Harmonic Maps of Nonorientable Surfaces into Complex Grassmann Manifolds

By

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Abstract

We study harmonic maps of nonorientable surfaces into complex Grassmannians. J. C. Wood coded a factorable harmonic maps of a Riemann surfaces into a complex Grassmannian by a sequence of holomorphic maps of the surface into Grassmannians. We investigate codes of harmonic maps of nonorientable surfaces. Especially the degrees of codes are studies.

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§1. Introduction

The construction of all harmonic maps from the two-sphere to a complex Grassmannian was discussed by many authors (see, for example, [3,5,9,10, 11, 12]). Especially, J. C. Wood [12] gave the explicit construction of all harmonic maps $S^2 \to G_k(C^n)$. In the present paper, we shall investigate harmonic maps of nonorientable surfaces into a complex Grassmannian manifold $G_k(C^n)$. We deal with a nonorientable surface $M$ which is a quotient of a Riemann surface $M$ by the equivalence relation $z \sim w$ if and only if $w = I(z)$, where $I$ is an antiholomorphic involution of $M$ with no fixed point. Let $\pi : M \to N$ be the natural projection. A necessary and sufficient condition for a map $\phi$ of $M$ into a manifold $N$ to be factored as $\phi = \hat{\phi} \cdot \pi$, where $\hat{\phi}$ is a map of $M$ into $N$, is that, $\phi(I(p)) = \phi(p)$ for each $p \in M$. Let $g$ be a Riemannian metric compatible with the conformal structure of $M$. Then there exists a Riemannian structure $g$ on $M$ such that $\pi$ is locally isometric. The assignment $\phi \to \hat{\phi}$ is a bijective correspondence between harmonic maps $\phi : M \to N$ with $\phi \cdot I = \phi$ and harmonic maps $\hat{\phi} : M \to N$. We study harmonic maps $\phi : M \to G_k(C^n)$ with $\phi \cdot I = \phi$.

If $M = S^2$, we identify $S^2$ with $CU\{\infty\}$. The antipodal map is an involution given by $I(z) = -1/z$. The quotient space is the projective plane $P^2$. If a harmonic map $\phi : S^2 \to S^{2m}$ satisfies $\phi \cdot I = \pm \phi$, there corresponds the map $\tilde{\phi} : P^2 \to P^{2m}$. N. Egiri [6] studied these maps.

Let $\phi : M \to G_k(C^n)$ be a harmonic map and $\tilde{\phi}$ be obtained by some reduction/extension. In §2, we show $\tilde{\phi} \cdot I$ is obtained from $\phi \cdot I$ by the natu-
rally corresponding reduction/extension. J. C. Wood [12] coded a factorable harmonic map \( \phi : M \to G_k(C^n) \) by a sequence of holomorphic maps from \( M \) into complex Grassmannians. In §3, we shall get the necessary and sufficient conditions about codes under which harmonic maps \( \phi \) satisfy \( \phi \cdot I = \phi \). In the final section, we investigate the degrees of the codes of harmonic maps \( \phi \) with \( \phi \cdot I = \phi \).

§2. Preliminaries

For the definition and basic properties of harmonic maps into a complex Grassmannian, see [4, 11, 12]. For any integers \( n, k \) with \( 0 \leq k \leq n \), let \( G_k(C^n) \) be the Grassmannian of complex \( k \)-dimensional subspaces of \( C^n \) with its standard Kahler structure. Let \( M \) be a Riemann surface with antiholomorphic involution \( I \). We identify a smooth map \( \phi : M \to G_k(C^n) \) with the smooth complex subbundle \( \mathfrak{h} \) of rank \( k \) of the trivial bundle \( C^n \). Denote by \( \phi \cdot I \) the subbundle with fibre \( \phi \cdot I_x = \phi_{I(x)} \). Let \( \phi^\perp \) be the orthogonal complement of \( \phi \). It is evident that \((\phi \cdot I)^\perp = \phi^\perp \cdot I \). Let \( \partial \) denote the flat connection on \( C^n \) and \( <,> \) the standard Hermitian metric. A subbundle \( \phi \) inherits a metric \( <,> \), and connection \( \nabla_\phi \) by \( <v,w> = <v,w>, v,w \in \phi_x, x \in M \) and \( (\nabla_\phi)_w V = \pi_\phi \cdot \partial_w V, w \in TM, V \in C^\infty(\phi) \). We give \( \phi \) its Koszul-Malgrange holomorphic structure. It is known that \( \phi \) is a holomorphic (resp. antiholomorphic) subbundle of \( C^n \) if and only if \( \phi \) is a holomorphic (resp. antiholomorphic) map. Moreover, it is called harmonic if \( \phi \) is a harmonic map.

