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Relative Minimizer of Prescribed Mean Curvature Equation

Dedicated to Professor Yoshihiro Ichijyō on his 65th birthday

By

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Abstract

We consider a relative minimizer of the H-system where H is not necessarily a constant. It is known now that for H in a neighborhood of some appropriate constant H_0, there exists a relative minimizer $\underline{X}$ of the functional E_H. In this paper, we show some properties of a relative minimizer $\underline{X}$, especially that there exists some neighborhood U of $\underline{X}$ outside of which every critical value is greater than that of $\underline{X}$.

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§1. Introduction

We consider the Dirichlet problem for the equation of prescribed mean curvature which is not necessarily constant.

Let Ω be the unit disk in $\boldsymbol{R}^2$;

$$\Omega = \{w = (u, v); u^2 + v^2 < 1\}.$$

The Dirichlet problem for the equation of prescribed mean curvature H is expressed as

$$\Delta X = 2H(X)X_u \wedge X_v, \quad \text{in } \Omega, \tag{1.1}$$

$$X = X_D, \quad \text{on } \partial\Omega. \tag{1.2}$$

Here, we denote $X_u = \dfrac{\partial}{\partial u}X$ and $X_v = \dfrac{\partial}{\partial v}X$, and $\wedge$ is the exterior product in $\boldsymbol{R}^3$. $H : \boldsymbol{R}^3 \to \boldsymbol{R}$ is a given function and X_D is a given function mainly of class $C^2(\overline{\Omega}; \boldsymbol{R}^3)$.

Then we consider a functional E_H on $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$, where $E_H(X) = D(X) + 2V_H(X)$. Here

$$D(X) = \frac{1}{2}\int_\Omega |\nabla X|^2\, dw \tag{1.3}$$

is the Dirichlet integral and

$$V_H(X) = \frac{1}{3}\int_\Omega Q(X)\cdot X_u \wedge X_v\, dw \tag{1.4}$$

is the Q-volume introduced by Hildebrandt, where $Q(X)$ is defined through $H(X)$

$$Q(x_1,x_2,x_3) = \left(\int_0^{x_1} H(s,x_2,x_3)\,ds, \int_0^{x_2} H(x_1,s,x_3)\,ds, \int_0^{x_3} H(x_1,x_2,s)\,ds\right). \tag{1.5}$$

When $H \equiv H_0 \in \boldsymbol{R}$, we have $E_{H_0}(X) = D(X) + 2H_0V(X)$, where

$$V(X) = \frac{1}{3}\int_\Omega X\cdot X_u \wedge X_v\, dw \tag{1.6}$$

is the algebraic volume of a surface X.

We summarize basic results here. Note that solutions to the Dirichlet problem (1.1), (1.2) are characterized as critical points of $E_H(X)$.

First, in the case of $H \equiv H_0 \in \boldsymbol{R}$, following two theorems are fundamental and now well-known. For their proofs and for further references, see for example Struwe [7], [9], or Brezis-Coron [1].

Theorem 1.1. *Suppose $H \equiv H_0 \in \boldsymbol{R}$ and let $X_D \in L^\infty \cap H^1(\Omega;\boldsymbol{R}^3)$ be given. Assume that*

$$|H_0| \cdot \| X_D \|_\infty < 1$$

is satisfied. Then there is a solution $\underline{X}_0 \in \{X_D\} + H_0^1(\Omega;\boldsymbol{R}^3)$ to (1.1), (1.2). Moreover $\underline{X}_0$ is characterized as a strict relative minimizer of E_{H_0} in this space.

Remark 1.2. The fact that a relative minimizer is also a strict relative minimizer is originally due to Brezis-Coron [1]. We give this result in Proposition 2.3 following Struwe [7].

When $H_0 \neq 0$ and X_D is non-constant, there exists a second solution.

