Relative Minimizer of Prescribed Mean Curvature Equation

Dedicated to Professor Yoshihiro Ichijō on his 65th birthday

By

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Abstract

We consider a relative minimizer of the $H$-system where $H$ is not necessarily a constant. It is known now that for $H$ in a neighborhood of some appropriate constant $H_0$, there exists a relative minimizer $\tilde{X}$ of the functional $E_H$. In this paper, we show some properties of a relative minimizer $\tilde{X}$, especially that there exists some neighborhood $U$ of $\tilde{X}$ outside of which every critical value is greater than that of $\tilde{X}$.

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§1. Introduction

We consider the Dirichlet problem for the equation of prescribed mean curvature which is not necessarily constant.

Let $\Omega$ be the unit disk in $\mathbb{R}^2$;
$$\Omega = \{ w = (u, v); u^2 + v^2 < 1 \}.$$  

The Dirichlet problem for the equation of prescribed mean curvature $H$ is expressed as

\begin{align*}
\Delta X &= 2H(X)X_u \wedge X_v, \quad \text{in } \Omega, \\
X &= X_D, \quad \text{on } \partial \Omega.
\end{align*}

(1.1) \hspace{1cm} (1.2)

Here, we denote $X_u = \frac{\partial}{\partial u} X$ and $X_v = \frac{\partial}{\partial v} X$, and $\wedge$ is the exterior product in $\mathbb{R}^3$. $H: \mathbb{R}^3 \to \mathbb{R}$ is a given function and $X_D$ is a given function mainly of class $C^2(\overline{\Omega}, \mathbb{R}^3)$. 
Then we consider a functional $E_H$ on $\{X_D\} + H_0^1(\Omega; \mathbb{R}^3)$, where $E_H(X) = D(X) + 2V_H(X)$. Here
\begin{equation}
D(X) = \frac{1}{2} \int_\Omega |\nabla X|^2 \, dw
\end{equation}
is the Dirichlet integral and
\begin{equation}
V_H(X) = \frac{1}{3} \int_\Omega Q(X) \cdot X_u \wedge X_v \, dw
\end{equation}
is the $Q$-volume introduced by Hildebrandt, where $Q(X)$ is defined through $H(X)$
\begin{equation}
Q(x_1, x_2, x_3) = \left( \int_0^{x_1} H(s, x_2, x_3) \, ds, \int_0^{x_2} H(x_1, s, x_3) \, ds, \int_0^{x_3} H(x_1, x_2, s) \, ds \right).
\end{equation}
When $H \equiv H_0 \in \mathbb{R}$, we have $E_{H_0}(X) = D(X) + 2H_0V(X)$, where
\begin{equation}
V(X) = \frac{1}{3} \int_\Omega X \cdot X_u \wedge X_v \, dw
\end{equation}
is the algebraic volume of a surface $X$.

We summarize basic results here. Note that solutions to the Dirichlet problem (1.1), (1.2) are characterized as critical points of $E_H(X)$.

First, in the case of $H \equiv H_0 \in \mathbb{R}$, following two theorems are fundamental and now well-known. For their proofs and for further references, see for example Struwe [7], [9], or Brezis-Coron [1].

**Theorem 1.1.** Suppose $H \equiv H_0 \in \mathbb{R}$ and let $X_D \in L^\infty \cap H^1(\Omega; \mathbb{R}^3)$ be given. Assume that
\[ |H_0| \cdot ||X_D||_{\text{Lin}} < 1 \]
is satisfied. Then there is a solution $X_0 \in \{X_D\} + H_0^1(\Omega; \mathbb{R}^3)$ to (1.1), (1.2). Moreover $X_0$ is characterized as a strict relative minimizer of $E_{H_0}$ in this space.

**Remark 1.2.** The fact that a relative minimizer is also a strict relative minimizer is originally due to Brezis-Coron [1]. We give this result in Proposition 2.3 following Struwe [7].

When $H_0 \neq 0$ and $X_D$ is non-constant, there exists a second solution.