Let \( \phi \) and \( \psi \) be mutually orthogonal subbundle of \( C^n \). Denote by \( \phi \oplus \psi \) the subbundle with fibre \( \phi_x \oplus \psi_x \) at \( x \in M \). Let \( \tilde{A}'_{\phi,\psi} : T^{1,0}M \otimes \phi \to \psi \) be the global \( \partial' \)-second fundamental form of \( \phi \) in \( \phi \oplus \psi \). Then, \( (\tilde{A}'_{\phi,\psi})_w v = \pi_\phi \cdot \partial_w V \), where \( w \in T^{1,0}M, v \in \phi_x \), and \( V \) is a smooth extension of \( v \). The global \( \partial'' \)-second fundamental form \( \tilde{A}''_{\phi,\psi} : T^{0,1}M \otimes \phi \to \psi \) is defined similarly. Choose a local holomorphic vector field \( Z \) on \( M \), for example \( Z = \partial/\partial z \) for some local complex coordinate \( z \) and denote the representatives \( (\tilde{A}'_{\phi,\psi})_Z \) and \( (\tilde{A}''_{\psi,\psi})_Z \) by \( A'_\phi \psi \) and \( A''\phi,\psi \) respectively, which are again called \( \partial' \)- and \( \partial'' \)-second fundamental forms. Particularly, the second fundamental forms of \( \phi \) in \( C^n \), \( \tilde{A}'_{\phi} = \tilde{A}'_{\phi,\phi} \) and \( \tilde{A}''_{\phi} = \tilde{A}''_{\phi,\phi} \) are important. These are called the fundamental collineations of \( \phi \) in [5, 11]. We also put \( A'_\phi = (\tilde{A}'_{\phi})_Z \) and \( A''_\phi = (\tilde{A}''_{\phi})_Z \). Note that \( \phi : M \to G_k(C^n) \) is holomorphic (resp. antiholomorphic) if and only if \( A'_\phi = 0 \) (resp. \( A''_\phi = 0 \)). Moreover, \( \phi \) is harmonic if and only if \( A'_\phi : \phi \to \phi \) is holomorphic or \( A''_\phi : \phi \to \phi^\perp \) is antiholomorphic (see [4,12]).

Suppose that \( \phi : M \to G_k(C^n) \) is harmonic. Then the \( \partial' \)-Gauss (resp. \( \partial'' \)-Gauss) bundle \( G'(\phi) \) (resp. \( G''(\phi) \)) is the holomorphic subbundle \( \text{Im}A'_\phi \) of \( \phi \) (resp. antiholomorphic subbundle \( \text{Im}A''_\phi \) of \( \phi \)). These bundles are harmonic. We define \( G^{(i)}(\phi), (i \in Z) \) by \( G^{(0)}(\phi) = \phi \), \( G^{(i)} = G'(G^{(i-1)})(\phi), (i \geq 1) \).
\( C^\alpha (C^{(-i+1)}) \), \( i \geq 1 \).

If \( \phi, \psi \) are subbundle of \( C^\alpha \) with \( \psi \subset \phi \). Then \( \psi \subset \phi \) is denoted by \( \phi \oplus \psi \). Let \( \phi : M \to G_k (C^n) \) be harmonic. Assume that \( \alpha \subset \phi \) and \( \beta \subset \phi \) satisfy the \( \delta' \)-replacement (resp. \( \delta'' \)-replacement) conditions, that is, \( \alpha \) is holomorphic (resp. antiholomorphic) subbundle of \( \phi \) and \( \beta \) is holomorphic (resp. antiholomorphic) subbundle of \( \phi \perp \) with \( A'_\phi (\alpha) \subset \beta \) and \( A''_\phi \subset \alpha \) (resp. \( A''_\phi (\alpha) \subset \beta \) and \( A''_\phi \subset \alpha \). Then \( \tilde{\phi} = (\phi \oplus \alpha) \oplus \beta \) is harmonic (see Proposition 2.1 in [12]). The transformation \( \phi \to \tilde{\phi} \) is called the \( \delta' \)-replacement (resp. \( \delta'' \)-replacement) of the holomorphic (resp. antiholomorphic) subbundle \( \alpha \) by \( \beta \).

