence we get

Proposition 4.1. *Let $\phi : M \to G_k(C^n)$ be a smooth map with $\phi \cdot I = \phi$. Then $E'(\phi) = E''(\phi)$ and $deg\phi = 0$.*

Let $\tilde{\phi}$ be a holomorphic extension of $G'(\phi)$ coded by a holomorphic map f. Then $G'(\phi) \to \tilde{\phi}$ is the ∂'-replacement of $\alpha = 0$ by $\beta = f$ (see Lemma 2.11 in [12]), $\phi \to G'(\phi)$ is the ∂'-replacement of $\alpha = \phi$ by $\beta = G'(\phi)$. Hence, using (2), we obtain

Lemma 4.2. *Let $\phi : M \to G_k(C^n)$ be a harmonic map. Let $\underline{\tilde{\phi}}$ be a holomorphic extension of $G'(\phi)$coded by a holomorphic map f. Then we have*

$$E'(\tilde{\phi}) = E'(\phi) - 4\pi c_1(G'(\phi)) - 4\pi c_1(\underline{f}), \; E''(\tilde{\phi}) = E''(\phi) - 4\pi c_1(\phi).$$

Let ϕ be ∂'-irreducible. We consider the fundamental collineation $A'_\phi : \phi \to G'(\phi)$. Taking the k-th exterior power of each bundle, we get the holomorhic bundle map $detA'_\phi : \bigwedge^k \phi \to \bigwedge^k G'(\phi)$ of line bundles. Then $detA'_\phi$ has only isolated zeros. The nunber of its zeros, counted according to multiplicity, is called the ramification index of $detA'_\phi$ and is denoted $r(detA'_\phi)$. In [11], J. Wolfson obtained the Plucker formula for harmonic maps of M to $G_k(C^n)$

$$c_1(G'(\phi)) = c_1(\phi) + r(detA'_\phi) - k(2g-2), \tag{4}$$

where g is the genus of M.

Let a harmonic map $\phi : M \to G_k(C^n)$ be coded by $(f_1, \cdots, f_\ell)$ of holomorphic maps. Let $\underline{\phi}_i$ $(i = 0, 1, \cdots, \ell)$ be the harmonic bundles given iteratively by :$\underline{\phi}_0 = 0$, for $i \geq 1$, ϕ_i is the holomorphic extension of $G'(\underline{\phi}_{i-1})$ coded by f_i, and $\phi = \phi_\ell$. Let r_1 be the ramification index of $detA'_{\phi_i} : \bigwedge^k \phi_i \to \bigwedge^k G'(\phi_i)$. Then using Lemma 4.2 and the formula (4) iteratively, we get

$$E'(\phi) = -4\pi(\sum_{i=1}^{\ell-1} c_1(\underline{\phi}_i) + \sum_{i=1}^{\ell} c_1(\underline{f}_i) + \sum_{i=1}^{\ell-1} r_i - k(\ell-1)(2g-2)), \tag{5}$$

$$E''(\phi) = -4\pi \sum_{i=1}^{\ell-1} c_1(\underline{\phi}_i). \tag{6}$$

We shall call $r = \sum_{i=1}^{\ell-1} r_i$ the total ramification index of ϕ. If $r = 0$, ϕ is said to be totally unramified. Using the formula (4) again iteratively, we have

$$c_1(\underline{\phi}_i) = \sum_{j=1}^{i} c_1(\underline{f}_j) + \sum_{j=1}^{j-1} r_j - k(i-1)(2g-2). \tag{7}$$

Hence, it follows

$$\sum_{i=1}^{\ell-1} c_1(\underline{\phi}_i) = \sum_{i=1}^{\ell-1} (\ell-i) c_1(\underline{f}_i) + \sum_{i=1}^{\ell-2} (\ell-i-1) r_i - k(\ell-1)(\ell-2)(g-1). \tag{8}$$

By taking account of Proposition 4..1, we obtain from(5) and (6)

Theorem 4.3. *Let a harmonic map $\phi : M \to G_k(C^n)$ be coded by $(f_1, \cdots, f_\ell)$ of holomorphic maps. if ϕ satisfies $\phi \cdot I = \phi$, it holds*

$$\sum_{i=1}^{\ell} deg(\underline{f}_i) = r - k(\ell-1)(2g-2).$$

where r is the total ramification index of ϕ and g is the genus of M.

Corollary 4.4. *Let $\phi : S^2 \to CP^{2m}$ is a harmonic map with $\phi \cdot I = \phi$. Let f be the directrix curve of ϕ. Then $deg f = 2m + r$.*

Hence if ϕ is totally unramified, $deg f = 2m$. N.Egiri determined the harmonic map $\phi : S^2 \to S^{2m}$ with $\phi \cdot I = \pm\phi$ whose directrix curve f satisfies $deg f = 2m$ (see Corollary 4.1 in [8]). In this case, we have the corresponding harmonic map $\underline{\phi} : P^2 \to P^{2m}(1)$.

Corollary 4.5(Egiri [6]). *Let $\phi : S^2 \to S^{2m}$ be a totally unramified harmonic map with $\phi \cdot I = \pm\phi$. Then the corresponding map $\underline{\phi} : P^2 \to P^{2m}(1)$ is the standard minimal immersion of P^2 into $P^{2m}(1)$.*

References

[1] J. L. M. Barbosa, On minimal immersions of S^2 into S^{2m}, Trans. Amer. Math. Soc. **210**(1975), 75-106.

[2] J. Bolton, G. R. Jensen, M. Rigol and M. Woodward, On conformal minimal immersions of S^2 into CP^n, Math. Ann. **279** (1988), 599-620.

[3] J. Bolton, L. Vrancken and M. Woodward, Minimal immersions of S^2 and CP^2 into CP^n with few higher order , preprint.

[4] F. E. Burstall and J. C. Wood, The construction of harmonic maps into complex Grassmannians, J. Diff. Geom. **23**(1986),255-298.

[5] S. S. Chern and J. Wolfson, Harmonic maps of the two-sphere into a complex Grassmann manifold II, Ann. of Math. **125**(1987), 301-336.

[6] N. Egiri, Equivariant minimal immersions of S^2 into $S^{2m}(1)$, Trans. Amer. Math. Soc. **279**(1986), 105-124.

[7] J. Eells and J. C. Wood, Harmonic maps from surfaces to complex projective spaces, Adv. in Math. **49**(1983), 217-263.

[8] T. Ishihara, Complete non-orientable minimal surfaces, Trans. Amer. Math. Soc. **333**(1992), 889-901.

[9] K. Uhlenbeck, Harmonic maps into Lie groups(Classical solutions of Chiral model), J. Diff. Geom. **30**(1989), 1-50.

[10] G. Valli, On the energy spectrum of harmonic 2-spheres in unitary groups, Topology **27**(1988), 129-136.

[11] J. G. Wolfson, Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds, J. Diff. Geom. **27**(1988), 161-178.

[12] J. C. Wood, The explicit construction and parametrization of all harmonic maps from two-sphere to complex Grassmannian, J. Reine Angew. Math. **386**(1988),1-31.