**Theorem 1.3.** Suppose $H \equiv H_0 \neq 0$ and let $X_D \in L^\infty \cap H^1(\Omega; \mathbb{R}^3)$ be non-constant. Assume moreover that $E_{H_0}$ admits a local minimum $X_0$ in the class $\{X_D\} + H_0^1(\Omega; \mathbb{R}^3)$. Then there exists a solution $X \in \{X_D\} + H_0^1(\Omega; \mathbb{R}^3)$ of (1.1), (1.2) different from $X_0$. Moreover $X$ satisfies
\begin{equation}
E_{H_0}(X_0) < E_{H_0}(X) = \inf_{p \in P} \sup_{X \in \mathcal{E}} E_{H_0}(X) < E_{H_0}(X_0) + \beta_0,
\end{equation}
where $\beta_0 = \frac{4\pi}{3H_0}$ and $P = \{p \in C^0([0,1]; \{X_D\} + H_0^1): p(0) = X_0, E_{H_0}(p(1)) < E_{H_0}(X_0)\}$.

For variable curvature function $H$, similar results were obtained. The following result is due to Hildebrandt [2, Satz 2].
Theorem 1.4. Suppose $H$ is of class $C^1$ and let $X_D \in L^\infty \cap H^1(\Omega; \mathbb{R}^3)$ be given with $\| X_D \|_\infty < 1$. Then if
\[ h = \text{ess sup}_{|X| \leq 1} |H(X)| < 1, \]
there exists a solution $X \in \{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$ to (1.1), (1.2) such that
\[ X = \inf\{ E_H(X); X \in \{X_D\} + H^1_0(\Omega; \mathbb{R}^3), \| X \|_\infty \leq 1 \}. \]

Recently, a second solution was also obtained by Struwe [8] and Wang [10]. To state the results, introduce a metric
\[ [H - H_0] = \text{ess sup}_{X \in \mathbb{R}^3} \{(1 + |X|)(|H(X) - H_0| + |\nabla H(X)|) \]
\[ + |Q(X) - H_0X| + |dQ(X) - H_0id| \}, \]
and denote an $\alpha$-neighborhood of $H_0$ as $\mathcal{H}_\alpha = \{H; |H - H_0| < \alpha \}$. Then the following results hold.

First, Struwe [8, Theorem 1.3] proved the following.

Theorem 1.5. Suppose $X_D \in C^1(\overline{\Omega}; \mathbb{R}^3)$ is non-constant and for $H_0 \in \mathbb{R} \setminus \{0\}$ the functional $E_{H_0}$ admits a relative minimizer in $\{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$. Then there exists a number $\alpha > 0$ such that for a dense set $\Lambda$ of curvature functions $H$ in $\mathcal{H}_\alpha$, the Dirichlet problem (1.1), (1.2) admits at least two distinct regular solutions in $\{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$.

Then Wang [10, Theorem 1.6] extended the above result to the full $\alpha$-neighborhood and obtained the following.

Theorem 1.6. Suppose $X_D \in C^2(\overline{\Omega}; \mathbb{R}^3)$ is non-constant and for $H_0 \in \mathbb{R} \setminus \{0\}$ the functional $E_{H_0}$ admits a relative minimizer in $\{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$. Then there exists a number $\alpha > 0$ such that for a curvature function $H$ in the full $\alpha$-neighborhood $\mathcal{H}_\alpha$ of $H_0$, the Dirichlet problem (1.1), (1.2) admits at least two distinct solutions in $\{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$.

Among the solutions obtained in the above theorems, one is a relative minimizer and the other is of unstable type of $E_H$. Following Wang [10], we call the former S-solution and the latter L-solution. He showed also that the S-solution is a “strict” relative minimizer of $E_H$ in the sense
\[ E_H(X) < E_H(\overline{X}), \]
where $X$ is the S-solution and $\overline{X}$ is the L-solution.