**Proposition 2.1.** Let \( \phi : M \to G_k (C^n) \) be harmonic. Put \( \psi = \phi \cdot I \). Then \( \psi \) is also harmonic. If \( \alpha (\alpha \subset \phi) \) and \( \beta (\beta \subset \phi \perp) \) satisfy the \( \delta' \)-replacement (resp. \( \delta'' \)-replacement) conditions, then \( \alpha \cdot I (\alpha \cdot I \subset \phi) \) and \( \beta (\cdot I \subset \phi \perp) \) satisfy the \( \delta'' \)-replacement (resp. \( \delta' \)-replacement) conditions, and \( \tilde{\phi} = (\phi \oplus \alpha) \oplus \beta \) and \( \tilde{\psi} = (\psi \oplus \alpha \cdot I) \oplus \beta \cdot I \) satisfy \( \tilde{\psi} = \phi \cdot I \).

**Proof.** Assume that \( \alpha \) and \( \beta \) satisfy the \( \delta' \)-replacement conditions. For a local holomorphic vector field \( Z \), we can put \( dI (Z) = aZ \) and \( dI (\tilde{Z}) = a\tilde{Z} \). For \( v \in \alpha \cdot I_x \),

\[
A''_\psi (v) = \pi_{\psi \cdot I (x) \cdot I} \cdot \partial_2 V = a\pi_{\phi \cdot I (x)} \cdot \partial_2 V = aA'_\phi (v),
\]

where \( V \in C^\alpha (\alpha \cdot I) \) is a smooth extension of \( v \) which is also regarded as an element of \( \alpha _{I (x)} \). As \( A'_\phi (v) \in \beta \cdot I_x \), we show that \( A''_\psi (\alpha \cdot I) \subset \beta \cdot I \). Similarly, we have \( A''_\psi (\beta \cdot I) \subset \alpha \cdot I \). It is evident that \( \tilde{\psi} = \phi \cdot I \). We can show the dual statement similarly.

Let \( \alpha \) is a holomorphic subbundle of \( ker A'_{\phi \perp} \). Then with \( \beta = 0 \), it satisfies the replacement conditions. The resulting harmonic bundles \( \tilde{\phi} = \phi \oplus \alpha \) and \( \tilde{\phi} = \phi \perp \oplus \alpha \) are said to be the extension of \( \phi \) by the holomorphic subbundle \( \phi \perp \) by the holomorphic subbundle \( \alpha \). Similarly for a holomorphic subbundle \( \alpha \) of \( ker A'_{\phi} \), we have harmonic bundles \( \tilde{\phi} = \phi \oplus \alpha \) and \( \tilde{\phi} = \phi \perp \oplus \alpha \), the reduction/extension by a holomorphic subbundle \( \alpha \). There a dual notion of reduction/extension by an antiholomorphic subbundle. J. C. Wood[12] codes these eight types of reduction/extension as follows:

1. \( \tilde{\phi} = \phi \perp \oplus \alpha \)
2. \( \tilde{\phi} = \phi \oplus \alpha \)
3. \( \tilde{\phi} = \phi \perp \oplus \alpha \)
4. \( \tilde{\phi} = \phi \oplus \alpha \)
5. \( \tilde{\phi} = \phi \oplus \alpha \)
6. \( \tilde{\phi} = \phi \perp \oplus \alpha \)
7. \( \tilde{\phi} = \phi \oplus \alpha \)
8. \( \tilde{\phi} = \phi \perp \oplus \alpha \)

Moreover, J. C. Wood gave the one-to-one correspondence between holomorphic maps \( f : M \to G_s (C^I) \) and reductions/extensions \( \phi \to \tilde{\phi} \) of type \( \zeta \).
and said that $\tilde{\psi}$ is obtained by the reduction/extension coded by $(f, \zeta)$. If $\alpha$ is a holomorphic subbundle of $\ker A'_{\phi^+}$ (resp. $\ker A'_{\phi^+} \cdot I$) is an antiholomorphic subbundle of $\ker A''_{\phi^+} \cdot I$ (resp. $\ker A''_{\phi^+} \cdot I$). The dual fact is also true. Hence we have

**Lemma 2.2.** Let $\phi : M \to G_k(C^n)$ be harmonic. Put $\psi = \phi \cdot I$. Let $\tilde{\psi}$ (resp. $\tilde{\psi}$) be obtained by reduction/extension coded by $(f, \zeta)$ (resp. $(f \cdot I, \zeta^*)$), where $1^* = 3, 2^* = 4, 3^* = 1, 4^* = 2, 5^* = 7, 6^* = 8, 7^* = 5, 8^* = 6$. Then $\tilde{\psi} = \tilde{\phi} \cdot I$.