Theorem 1.3. *Suppose $H \equiv H_0 \neq 0$ and let $X_D \in L^\infty \cap H^1(\Omega;\boldsymbol{R}^3)$ be non-constant. Assume moreover that E_{H_0} admits a local minimum $\underline{X}_0$ in the class $\{X_D\} + H_0^1(\Omega;\boldsymbol{R}^3)$. Then there exists a solution $\overline{X} \in \{X_D\} + H_0^1(\Omega;\boldsymbol{R}^3)$ of (1.1), (1.2) different from $\underline{X}_0$. Moreover $\overline{X}$ satisfies*

$$E_{H_0}(\underline{X}_0) < E_{H_0}(\overline{X}) = \inf_{p\in P}\sup_{X\in p} E_{H_0}(X) < E_{H_0}(\underline{X}_0) + \beta_0, \tag{1.7}$$

where $\beta_0 = \frac{4\pi}{3H_0^2}$ and $P = \{p \in C^0([0,1]; \{X_D\} + H_0^1); p(0) = \underline{X}_0,\ E_{H_0}(p(1)) < E_{H_0}(\underline{X}_0)\}$.

For variable curvature function H, similar results were obtained. The following result is due to Hildebrandt [2, Satz 2].

Theorem 1.4. *Suppose H is of class C^1 and let $X_D \in L^\infty \cap H^1(\Omega; \boldsymbol{R}^3)$ be given with $\| X_D \|_\infty < 1$. Then if*

$$h = \operatorname*{ess\,sup}_{|X|\leq 1} | H(X) | < 1,$$

there exists a solution $\underline{X} \in \{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$ to (1.1), (1.2) *such that*

$$\underline{X} = \inf\{E_H(X); X \in \{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3), \ \| X \|_\infty \leq 1\}.$$

Recently, a second solution was also obtained by Struwe [8] and Wang [10]. To state the results, introduce a metric

$$(1.8) \qquad [H - H_0] = \operatorname*{ess\,sup}_{X \in \boldsymbol{R}^3}\{(1+ | X |)(| H(X) - H_0 | + | \nabla H(X) |) + | Q(X) - H_0 X | + | dQ(X) - H_0 id |\},$$

and denote an α-neighborhood of H_0 as $\mathcal{H}_\alpha = \{H; [H - H_0] < \alpha\}$. Then the following results hold.

First, Struwe [8, Theorem 1.3] proved the following.

Theorem 1.5. *Suppose $X_D \in C^2(\overline{\Omega}; \boldsymbol{R}^3)$ is non-constant and for $H_0 \in \boldsymbol{R} \setminus \{0\}$ the functional E_{H_0} admits a relative minimizer in $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$. Then there exists a number $\alpha > 0$ such that for a dense set $\mathcal{A}$ of curvature functions H in $\mathcal{H}_\alpha$, the Dirichlet problem* (1.1), (1.2) *admits at least two distinct regular solutions in $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$.*

Then Wang [10, Theorem 1.6] extended the above result to the full α-neighborhood and obtained the following.

Theorem 1.6. *Suppose $X_D \in C^2(\overline{\Omega}; \boldsymbol{R}^3)$ is non-constant and for $H_0 \in \boldsymbol{R} \setminus \{0\}$ the functional E_{H_0} admits a relative minimizer in $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$. Then there exists a number $\alpha > 0$ such that for a curvature function H in the full α-neighborhood $\mathcal{H}_\alpha$ of H_0, the Dirichlet problem* (1.1), (1.2) *admits at least two distinct solutions in $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$.*

Among the solutions obtained in the above theorems, one is a relative minimizer and the other is of unstable type of E_H. Following Wang [10], we call the former S-solution and the latter L-solution. He showed also that the S-solution is a "strict" relative minimizer of E_H in the sense

$$(1.9) \qquad E_H(\underline{X}) < E_H(\overline{X}),$$

where $\underline{X}$ is the S-solution and $\overline{X}$ is the L-solution.

In this note, we want to study a relation between $E_H(\underline{X})$ and $E_H(X)$ for arbitrary $X \in \{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$ (see Theorem 3.1), and to show an inequality (1.9) as a special case.

§2. Preliminary Results

We list two lemmas which are necessary in the following proofs.

Lemma 2.1. *Let α and $\mathcal{A}$ be as in Theorem* 1.5. *Then there exists a constant $c > 0$ independent of α such that if $H \in \mathcal{A}$,*

$$D(\overline{X} - \underline{X}) > c, \tag{2.1}$$

where $\underline{X}$ (resp. $\overline{X}$) is the S-solution (resp. L-solution) to (1.1), (1.2).