In this note, we want to study a relation between $E_H(X)$ and $E_H(\overline{X})$ for arbitrary $X \in \{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$ (see Theorem 3.1), and to show an inequality (1.9) as a special case.

§2. Preliminary Results

We list two lemmas which are necessary in the following proofs.
Lemma 2.1. Let \( \alpha \) and \( A \) be as in Theorem 1.5. Then there exists a constant \( c > 0 \) independent of \( \alpha \) such that if \( H \in A \),

\[
D(\overline{X} - \overline{X}) > c,
\]

where \( \overline{X} \) (resp. \( \overline{X} \)) is the S-solution (resp. L-solution) to (1.1), (1.2).

See Wang [10, Lemma 2.6].

Lemma 2.2. Let \( \alpha, X_D, \) and \( H_0 \) be as in Theorem 1.5. Then for any \( \epsilon > 0 \), there exists a constant \( \alpha > 0 \) with the property that for any curvature function \( H \in \mathcal{H}_\alpha \), if \( \overline{X} \) is the S-solution to (1.1), (1.2), then

\[
\| \overline{X} - X_0 \| \leq \epsilon,
\]

where \( X_0 \) is a relative minimizer of \( E_{H_0} \).

See Wang [10, Lemma 3.2].

As we note already in Remark 1.2, Brezis-Coron [1] proved that a relative minimizer is also a strict relative minimizer. For the sake of completeness, we give the proof of this result following Struwe [7, Lemma IV.1.2].

Proposition 2.3. For \( H_0 \in \mathcal{R} \setminus \{0\} \), suppose that \( E_{H_0} \) admits a relative minimizer \( X_0 \) in the space \( \{X_D\} + L^\infty \cap H_0^1(\Omega; \mathbb{R}^3) \). Then \( X_0 \) is a strict relative minimizer of \( E_{H_0} \) in \( \{X_D\} + H_0^1(\Omega; \mathbb{R}^3) \), and there is a constant \( \delta > 0 \) such that

\[
\int_\Omega |\nabla \varphi|^2 \, dw + 4H_0 \int_\Omega X_0 : \varphi_a \wedge \varphi_a \, dw \geq \delta \int_\Omega |\nabla \varphi|^2 \, dw, \quad \text{for all } \varphi \in H_0^1.
\]

PROOF. It is evident that

\[
D^2E_{H_0}(X_0)(\varphi, \varphi) = \int_\Omega |\nabla \varphi|^2 \, dw + 4H_0 \int_\Omega X_0 : \varphi_a \wedge \varphi_a \, dw.
\]

Note that \( C_0^\infty \subset L^\infty \cap H_0^1 \) and \( C_0^\infty \) is dense in \( H_0^1 \), so the following inequality holds trivially.

\[
\delta = \inf \{ D^2E_{H_0}(X_0)(\varphi, \varphi) ; \varphi \in H_0^1, \ D(\varphi) = 1 \} \geq 0.
\]

Now we must show \( \delta > 0 \).

If \( \delta = 0 \), then a minimizing sequence for \( \delta \) is relatively compact in \( H_0^1(\Omega; \mathbb{R}^3) \) because \( D^2V(X_0) \) is compact (see for example Struwe [7, Theorem III.2.3]). So we have \( \varphi \in H_0^1(\Omega; \mathbb{R}^3) \) such that \( D(\varphi) = 1 \) and

\[
\delta = D^2E_{H_0}(X_0)(\varphi, \varphi) = 0.
\]

Then \( \varphi \) satisfies

\[
\Delta \varphi = 2H_0(\overline{X}_t \wedge \overline{\varphi}_t + \varphi_t \wedge \overline{X}_t),
\]

and it follows that \( \varphi \in L^\infty \) (see for example Struwe [7, Theorem III.5.1]).