§3. Factorizations

If a harmonic map $\phi : M \to G_k(C^n)$ is factorable by reduction and extensions (see Definition 3.1 in [12]), J. C. Wood found a sequence $((f_1, \zeta_1), \cdots, (f_r, \zeta_r))$ of holomorphic maps $f_i : M \to G_k(C^{r_i})$ and integers $i \in \{1, \cdots, 8\}$ such that subbundles $\phi_0, \cdots, \phi_r = \phi$ are given iteratively as follows: $\phi_0 = 0$, for $i \geq 1$, $\phi_i$ is obtained from $\phi_{i-1}$ or $\phi_{i-1}^*$ by performing the reduction/extension coded by $(f_i, \zeta_i)$ (see Theorem 5.1 in [12]). We shall say that $\phi$ is coded by the sequence $((f_1, \zeta_1), \cdots, (f_r, \zeta_r))$. From Lemma 2.2, we obtain

**Proposition 3.1.** Let $\phi : M \to G_k(C^n)$ be harmonic which is factorable by reduction and extensions. Assume that $\phi$ is coded by a sequence $((f_1, \zeta_1), \cdots, (f_r, \zeta_r))$. Then $\phi$ satisfies $\phi \cdot I = \phi$ if and only if $\phi$ is also coded by the sequence $((f_1, \zeta_1^*), \cdots, (f_r, \zeta_r^*))$.

Let $\phi : M \to G_k(C^n)$ be harmonic. Let $\tilde{\phi}$ be a holomorphic extension of $G'_{\phi}$ of a subbundle of rank $s$. (see Example 2.7 in [12]). It is also a reduction of $\phi$ of type 3 (see Lemma 2.11 in [12]). Put $t = \text{rank } \ker A''_{\phi}$. For any $s \in \{0, 1, \cdots, t\}$, there is a canonical one-to-one correspondence between holomorphic maps $f : M \to G_s(C^t)$ and holomorphic extensions $\tilde{\phi}$ of $G'_{\phi}$ by subbundle of rank $s$ (see Lemma 4.13 in [12]). In this case, $\phi$ is said to be the holomorphic extension of $G'_{\phi}$ coded by $f$. $\phi$ is a harmonic map of finite $\partial^n$-order if $G^{(-r)}_{\phi} = 0$ for some positive integer $r$ (see Definition 3.7 in [12]). Let $S_0$ be the set of all sequences $(f_1, \cdots, f_t)$ of holomorphic maps $f_i : M \to G_{s_i}(C^t)$, where $0 \leq s_i \leq t_i$, which give harmonic subbundles $\phi_i$, $0 \leq i \leq t$ of $C^n$ iteratively by $\phi_0 = 0$, $\phi_i$ obtained from $\phi_{i-1}$ as the holomorphic extension of $G'_{\phi_{i-1}}$ coded by $f_i$ and $\ker A'_{\phi_{i-1}} = 0, 0 \leq i \leq t$. Then the assignment $(f_1, \cdots, f_t) \mapsto \phi = \phi_t$ is a bijection between $S_0$ and the set of all harmoni maps $\phi : M \to G_k(C^n)$ of finite $\partial^n$-order (see Theorem 5.2 in [12]). The harmonic map is said to be coded by the sequence $(f_1, \cdots, f_t)$ of holomorphic maps. Dually, we have an antiholomorphic extension of $G''_{\phi}$ of rank $s'$ coded by an antiholomorphic map $g : M \to G_{s'}(C^t)$, where $t' = \text{rank } \ker A'_{\phi}$. Moreover, dually all harmonic maps of finite $\partial'$-order are coded by unique sequences $(g_1, \cdots, g_{t'})$ of antiholomorphic maps. Let $\phi : M \to G_k(C^n)$ is harmonic map of finite $\partial$-order and $\partial^n$-order. Let $\phi$ be coded by a sequence $(f_1, \cdots, f_{t'})$ of holomorphic maps and coded by a sequence $(g_1, \cdots, g_{t'})$.
of antiholomorphic maps. We shall call \((g_1, \ldots, g_{2})\) the polar of \((f_1, \ldots, f_{2})\). If the polar of \((f_1, \ldots, f_{2})\) is \((f_1 \cdot I, \ldots, f_{2} \cdot I)\), \((f_1, \ldots, f_{2})\) is said to be symmetric with respect to \(I\). Let \(S_1\) be the subset of \(S_0\) whose elements are all sequences \((f_1, \ldots, f_{2})\) symmetric with respect to \(I\). We can see with ease that \(\tilde{f}\) is a holomorphic extension of \(G'(\phi)\) coded by \(f\) if and only if \(\tilde{f} \cdot I\) is a holomorphic extension of \(G''(\phi \cdot I)\) coded by \(f \cdot I\). Assume \(\phi : M \rightarrow G_k(C^n)\) satisfies \(\phi \cdot I = \phi\). Then \(\phi\) is a harmonic map of finite \(\partial'\)-order if and only if it is of \(\partial''\)-order. In this case, \(\phi\) is coded by the sequence \((f_1, \ldots, f_{2})\) symmetric with respect to \(I\). Conversely, if \(\phi\) is coded by the sequence \((f_1, \ldots, f_{2})\) symmetric with respect to \(I\), it satisfies \(\phi \cdot I = \phi\). Thus we have