See Wang [10, Lemma 2.6].

Lemma 2.2. *Let α, X_D, and H_0 be as in Theorem* 1.5. *Then for any $\epsilon > 0$, there exists a constant $\alpha > 0$ with the property that for any curvature function $H \in \mathcal{H}_\alpha$, if $\underline{X}$ is the S-solution to* (1.1), (1.2), *then*

$$\| \underline{X} - \underline{X}_0 \|_\infty < \epsilon, \tag{2.2}$$

where $\underline{X}_0$ is a relative minimizer of E_{H_0}.

See Wang [10, Lemma 3.2].

As we note already in Remark 1.2, Brezis-Coron [1] proved that a relative minimizer is also a strict relative minimizer. For the sake of completeness, we give the proof of this result following Struwe [7, Lemma IV.1.2].

Proposition 2.3. *For $H_0 \in \boldsymbol{R} \setminus \{0\}$, suppose that E_{H_0} admits a relative minimizer $\underline{X}_0$ in the space $\{X_D\} + L^\infty \cap H_0^1(\Omega; \boldsymbol{R}^3)$. Then $\underline{X}_0$ is a strict relative minimizer of E_{H_0} in $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$, and there is a constant $\delta > 0$ such that*

$$\int_\Omega |\nabla\varphi|^2 \, dw + 4H_0 \int_\Omega \underline{X}_0 \cdot \varphi_u \wedge \varphi_v \, dw \geq \delta \int_\Omega |\nabla\varphi|^2 \, dw, \quad \textit{for all } \varphi \in H_0^1. \tag{2.3}$$

PROOF. It is evident that

$$D^2 E_{H_0}(\underline{X}_0)(\varphi, \varphi) = \int_\Omega |\nabla\varphi|^2 \, dw + 4H_0 \int_\Omega \underline{X}_0 \cdot \varphi_u \wedge \varphi_v \, dw. \tag{2.4}$$

Note that $C_0^\infty \subset L^\infty \cap H_0^1$ and C_0^∞ is dense in H_0^1, so the following inequality holds trivially.

$$\delta = \inf\{D^2 E_{H_0}(\underline{X}_0)(\varphi, \varphi); \varphi \in H_0^1, \ D(\varphi) = 1\} \geq 0. \tag{2.5}$$

Now we must show $\delta > 0$.

If $\delta = 0$, then a minimizing sequence for δ is relatively compact in $H_0^1(\Omega; \boldsymbol{R}^3)$ because $D^2V(\underline{X}_0)$ is compact (see for example Struwe [7, Theorem III.2.3]). So we have $\varphi \in H_0^1(\Omega; \boldsymbol{R}^3)$ such that $D(\varphi) = 1$ and

$$\delta = D^2 E_{H_0}(\underline{X}_0)(\varphi, \varphi) = 0. \tag{2.6}$$

Then φ satisfies

$$\Delta\varphi = 2H_0(\underline{X}_{0u} \wedge \varphi_v + \varphi_u \wedge \underline{X}_{0v}), \tag{2.7}$$

and it follows that $\varphi \in L^\infty$ (see for example Struwe [7, Theorem III.5.1]).

Hence by minimality of $\underline{X}_0$, for small $|t|$, we have

$$E_{H_0}(\underline{X}_0) \leq E_{H_0}(\underline{X}_0 + t\varphi) = E_{H_0}(\underline{X}_0) + 2H_0 t^3 V(\varphi), \tag{2.8}$$

so it follows $V(\varphi) = 0$. Then $E_{H_0}(\underline{X}_0 + t\varphi) = E_{H_0}(\underline{X}_0)$ for any $t \in \boldsymbol{R}$, and this implies $\underline{X}_0 + t\varphi$ is a relative minimizer of E_{H_0} for small $|\, t \,|$.

Therefore we have

$$(2.9) \qquad \Delta(\underline{X}_0 + t\varphi) = 2H_0(\underline{X}_0 + t\varphi)_u \wedge (\underline{X}_0 + t\varphi)_v \qquad \text{for small } |\, t \,|,$$

and there results $0 = \varphi_u \wedge \varphi_v$.