Hence by minimality of \( X_0 \), for small \( |t| \), we have

\[
E_{H_0}(X_0) - E_{H_0}(X_0 + t\varphi) = E_{H_0}(X_0) + 2H_0t^2V(\varphi),
\]

where \( X_0 \) is the S-solution to (1.1), (1.2).
so it follows $V(\varphi) = 0$. Then $E_{H_0}(X_0 + t\varphi) = E_{H_0}(X_0)$ for any $t \in \mathbb{R}$, and this implies $X_0 + t\varphi$ is a relative minimizer of $E_{H_0}$ for small $|t|$.

Therefore we have

\begin{equation}
\Delta(X_0 + t\varphi) = 2H_0(X_0 + t\varphi) \cdot \nabla(X_0 + t\varphi) \quad \text{for small } |t|, 
\end{equation}

and there results $0 = \varphi_u \wedge \varphi_v$.

Now we obtain

\begin{equation}
\delta = D^2E_{H_0}(X_0)(\varphi, \varphi) = 2, 
\end{equation}

but this contradicts (2.6), so we complete the proof of Proposition 2.3. \qed

Because $E_H$ is not differentiable on $\{X_D\} + H_0^1(\Omega; \mathbb{R}^3)$, we cannot consider $D^2E_H(X)$. But in some sense, the following result can be used in place of positive-definiteness of $D^2E_H(X)$.

**Proposition 2.4.** Let $H_0 \neq 0$ be a constant with the property that $E_{H_0}$ admits a relative minimizer $X_0 \in \{X_D\} + H_0^1(\Omega; \mathbb{R}^3)$. Then there exists a constant $\alpha > 0$ such that if $H \in \mathcal{H}_\alpha$, there is a constant $\delta > 0$ depending only on $\alpha$ and $X_D$ for which the following inequality holds.

\begin{equation}
\int_{\Omega} |\nabla \varphi|^2 \, dw + 4H_0 \int_{\Omega} (X \cdot \varphi_u \wedge \varphi_v) \, dw \geq \delta \int_{\Omega} |\nabla \varphi|^2 \, dw, \quad \text{for all } \varphi \in H_0^1. 
\end{equation}

Here $X$ is the $S$-solution to (1.1), (1.2).

**PROOF.** Note first that $S$-solution of $E_H$ exists for any $H \in \mathcal{H}_\alpha$ (see the proof of Struwe [8, Theorem 1.3]). By Proposition 2.3, we have a constant $\delta_1 > 0$ such that for all $\varphi \in H_0^1$,

\begin{equation}
\int_{\Omega} |\nabla \varphi|^2 \, dw + 4H_0 \int_{\Omega} (X_0 \cdot \varphi_u \wedge \varphi_v) \, dw \geq \delta_1 \int_{\Omega} |\nabla \varphi|^2 \, dw.
\end{equation}

Hence for any $\varphi \in H_0^1$ we have

\begin{equation}
\int_{\Omega} |\nabla \varphi|^2 \, dw + 4H_0 \int_{\Omega} (X \cdot \varphi_u \wedge \varphi_v) \, dw 
= \int_{\Omega} |\nabla \varphi|^2 \, dw + 4H_0 \int_{\Omega} (X_0 \cdot \varphi_u \wedge \varphi_v) \, dw + 4H_0 \int_{\Omega} (X - X_0) \cdot \varphi_u \wedge \varphi_v \, dw 
\geq \delta_1 \int_{\Omega} |\nabla \varphi|^2 \, dw + 4H_0 \int_{\Omega} (X - X_0) \cdot \varphi_u \wedge \varphi_v \, dw.
\end{equation}

Therefore, Proposition 2.4 follows from Lemma 2.2. \qed

**Remark 2.5.** Proposition 2.4 is almost the same as the result of Wang [10, Lemma 3.1]. But, it seems natural to state in the above form.