**Theorem 3.2.** The assignment \((f_1, \cdots, f_{2}) \rightarrow \phi\) is the bijection between the subset \(S_1\) and the set of all harmonic maps \(\phi : M \rightarrow G_k(C^n)\) of finite \(\partial''\)-order (or \(\partial'\)-order) with \(\phi \cdot I = \phi\).

If \(M = S^2\), all harmonic maps \(\phi : M \rightarrow G_k(C^n)\) are of finite \(\partial''\)-order. Particularly, for a full harmonic map \(\phi : S^2 \rightarrow CP^2\), there is a unique holomorphic map, called the directrix curve such that \(\phi = G(i)(f)\), for some \(i\), \(0 \leq i \leq n\). \(G(n)(f)\) is the polar of \(f\) (see [7]). Thus we get

**Corollary 3.3.** Let \(\phi : S^2 \rightarrow CP^n\) be a full harmonic map. Let \(f\) be the directrix curve of \(\phi\). The polar of \(f\) is \(G(n)(f)\). The harmonic map \(\phi\) satisfies \(\phi \cdot I = \phi\) if and only if \(n\) is even, that is, \(n = 2m\), \(\phi = G(m)(f)\) and \(G(n)(f) = f \cdot I\).

**Remarks.** (1) The author was informed by N. Egiiri that the above corollary has been obtained by J. Bolton, L. Vancken and L. M. Woodward in [3]. (2) For a harmonic map \(\phi : M \rightarrow S^2\), N. Egiiri[6] showed that \(\phi\) satisfies \(\phi \cdot I = \pm \phi\) if and only if its directrix curve \(f\) satisfies \(f(z) = \pm z^{2m}f((l(z))\). (3) J. Bolton, G. Jensen, M. Rigoli and L. Woodward investigated harmonic maps of \(S^2\) into \(CP^n\) with induced metrics of constant curvature. They determined such harmonic maps \(\phi\) with \(\phi \cdot I = \phi\) explicitly (Theorems 5.2 and 5.4 in [2]).