Now we obtain

$$(2.10) \qquad \delta = D^2 E_{H_0}(\underline{X}_0)(\varphi, \varphi) = 2,$$

but this contradicts (2.6), so we complete the proof of Proposition 2.3. ▫

Because E_H is not differentiable on $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$, we can not consider $D^2E_H(\underline{X})$. But in some sense, the following result can be used in place of positive-definiteness of $D^2E_H(\underline{X})$.

Proposition 2.4. *Let $H_0 \neq 0$ be a constant with the property that E_{H_0} admits a relative minimizer $\underline{X}_0 \in \{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$. Then there exists a constant $\alpha > 0$ such that if $H \in \mathcal{H}_\alpha$, there is a constant $\delta > 0$ depending only on α and X_D for which the following inequality holds.*

$$(2.11) \quad \int_\Omega |\, \nabla\varphi \,|^2 \, dw + 4H_0 \int_\Omega \underline{X} \cdot \varphi_u \wedge \varphi_v \, dw \geq \delta \int_\Omega |\, \nabla\varphi \,|^2 \, dw, \quad \textit{for all } \varphi \in H_0^1.$$

Here $\underline{X}$ is the S-solution to (1.1), (1.2).

Proof. Note first that S-solution of E_H exists for any $H \in \mathcal{H}_\alpha$ (see the proof of Struwe [8, Theorem 1.3]). By Proposition 2.3, we have a constant $\delta_1 > 0$ such that for all $\varphi \in H_0^1$,

$$(2.12) \qquad \int_\Omega |\, \nabla\varphi \,|^2 \, dw + 4H_0 \int_\Omega \underline{X}_0 \cdot \varphi_u \wedge \varphi_v \, dw \geq \delta_1 \int_\Omega |\, \nabla\varphi \,|^2 \, dw.$$

Hence for any $\varphi \in H_0^1$ we have

$$\begin{aligned}(2.13) \int_\Omega |\, \nabla\varphi \,|^2 \, dw &+ 4H_0 \int_\Omega \underline{X} \cdot \varphi_u \wedge \varphi_v \, dw \\ &= \int_\Omega |\, \nabla\varphi \,|^2 \, dw + 4H_0 \int_\Omega \underline{X}_0 \cdot \varphi_u \wedge \varphi_v \, dw + 4H_0 \int_\Omega (\underline{X} - \underline{X}_0) \cdot \varphi_u \wedge \varphi_v \, dw \\ &\geq \delta_1 \int_\Omega |\, \nabla\varphi \,|^2 \, dw + 4H_0 \int_\Omega (\underline{X} - \underline{X}_0) \cdot \varphi_u \wedge \varphi_v \, dw.\end{aligned}$$

Therefore, Proposition 2.4 follows from Lemma 2.2. ∎

Remark 2.5. Proposition 2.4 is almost the same as the result of Wang [10, Lemma 3.1]. But, it seems natural to state in the above form.

§3. Relative Minimizer

Now we can show a relation between $E_H(X)$ for arbitrary $X \in \{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$ and $E_H(\underline{X})$.

Theorem 3.1. *Let α be as in Proposition* 2.4. *Then for any $X \in \{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$, if $H \in \mathcal{H}_\alpha$, we have*

$$(3.1)\quad E_H(X) = E_H(\underline{X}) + \frac{1}{2}\int_\Omega |\nabla\varphi|^2\, dw + 2H_0 \int_\Omega \underline{X}\cdot\varphi_u \wedge \varphi_v\, dw + 2V_H(\varphi) + O(\alpha)\left(\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2} + \int_\Omega |\nabla\varphi|^2\, dw\right),$$

where $\underline{X}$ is the S-solution to (1.1), (1.2), *and $\varphi = X - \underline{X} \in H_0^1(\Omega; \boldsymbol{R}^3)$.*

