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Theory of Fourier Microfunctions of Several Types (I)

Dedicated to Professor Yoshihiro Ichijyô on his 65th birthday

By

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Abstract

In this paper, we define the concept of Fourier microfunctions and investigate their structures. Thereby we obtain the decomposition of singularity of Fourier hyperfunctions. Then we can deduce the qualitative and quantitative property of Fourier hyperfunctions by examining only their singularity spectrums. We also investigate the vector-valued version.

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Introduction

This paper is the first part of the series of papers on the theory of Fourier microfunctions of several types, which is devided into three parts.

In this paper, we define the concept of Fourier microfunctions and vector-valued Fourier microfunctions, and study their fundamental properties. We can investigate these in a similar way to S.K.K. [19].

Let $\tilde{\boldsymbol{R}}^n = \boldsymbol{D}^n$ be the directional compactification of the n-dimensional Euclidean space $\boldsymbol{R}^n$ in the sense of Kawai [11] and put $\tilde{\boldsymbol{C}}^n = \tilde{\boldsymbol{R}}^n \times i\boldsymbol{R}^n$, $(i = \sqrt{-1})$. Put $M = \tilde{\boldsymbol{R}}^n$ and $X = \tilde{\boldsymbol{C}}^n$. Let $\tilde{\mathcal{O}}$ be the sheaf of slowly increasing and holomorphic functions over X. Then the sheaf $\tilde{\mathscr{B}}$ of Fourier hyperfunctions over M is defined by the relation

$$\tilde{\mathscr{B}} = \mathscr{H}^n_M(\tilde{\mathcal{O}}) = \mathrm{Dist}^n(M, \tilde{\mathcal{O}}),$$

and sections of $\tilde{\mathscr{B}}$ on an open set Ω in M are called Fourier hyperfunctions on Ω. Then, putting $\tilde{\mathscr{A}} = \tilde{\mathcal{O}}|_M$, a sheaf homomorphism $\tilde{\mathscr{A}} \to \tilde{\mathscr{B}}$ is defined and becomes and injection. Thereby, the concept of Fourier hyperfunctions can be considered as a generalization of the concept of slowly increasing and real-analytic functions. One purpose of this paper is to analyze the structure of the quotient sheaf $\tilde{\mathscr{B}}/\tilde{\mathscr{A}}$. We can analyze this structure by a similar way to the theory of Sato microfunctions. For Sato microfunctions, we refer the reader to Kaneko [7], [9], Kashiwara-Kawai-Kimura [10], Morimoto [12], [13], Sato [15], [16], [17], [18], and Sato-Kawai-Kashiwara [19]. The first target is to show that we can define the sheaf $\tilde{\mathscr{C}}$ of Fourier microfunctions over S^*M, which is the cosphere bundle over M, and we can have the fundamental exact sequence

$$0 \longrightarrow \tilde{\mathscr{A}} \longrightarrow \tilde{\mathscr{B}} \longrightarrow \pi_*\tilde{\mathscr{C}} \longrightarrow 0,$$

where $\pi: S^*M \to M$ is the projection and $\pi_*\tilde{\mathscr{C}}$ denotes the direct image of $\tilde{\mathscr{C}}$ with respect to π.

Further we investigate more precise structures of Fourier microfunctions. As to Fourier microfunctions, there exist another approaches by Kaneko [8], [9] and the flabbiness of the sheaf $\tilde{\mathscr{C}}$ has been proved in Kaneko [9]. But we have not yet proved the flabbiness of the sheaf $\tilde{\mathscr{C}}$ by our method.

The results of Fourier microfunctions were reported at Seminar on Real Analysis, 1989 (Hitotsubashi University, Tokyo) and at the short communication of ICM90 in Kyoto.

Next, we consider a similar construction of the theory of vector-valued Fourier microfunctions.