We shall construct some examples of harmonic maps \(\phi\) of \(S^2\) to \(CP^{2m}\) with \(\phi \cdot I = \phi\). Let \(\Xi : S^2 \rightarrow CP^{2m}\) be a full holomorphic curve. Consider \(S^2\) covered by isothermal coordinates given by stereographic projections. Then \(I(z) = -1/\bar{z}\). We can represent \(\Xi\) by a polynomial \(\xi : C \rightarrow C^{2m+1}\), given by \(\xi(z) = \sum a_i z^i\). Particularly, we deal with the polynomials considered by J. Barbosa [1]: \(\xi_{km}(z) = a_0 + \sum_{i=1}^{2m-1} a_{k-m+i} z^{k-m+i} + a_{2k} z^{2k}\), \(k \geq m\). Let denote by \(\xi_{km}^{(j)}\) be the \(j\)-th derivative of \(\xi_{km}\). Put \(\xi^*(z) = a_0 z^{2k} + \sum_{i=1}^{2m-1} a_{k-m+i} (-1)^{k-m+i} z^{k-m+i} + a_{2k}\). Let \(\Phi_{km}\) be the harmonic map such that its directrix curve is \(\xi\) and \(\Phi_{km} = G(m)(\xi_{km})\). Then \(\Phi_{km} \cdot I = \Phi\) if and only if \(\xi^*\) is the polar of \(\xi_{km}\), that is, for \(j = 0, k-m+1, k-m+2, \ldots, k+m-1, \langle \xi_{km}^{(j)}, \xi^* \rangle = 0\), where \(\langle, \rangle\) is the Hermitian inner product of \(C^{2m+1}\). From \(\langle \xi_{km}^{(k+m-1)}, \xi^* \rangle = 0\), we get \(\langle a_{k+m-1}, a_{2k} \rangle = 0\). Inductively, we have
\[ <a_i, a_j> = 0, i \neq j. \] Moreover, we get that \(k - m\) is even and that for \(i = 1, \cdots, 2m - 1, \|a_{k+m-i}\|^2 = \left(\frac{2k}{k-m+i} \right) \left(\frac{k-m+i-1}{k-m} \right) \|a_{2k}\|^2,\]
\[ \|a_0\|^2 = \left(\sum_{i=1}^{2m-1} (-1)^{i+1} \left(\frac{2k}{k-m+i} \right) \left(\frac{k-m+i-1}{k-m} \right) - 1\right) \|a_{2k}\|^2. \]

4. The energy and the degree

For any smooth map of a closed Riemann surface \(M \to G_k(C^n)\), we define the \((1,0)\) and \((0,1)\) energy integral (see [12])

\[ E'(\phi) = \int_M |\partial\phi|^2 dM, \quad E''(\phi) = \int_M |\overline{\partial}\phi|^2 dM. \]

Then they satisfy

\[ E'(\phi) - E''(\phi) = 4\pi \text{deg}\phi = -4\pi c_1(\phi). \]

Let \(\tilde{\phi}\) be obtained by replacing \(\alpha(C, \phi)\) by \(\beta(C, \phi)\) where \(\alpha, \beta\) satisfy the \(\partial\)'-replacement conditions. In [12], J. C. Wood got

\[ E'(\tilde{\phi}) - E'(\phi) = -4\pi c_1(\beta), \quad E''(\tilde{\phi}) - E''(\phi) = -4\pi C_1(\alpha). \]

Dually, if \(\alpha, \beta\) satisfy the \(\partial''\)-replacement conditions, then

\[ E'(\tilde{\phi}) - E'(\phi) = 4\pi c_1(\alpha), \quad E''(\tilde{\phi}) - E''(\phi) = 4\pi C_1(\beta). \]

As \(\phi^\perp\) is obtained by \(\partial\)'-replacing of \(\phi\) by \(\phi^\perp\), we have \(E'(\phi^\perp) = E'(\phi) + 4\pi c_1(\phi), E''(\phi^\perp) = E''(\phi) - 4\pi C_1(\phi).\) Since the antiholomorphic involution \(I\) is an isometry, we have \(E'(\phi \cdot I) = E'(\phi), E''(\phi \cdot I) = E''(\phi).\) Hence we get

**Proposition 4.1.** Let \(\phi : M \to G_k(C^n)\) be a smooth map with \(\phi \cdot I = \phi.\) Then \(E'(\phi) = E''(\phi)\) and \(\text{deg}\phi = 0.\)

Let \(\tilde{\phi}\) be a holomorphic extension of \(G'(\phi)\) coded by a holomorphic map \(f.\) Then \(G'(\phi) \to \tilde{\phi}\) is the \(\partial\)'-replacement of \(\alpha = 0\) by \(\beta = f\) (see Lemma 2.11 in [12]), \(\phi \to G'(\phi)\) is the \(\partial\)'-replacement of \(\alpha = \phi\) by \(\beta = G'(\phi)\). Hence, using (2), we obtain

**Lemma 4.2.** Let \(\phi : M \to G_k(C^n)\) be a harmonic map. Let \(\tilde{\phi}\) be a holomorphic extension of \(G'(\phi)\) coded by a holomorphic map \(f.\) Then we have

\[ E'(\tilde{\phi}) = E'(\phi) - 4\pi c_1(G'(\phi)) - 4\pi c_1(f), \quad E''(\tilde{\phi}) = E''(\phi) - 4\pi c_1(\phi). \]