§3. Relative Minimizer

Now we can show a relation between $E_H(X)$ for arbitrary $X \in \{X_D\} + H_0^1(\Omega; \mathbb{R}^3)$ and $E_H(X)$. 
Theorem 3.1. Let $\alpha$ be as in Proposition 2.4. Then for any $X \in \{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$, if $H \in \mathcal{H}_\alpha$, we have

\begin{align}
E_H(X) &= E_H(X) + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \, dw + 2H_0 \int_\Omega X \cdot \varphi_u + \varphi_v \, dw + 2V_H(\varphi) \\
&\quad + O(\alpha) \left( \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2} + \int_\Omega |\nabla \varphi|^2 \, dw \right),
\end{align}

where $X$ is the $S$-solution to (1.1), (1.2), and $\varphi = X - \bar{X} \in H^1_0(\Omega; \mathbb{R}^3)$.

Proof. Let $\varphi = X - \bar{X} \in H^1_0(\Omega; \mathbb{R}^3)$. Then, by the fact that $\bar{X}$ satisfies (1.1), we have

\begin{align}
E_H(X) &= E_H(X + \varphi) \\
&= \frac{1}{2} \int_\Omega |\nabla (X + \varphi)|^2 \, dw + \frac{2}{3} \int_\Omega Q(X + \varphi) \cdot (X + \varphi) \wedge (X + \varphi) \, dw \\
&= E_H(X) - 2 \int_\Omega H(X) \varphi \cdot X_u \wedge X_v \, dw + \frac{2}{3} \int_\Omega Q(X + \varphi) \cdot \varphi_u \wedge \varphi_v \, dw \\
&\quad + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \, dw + \frac{2}{3} \int_\Omega (Q(X + \varphi) - Q(X)) \cdot X_u \wedge X_v \, dw \\
&\quad + \frac{2}{3} \int_\Omega Q(X + \varphi) \cdot (X_u \wedge \varphi_u + \varphi_v \wedge X_v) \, dw.
\end{align}

Now we estimate terms in the right-hand side of the above equation. First, it is easy to see the following.

\begin{align}
\int_\Omega (Q(X + \varphi) - Q(X)) \cdot X_u \wedge X_v \, dw \\
&= \int_\Omega dQ(X) \varphi \cdot X_u \wedge X_v \, dw + O(\alpha) \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2}.
\end{align}

Next, by decomposing $Q(X + \varphi) = Q(\varphi) + Q(X + \varphi) - H_0(X + \varphi) + H_0X + H_0\varphi - Q(\varphi)$, and using the definition (1.8) of a metric, we have

\begin{align}
\int_\Omega Q(X + \varphi) \cdot \varphi_u \wedge \varphi_v \, dw \\
&= \int_\Omega Q(\varphi) \cdot \varphi_u \wedge \varphi_v \, dw + H_0 \int_\Omega X \cdot \varphi_u \wedge \varphi_v \, dw + O(\alpha) \int_\Omega |\nabla \varphi|^2 \, dw.
\end{align}

Finally, by similar calculations and the integration by parts,

\begin{align}
\int_\Omega Q(X + \varphi) \cdot (X_u \wedge \varphi_v + \varphi_u \wedge X_v) \, dw \\
&= \int_\Omega \varphi \cdot (X_u \wedge dQ(X) X_v + dQ(X) X_u \wedge X_v) \, dw + 2H_0 \int_\Omega X \cdot \varphi_u \wedge \varphi_v \, dw \\
&\quad + O(\alpha) \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2}.
\end{align}

Note further that by an algebraic formula, we have

\begin{align}
\int_\Omega \varphi \cdot (X_u \wedge dQ(X) X_v + dQ(X) X_u \wedge X_v) \, dw + \int_\Omega dQ(X) \varphi \cdot X_u \wedge X_v \, dw \\
&= 3 \int_\Omega H(X) \varphi \cdot X_u \wedge X_v \, dw.
\end{align}
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So from (3.2)-(3.6) we have

$$E_H(X) = E_H(X) + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \, dw + 2H_0 \int_\Omega X \cdot \varphi_u \wedge \varphi_v \, dw$$

$$+ 2V_H(\varphi) + O(\alpha) \left( \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2} + \int_\Omega |\nabla \varphi|^2 \, dw \right),$$

and we obtain Theorem 3.1. ■

**Proposition 3.2.** Let $\alpha$ be as in Proposition 2.4 and $H \in \mathcal{H}_\alpha$. Then for any $X$ which is a solution to (1.1), (1.2), the following holds.