At last we note that Fourier microfunctions and vector-valued Fourier microfunctions on an open set in $S^*\boldsymbol{R}^n$ are nothing else but Sato microfunctions and vector-valued Sato microfunctions, respectively, where $S^*\boldsymbol{R}^n$ is the cosphere bundle over $\boldsymbol{R}^n$.

In section 1, we mention a general framework necessary to the study of the theory of Fourier microfunctions of several types in this series of papers.

In section 2, we construct the theory of Fourier microfunctions.

In section 3, we construct the theory of vector-valued Fourier microfunctions.

1. General theory

1.1. Real-monoidal transformation and real-comonoidal transformation. In this subsection we recall some general results concerning real-monoidal transformation and real-comonoidal transformation following S.K.K. [19].

Let N and M be real-analytic manifolds and $f: M \to N$ be a real-analytic map. We denote by TN(resp. TM) the tangent vector bundle over N(resp. M) and by T^*N(resp. T^*M) the cotangent vector bundle over N(resp. M). We can define the following canonical homomorphism:

(1.1) $\quad 0 \longrightarrow TM \longrightarrow TN \underset{N}{\times} M \longrightarrow T_M N \to 0$ (when f is an embedding),

$$T^*M \longleftarrow T^*N \underset{N}{\times} M \longleftarrow T_M^*N \longleftarrow 0,$$

where $T_M N$(resp. $T_M^* N$) is the normal (resp. conormal) fiber space. We denote by SM(resp. S^*M, SN, S^*N, $S_M N$, $S_M^* N$) the spherical bundle $(TM - M)/\boldsymbol{R}^+$ (resp. $(T^*M - M)/\boldsymbol{R}^+, \cdots$), where $\boldsymbol{R}^+$ is the multiplicative group of strictly positive, real numbers which operate on the fibers. $S_M^* N$ is not necessarily a fiber bundle.

Then, we have an inclusion

$$S_M^* N \hookrightarrow S^*N \underset{N}{\times} M$$

and we have a projection

$$\rho : S^*N \underset{N}{\times} M - S_M^* N \longrightarrow S^*M. \tag{1.2}$$

Suppose moreover that $\iota: M \to N$ is an embedding. Then we can provide the disjoint union ${}^M\tilde{N} = (N - M) \sqcup S_M N$ with a structure of real-analytic manifold with boundary $S_M N$. Since this is constructed in the same way as monoidal transforms of complex manifolds, we call ${}^M\tilde{N}$ the real-monoidal transform of N with center M. Let $\tau: {}^M\tilde{N} \to N$ be the canonical projection. Then, $\tau^{-1}(M)$ is isomorphic to the normal spherical bundle $S_M N$, and seen to be the boundary of ${}^M\tilde{N}$. Moreover, τ gives an isomorphism ${}^M\tilde{N} - S_M N \to N - M$. For a tangent vector $\xi \in T_M N_x - \{0\}$, we denote by $x + \xi 0$ the corresponding point of $S_M N \subset {}^M\tilde{N}$. Similarly, for a cotangent vector $\eta \in T_M^* N_x - \{0\}$, we denote by $(x, \eta\infty)$ the corresponding point of $S_M^* N$.

$D_M N$ is a subset of the fiber product $S_M N \underset{N}{\times} S_M^* N$ defined by $\{(x + \xi 0, (x, \eta\infty)) \in S_M N \underset{N}{\times} S_M^* N; \langle \xi, \eta \rangle \geqq 0\}$. We define the topology on the set ${}^M\tilde{N}^+ = (N - M) \sqcup D_M N$ as follows: $N - M \subset {}^M\tilde{N}^+$ is an open set and the topology of $N - M$ induced from ${}^M\tilde{N}^+$ is the usual one, and for a point $\tilde{x} \in D_M N \subset {}^M\tilde{N}^+$, a neighborhood of $\tilde{x}$ is a subset U such that $U \cap D_M N$ is a neighborhood of $\tilde{x}$ with respect to the usual topology of $D_M N$ and that the image of U under the projection $\pi: {}^M\tilde{N}^+ \to {}^M\tilde{N}$ is a neighborhood of $\pi(\tilde{x})$. We note that the topology of ${}^M\tilde{N}^+$ is not Hausdorff.