Let \(\phi\) be \(\partial\)'-irreducible. We consider the fundamental collineation \(A'_\phi : \phi \to G'(\phi).\) Taking the k-th exterior power of each bundle, we get the holomorph bundle map \(\text{det}A'_\phi : \Lambda^k \phi \to \Lambda^k G'(\phi)\) of line bundles. Then \(\text{det}A'_\phi\) has only
isolated zeros. The number of its zeros, counted according to multiplicity, is called the ramification index of \( \det A'_\phi \) and is denoted \( r(\det A'_\phi) \). In [11], J. Wolfson obtained the Plucker formula for harmonic maps of \( M \) to \( G_k(C^n) \)

\[
(4) \quad c_1(G'(\phi)) = c_1(\phi) + r(\det A'_\phi) - k(2g - 2),
\]

where \( g \) is the genus of \( M \).

Let a harmonic map \( \phi : M \to G_k(C^n) \) be coded by \( (f_1, \ldots, f_\ell) \) of holomorphic maps. Let \( \phi_i \) \((i = 0, 1, \ldots, \ell)\) be the harmonic bundles given iteratively by \( \phi_0 = 0 \), for \( i \geq 1 \), \( \phi_i \) is the holomorphic extension of \( G'(\phi_{i-1}) \) coded by \( f_i \), and \( \phi = \phi_\ell \). Let \( r_i \) be the ramification index of \( \det A'_{\phi_i} : \bigwedge^k \phi_i \to \bigwedge^k G'(\phi_i) \). Then using Lemma 4.2 and the formula (4) iteratively, we get

\[
(5) \quad E'(\phi) = -4\pi \left( \sum_{i=1}^{\ell-1} c_1(\phi_i) + \sum_{i=1}^\ell c_1(f_i) + \sum_{i=1}^{\ell-1} r_i - k(\ell - 1)(2g - 2) \right),
\]

\[
(6) \quad E''(\phi) = -4\pi \sum_{i=1}^{\ell-1} c_1(\phi_i).
\]

We shall call \( r = \sum_{i=1}^{\ell-1} r_i \) the total ramification index of \( \phi \). If \( r = 0 \), \( \phi \) is said to be totally unramified. Using the formula (4) again iteratively, we have

\[
(7) \quad c_1(\phi_i) = \sum_{j=1}^i c_1(f_j) + \sum_{j=1}^{i-1} r_j - k(i - 1)(2g - 2).
\]

Hence, it follows

\[
(8) \quad \sum_{i=1}^{\ell-1} c_1(\phi_i) = \sum_{i=1}^{\ell-1} (\ell - i)c_1(f_i) + \sum_{i=1}^{\ell-2} (\ell - i - 1)r_i - k(\ell - 1)(\ell - 2)(g - 1).
\]

By taking account of Proposition 4.1, we obtain from (5) and (6)

**Theorem 4.3.** Let a harmonic map \( \phi : M \to G_k(C^n) \) be coded by \( (f_1, \ldots, f_\ell) \) of holomorphic maps. If \( \phi \) satisfies \( \phi \cdot I = \phi \), it holds

\[
\sum_{i=1}^\ell \deg(f_i) = r - k(\ell - 1)(2g - 2).
\]

where \( r \) is the total ramification index of \( \phi \) and \( g \) is the genus of \( M \).
Corollary 4.4. Let $\phi : S^2 \to CP^{2m}$ is a harmonic map with $\phi \cdot I = \phi$. Let $f$ be the directrix curve of $\phi$. Then $degf = 2m + r$.

Hence if $\phi$ is totally unramified, $degf = 2m$. N. Egiri determined the harmonic map $\phi : S^2 \to S^{2m}$ with $\phi \cdot I = \pm \phi$ whose directrix curve $f$ satisfies $degf = 2m$ (see Corollary 4.1 in [8]). In this case, we have the corresponding harmonic map $\phi : P^2 \to P^{2m}(1)$.

Corollary 4.5 (Egiri [6]). Let $\phi : S^2 \to S^{2m}$ be a totally unramified harmonic map with $\phi \cdot I = \pm \phi$. Then the corresponding map $\phi : P^2 \to P^{2m}(1)$ is the standard minimal immersion of $P^2$ into $P^{2m}(1)$.

References