$$2V_H(\varphi) = -\frac{1}{3} \int_\Omega |\nabla \varphi|^2 \, dw - \frac{4}{3} H_0 \int_\Omega X \cdot \varphi_u \wedge \varphi_v \, dw$$

$$+ O(\alpha) \left( \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2} + \int_\Omega |\nabla \varphi|^2 \, dw \right),$$

where $X$ is the $S$-solution to (1.1), (1.2) and $\varphi = X - \bar{X} \in H_0^1(\Omega; \mathbb{R}^3)$.

**Proof.** Because $X = X + \varphi$ and $X$ satisfy (1.1), we have

$$0 = \int_\Omega \nabla \varphi \cdot \nabla (X + \varphi) \, dw + 2 \int_\Omega H(X + \varphi) \varphi \cdot (X + \varphi)_u \wedge (X + \varphi)_v \, dw$$

$$= \int_\Omega |\nabla \varphi|^2 \, dw + 2 \int_\Omega (H(X + \varphi) - H(X))\varphi \cdot X_u \wedge X_v \, dw$$

$$+ 2 \int_\Omega H(X + \varphi)\varphi \cdot (X_u \wedge \varphi_v + \varphi_u \wedge X_v) \, dw$$

$$+ 2 \int_\Omega H(X + \varphi)\varphi \cdot \varphi_v \, dw.$$  

By similar calculations as those in the proof of Theorem 3.1, it is easy to see

$$\int_\Omega (H(X + \varphi) - H(X))\varphi \cdot X_u \wedge X_v \, dw = O(\alpha) \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2},$$

and

$$\int_\Omega H(X + \varphi)\varphi \cdot (X_u \wedge \varphi_v + \varphi_u \wedge X_v) \, dw$$

$$= 2H_0 \int_\Omega X \cdot \varphi_u \wedge \varphi_v \, dw + O(\alpha) \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2}.$$  

Moreover from $H(X + \varphi)\varphi = (H(X + \varphi) - H_0)(X + \varphi) + (H_0 - H(X + \varphi))X + (H_0 \varphi - Q(\varphi)) + Q(\varphi)$, we have

$$\int_\Omega H(X + \varphi)\varphi \cdot \varphi_v \, dw = \int_\Omega Q(\varphi) \cdot \varphi_u \wedge \varphi_v \, dw$$

$$+ O(\alpha) \left( \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2} + \int_\Omega |\nabla \varphi|^2 \, dw \right).$$
Now from (3.8)-(3.11), we obtain
\[
(3.12) \quad 0 = \int_\Omega |\nabla \varphi|^2 \, dw + 4H_0 \int_\Omega X \cdot \varphi_u \wedge \varphi_v \, dw + 2 \int_\Omega Q(\varphi) \cdot \varphi_u \wedge \varphi_v \, dw \\
+ O(\alpha) \left( \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2} + \int_\Omega |\nabla \varphi|^2 \, dw \right),
\]
and the proof is completed. ■

From Theorem 3.1 and Proposition 3.2, we obtain the following Corollary.

**Corollary 3.3.** If \( H \in A \) and \( \overline{X} \) is the \( L \)-solution to (1.1), (1.2), then for sufficiently small \( \alpha \), we have \( E_H(\overline{X}) > E_H(X) \), where \( X \) is the \( S \)-solution to (1.1), (1.2).