PROOF. Let $\varphi = X - \underline{X} \in H_0^1(\Omega; \boldsymbol{R}^3)$. Then, by the fact that $\underline{X}$ satisfies (1.1), we have

$$\begin{aligned}(3.2)\ E_H(X) &= E_H(\underline{X}+\varphi)\\ &= \frac{1}{2}\int_\Omega |\nabla(\underline{X}+\varphi)|^2\, dw + \frac{2}{3}\int_\Omega Q(\underline{X}+\varphi)\cdot(\underline{X}+\varphi)_u \wedge (\underline{X}+\varphi)_v\, dw\\ &= E_H(\underline{X}) - 2\int_\Omega H(\underline{X})\varphi\cdot \underline{X}_u \wedge \underline{X}_v\, dw + \frac{2}{3}\int_\Omega Q(\underline{X}+\varphi)\cdot\varphi_u\wedge\varphi_v\, dw\\ &\quad + \frac{1}{2}\int_\Omega |\nabla\varphi|^2\, dw + \frac{2}{3}\int_\Omega (Q(\underline{X}+\varphi) - Q(\underline{X}))\cdot \underline{X}_u\wedge\underline{X}_v\, dw\\ &\quad + \frac{2}{3}\int_\Omega Q(\underline{X}+\varphi)\cdot(\underline{X}_u\wedge\varphi_v + \varphi_u\wedge\underline{X}_v)\, dw.\end{aligned}$$

Now we estimate terms in the right-hand side of the above equation. First, it is easy to see the following.

$$(3.3)\quad \int_\Omega (Q(\underline{X}+\varphi) - Q(\underline{X}))\cdot\underline{X}_u\wedge\underline{X}_v\, dw = \int_\Omega dQ(\underline{X})\varphi\cdot\underline{X}_u\wedge\underline{X}_v\, dw + O(\alpha)\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2}.$$

Next, by decomposing $Q(\underline{X}+\varphi) = Q(\varphi) + Q(\underline{X}+\varphi) - H_0(\underline{X}+\varphi) + H_0\underline{X} + H_0\varphi - Q(\varphi)$, and using the definition (1.8) of a metric, we have

$$(3.4)\quad \int_\Omega Q(\underline{X}+\varphi)\cdot\varphi_u\wedge\varphi_v\, dw = \int_\Omega Q(\varphi)\cdot\varphi_u\wedge\varphi_v\, dw + H_0\int_\Omega \underline{X}\cdot\varphi_u\wedge\varphi_v\, dw + O(\alpha)\int_\Omega |\nabla\varphi|^2\, dw.$$

Finally, by similar calculations and the integration by parts,

$$(3.5)\ \int_\Omega Q(\underline{X}+\varphi)\cdot(\underline{X}_u\wedge\varphi_v + \varphi_u\wedge\underline{X}_v)\, dw = \int_\Omega \varphi\cdot(\underline{X}_u\wedge dQ(\underline{X})\underline{X}_v + dQ(\underline{X})\underline{X}_u\wedge\underline{X}_v)\, dw + 2H_0\int_\Omega \underline{X}\cdot\varphi_u\wedge\varphi_v\, dw + O(\alpha)\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2}.$$

Note further that by an algebraic formula, we have

$$(3.6)\quad \int_\Omega \varphi\cdot(\underline{X}_u\wedge dQ(\underline{X})\underline{X}_v + dQ(\underline{X})\underline{X}_u\wedge\underline{X}_v)\, dw + \int_\Omega dQ(X)\varphi\cdot\underline{X}_u\wedge\underline{X}_v\, dw = 3\int_\Omega H(\underline{X})\varphi\cdot\underline{X}_u\wedge\underline{X}_v\, dw.$$

So from (3.2)-(3.6) we have

$$\begin{aligned} E_H(X) &= E_H(\underline{X}) + \frac{1}{2}\int_\Omega |\nabla\varphi|^2\, dw + 2H_0\int_\Omega \underline{X}\cdot\varphi_u\wedge\varphi_v\, dw \\ &\quad + 2V_H(\varphi) + O(\alpha)\left(\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2} + \int_\Omega |\nabla\varphi|^2\, dw\right), \end{aligned}$$

and we obtain Theorem 3.1. ∎

Proposition 3.2. *Let* α *be as in Proposition* 2.4 *and* $H \in \mathcal{H}_\alpha$. *Then for any* X *which is a solution to* (1.1), (1.2), *the following holds.*

$$\begin{aligned} 2V_H(\varphi) &= -\frac{1}{3}\int_\Omega |\nabla\varphi|^2\, dw - \frac{4}{3}H_0\int_\Omega \underline{X}\cdot\varphi_u\wedge\varphi_v\, dw \\ &\quad + O(\alpha)\left(\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2} + \int_\Omega |\nabla\varphi|^2\, dw\right), \end{aligned} \tag{3.7}$$

where $\underline{X}$ *is the S-solution to* (1.1), (1.2) *and* $\varphi = X - \underline{X} \in H_0^1(\Omega;\mathbb{R}^3)$.