**Proof.** Let \( \varphi = \overline{X} - X \). Then \( X = \overline{X} \) satisfies (3.1) and (3.7), so we obtain
\[
(3.13) \quad E_H(\overline{X}) - E_H(X) \geq \frac{1}{6} \left( \int_\Omega |\nabla \varphi|^2 \, dw + 4H_0 \int_\Omega X \cdot \varphi_u \wedge \varphi_v \, dw \right) \\
- c\alpha \left( \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2} + \int_\Omega |\nabla \varphi|^2 \, dw \right).
\]
But, by Proposition 2.4, there exists \( \delta > 0 \) such that
\[
E_H(\overline{X}) - E_H(X) \geq \delta \int_\Omega |\nabla \varphi|^2 \, dw - c\alpha \left( \left( \int_\Omega |\nabla \varphi|^2 \, dw \right)^{1/2} + \int_\Omega |\nabla \varphi|^2 \, dw \right).
\]
Now from Lemma 2.1, we have \( D(\overline{X} - X) > c \) where \( c \) is independent of \( \alpha \), so we obtain Corollary 3.3. ■

§4. Some Additional Results

We show here some applications of the results obtained in the previous section.

When \( H \equiv H_0 \in R \), we can show the uniqueness of the relative minimizer. First we need the following results corresponding to Theorem 3.1 and Proposition 3.2 in the case of \( H \equiv H_0 \in R \).

**Proposition 4.1.** Let \( H \equiv H_0 \in R \) be as in Theorem 1.5, then we have
\[
(4.1) \quad E_{H_0}(X) = E_{H_0}(X_0) + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \, dw + 2H_0 \int_\Omega X_0 \cdot \varphi_u \wedge \varphi_v \, dw + 2V(\varphi),
\]
for any \( X \in \{X_\beta \} + H_0^0(\Omega; R^3) \), where \( X_0 \) is the \( S \)-solution to (1.1), (1.2) and \( \varphi = X - X_0 \in H_0^0(\Omega; R^3) \).

The above result is, of course, Taylor expansion of \( E_{H_0}(X) \).

**Proposition 4.2.** Let \( H \equiv H_0 \in R \) be as in Theorem 1.5, then for any \( X \) which is a solution to (1.1), (1.2), the following equality holds.
\[
(4.2) \quad 2V(\varphi) = -\frac{1}{3} \int_\Omega |\nabla \varphi|^2 \, dw - \frac{4}{3} H_0 \int_\Omega X_0 \cdot \varphi_u \wedge \varphi_v \, dw,
\]
where \( X_0 \) is the \( S \)-solution to (1.1), (1.2) and \( \varphi = X - X_0 \in H_0^0(\Omega; R^3) \).
By the above Propositions, we can show the following well-known uniqueness result (see Struwe [7, Corollary IV.1.3]).

**Corollary 4.3.** If $X_1$ and $X_2$ are relative minimizers of $E_{H_0}$ on $\{X_D\} + H^1_0(\Omega; \mathbb{R}^3)$, then $X_1 = X_2$.

**Proof.** Let $\varphi = X_2 - X_1 \in H^1_0(\Omega; \mathbb{R}^3)$. Then, by Proposition 4.1, 4.2, it is easy to see

\[(4.3) \quad E_{H_0}(X_2) = E_{H_0}(X_1) + \frac{1}{6} \left( \int_{\Omega} |\varphi|^2 \, dw + 4H_0 \int_{\Omega} X_1 \cdot \varphi_u \wedge \varphi_v \, dw \right).\]

So if $\varphi \neq 0$, by Proposition 2.3, we have

\[(4.4) \quad E_{H_0}(X_2) > E_{H_0}(X_1).\]

By the same way, we have also

\[(4.5) \quad E_{H_0}(X_1) > E_{H_0}(X_2).\]

This contradiction gives $X_2 - X_1 = 0$ and we obtain Corollary 4.3. □

In the case of variable curvature function $H$, we do not know the uniqueness of the relative minimizer, but we can obtain information about the relative minimizer.

**Proposition 4.4.** Let $X$ be the $S$-solution and $X$ any solution to (1.1), (1.2). Moreover we denote some neighborhood of $X$ in $H^1_0$ as $U$ which depends only on $\alpha$. Then if $X$ lies outside $U$, we have

\[(4.6) \quad E_H(X) > E_H(\overline{X}).\]

The proof is the same as that of Corollary 3.3.

**References**