Proof. Because $X = \underline{X} + \varphi$ and $\underline{X}$ satisfy (1.1), we have

$$\begin{aligned} 0 &= \int_\Omega \nabla\varphi\cdot\nabla(\underline{X}+\varphi)\, dw + 2\int_\Omega H(\underline{X}+\varphi)\varphi\cdot(\underline{X}+\varphi)_u\wedge(\underline{X}+\varphi)_v\, dw \\ &= \int_\Omega |\nabla\varphi|^2\, dw + 2\int_\Omega (H(\underline{X}+\varphi) - H(\underline{X}))\varphi\cdot\underline{X}_u\wedge\underline{X}_v\, dw \\ &\quad + 2\int_\Omega H(\underline{X}+\varphi)\varphi\cdot(\underline{X}_u\wedge\varphi_v + \varphi_u\wedge\underline{X}_v)\, dw \\ &\quad + 2\int_\Omega H(\underline{X}+\varphi)\varphi\cdot\varphi_u\wedge\varphi_v\, dw. \end{aligned} \tag{3.8}$$

By similar calculations as those in the proof of Theorem 3.1, it is easy to see

$$\int_\Omega (H(\underline{X}+\varphi) - H(\underline{X}))\varphi\cdot\underline{X}_u\wedge\underline{X}_v\, dw = O(\alpha)\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2}, \tag{3.9}$$

and

$$\begin{aligned} &\int_\Omega H(\underline{X}+\varphi)\varphi\cdot(\underline{X}_u\wedge\varphi_v + \varphi_u\wedge\underline{X}_v)\, dw \\ &\qquad = 2H_0\int_\Omega \underline{X}\cdot\varphi_u\wedge\varphi_v\, dw + O(\alpha)\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2}. \end{aligned} \tag{3.10}$$

Moreover from $H(\underline{X}+\varphi)\varphi = (H(\underline{X}+\varphi) - H_0)(\underline{X}+\varphi) + (H_0 - H(\underline{X}+\varphi))\underline{X} + (H_0\varphi - Q(\varphi)) + Q(\varphi)$, we have

$$\begin{aligned} \int_\Omega H(\underline{X}+\varphi)\varphi\cdot\varphi_u\wedge\varphi_v\, dw &= \int_\Omega Q(\varphi)\cdot\varphi_u\wedge\varphi_v\, dw \\ &\quad + O(\alpha)\left(\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2} + \int_\Omega |\nabla\varphi|^2\, dw\right). \end{aligned} \tag{3.11}$$

Now from (3.8)-(3.11), we obtain

$$(3.12)\quad 0 = \int_\Omega |\nabla\varphi|^2\, dw + 4H_0 \int_\Omega \underline{X}\cdot\varphi_u\wedge\varphi_v\, dw + 2\int_\Omega Q(\varphi)\cdot\varphi_u\wedge\varphi_v\, dw + O(\alpha)\left(\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2} + \int_\Omega |\nabla\varphi|^2\, dw\right),$$

and the proof is completed. ∎

From Theorem 3.1 and Proposition 3.2, we obtain the following Corollary.

Corollary 3.3. *If $H \in \mathcal{A}$ and $\overline{X}$ is the L-solution to* (1.1), (1.2), *then for sufficiently small α, we have $E_H(\overline{X}) > E_H(\underline{X})$, where $\underline{X}$ is the S-solution to* (1.1), (1.2).

PROOF. Let $\varphi = \overline{X} - \underline{X}$. Then $X = \overline{X}$ satisfies (3.1) and (3.7), so we obtain

$$(3.13)\quad E_H(\overline{X}) - E_H(\underline{X}) \geq \frac{1}{6}\left(\int_\Omega |\nabla\varphi|^2\, dw + 4H_0\int_\Omega \underline{X}\cdot\varphi_u\wedge\varphi_v\, dw\right) - c\alpha\left(\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2} + \int_\Omega |\nabla\varphi|^2\, dw\right).$$

But, by Proposition 2.4, there exists $\delta > 0$ such that

$$E_H(\overline{X}) - E_H(\underline{X}) \geq \delta\int_\Omega |\nabla\varphi|^2\, dw - c\alpha\left(\left(\int_\Omega |\nabla\varphi|^2\, dw\right)^{1/2} + \int_\Omega |\nabla\varphi|^2\, dw\right).$$

Now from Lemma 2.1, we have $D(\overline{X} - \underline{X}) > c$ where c is independent of α, so we obtain Corollary 3.3. ∎

§4. Some Additional Results

We show here some applications of the results obtained in the previous section.

When $H \equiv H_0 \in \boldsymbol{R}$, we can show the uniqueness of the relative minimizer. First we need the following results corresponding to Theorem 3.1 and Proposition 3.2 in the case of $H \equiv H_0 \in \boldsymbol{R}$.

Proposition 4.1. *Let $H \equiv H_0 \in \boldsymbol{R}$ be as in Theorem* 1.5, *then we have*

$$(4.1)\quad E_{H_0}(X) = E_{H_0}(\underline{X}_0) + \frac{1}{2}\int_\Omega |\nabla\varphi|^2\, dw + 2H_0\int_\Omega \underline{X}_0\cdot\varphi_u\wedge\varphi_v\, dw + 2V(\varphi),$$

for any $X \in \{X_D\} + H_0^1(\Omega;\boldsymbol{R}^3)$, where $\underline{X}_0$ is the S-solution to (1.1), (1.2) *and $\varphi = X - \underline{X}_0 \in H_0^1(\Omega;\boldsymbol{R}^3)$.*

The above result is, of course, Taylor expansion of $E_{H_0}(X)$.

Proposition 4.2. *Let $H \equiv H_0 \in \boldsymbol{R}$ be as in Theorem* 1.5, *then for any X which is a solution to* (1.1), (1.2), *the following equality holds.*

$$(4.2)\quad 2V(\varphi) = -\frac{1}{3}\int_\Omega |\nabla\varphi|^2\, dw - \frac{4}{3}H_0\int_\Omega \underline{X}_0\cdot\varphi_u\wedge\varphi_v\, dw,$$

where $\underline{X}_0$ is the S-solution to (1.1), (1.2) *and $\varphi = X - \underline{X}_0 \in H_0^1(\Omega;\boldsymbol{R}^3)$.*

By the above Propositions, we can show the following well-known uniqueness result (see Struwe [7, Corollary IV.1.3]).

Corollary 4.3. *If* X_1 *and* X_2 *are relative minimizers of* E_{H_0} *on* $\{X_D\} + H_0^1(\Omega; \boldsymbol{R}^3)$, *then* $X_1 = X_2$.

PROOF. Let $\varphi = X_2 - X_1 \in H_0^1(\Omega; \boldsymbol{R}^3)$. Then, by Proposition 4.1, 4.2, it is easy to see

$$E_{H_0}(X_2) = E_{H_0}(X_1) + \frac{1}{6}\left(\int_\Omega |\, \varphi \,|^2 \; dw + 4H_0 \int_\Omega X_1 \cdot \varphi_u \wedge \varphi_v \, dw\right). \tag{4.3}$$

So if $\varphi \neq 0$, by Proposition 2.3, we have

$$E_{H_0}(X_2) > E_{H_0}(X_1). \tag{4.4}$$

By the same way, we have also

$$E_{H_0}(X_1) > E_{H_0}(X_2). \tag{4.5}$$

This contradiction gives $X_2 - X_1 = 0$ and we obtain Corollary 4.3. ∎

In the case of variable curvature function H, we do not know the uniqueness of the relative minimizer, but we can obtain information about the relative minimizer.

Proposition 4.4. *Let* $\underline{X}$ *be the S-solution and* X *any solution to* (1.1), (1.2). *Moreover we denote some neighborhood of* $\underline{X}$ *in* H_0^1 *as* U *which depends only on* α. *Then if* X *lies outside* U, *we have*

$$E_H(X) > E_H(\underline{X}) \tag{4.6}$$

The proof is the same as that of Corollary 3.3.

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