Let ${}^M\tilde{N}^*$ be a disjoint union of $(N - M)$ and $S_M^* N$, and $\tau: {}^M\tilde{N}^+ \to {}^M\tilde{N}^*$, and $\pi: {}^M\tilde{N}^* \to N$ be the canonical projections. ${}^M\tilde{N}^*$ is equipped with the quotient topology of ${}^M\tilde{N}^+$ under τ.

In this way we obtain a diagram of maps of topological spaces:

$$\begin{array}{ccccccc}
 & & {}^M\tilde{N}^+ & \hookleftarrow & D_M N & & \\
 & \pi\swarrow & & \tau\searrow\!\!\swarrow\pi & & \searrow\tau & \\
{}^M\tilde{N} & \hookleftarrow & S_M N & & {}^M\tilde{N}^* & \hookleftarrow & S_M^* N \\
 & \tau\searrow & & \tau\searrow\!\!\swarrow\pi & & \nearrow\pi & \\
 & & N & \hookleftarrow & M & &
\end{array} \tag{1.3}$$

Note that

1) All horizontal inclusions are closed embedding;
2) ${}^{M}\tilde{N}^{+}$ can be considered as a closed subspace of ${}^{M}\tilde{N} \underset{N}{\times} {}^{M}\tilde{N}^{*}$;
3) The maps ${}^{M}\tilde{N} \to N$ and ${}^{M}\tilde{N}^{+} \to {}^{M}\tilde{N}^{*}$ are proper and separated.

The following two general lemmas are used frequently in this paper.

Lemma 1.1. *Let Y be a d-codimensional submanifold of a topological manifold X of dimension n. Then, for every sheaf (or complex of sheaves) $\mathscr{F}$ on X, we can define the following homomorphism*

$$\mathscr{F}|_{Y} \longrightarrow \boldsymbol{R}\Gamma_{Y}(\mathscr{F})[d],$$

where $\boldsymbol{R}$ and Γ_Y denote respectively the derived functor in the derived category and the functor of taking the subsheaf with support in Y of Hatshorne [1].

Lemma 1.2. *Let X and Y be two topological spaces, and $f: X \to Y$ a separated and proper map, and $\mathscr{F}$ a sheaf over X. Then, for every point y of Y, the homomorphism*

$$R^{k}f_{*}(\mathscr{F})_{y} \longrightarrow H^{k}(f^{-1}(y), \mathscr{F}|_{f^{-1}(y)})$$

is isomorphic for every integer k.

In the sequel, the notion of the derived category is of constant use. We refer to Hartshorne [1] as to the derived category. We do not distinguish the sheaf, the complex of sheaves and the corresponding object of the derived category.

Proposition 1.3. *Let $\mathscr{F}$ be a complex of sheaves on N (or more precisely an object of the derived category of sheaves on N). Then we have an isomorphism*

$$\boldsymbol{R}\tau_{*}\pi^{-1}\boldsymbol{R}\Gamma_{S_{M}N}(\tau^{-1}\mathscr{F}) \cong \boldsymbol{R}\Gamma_{S_{M}^{*}N}(\pi^{-1}\mathscr{F}).$$

Proposition 1.4. *Let $\mathscr{F}$ be a sheaf on N. Then, for every point $x \in S_{M}^{*}N$, we have*

$$\mathscr{H}^{k}_{S_{M}^{*}N}(\pi^{-1}\mathscr{F})_{x} \cong \lim\ \mathrm{ind}_{Z}\, H^{k}_{Z}(N, \mathscr{F}),$$

where Z runs through a family of locally closed sets of N such that

1) *$Z \cap M$ is a neighborhood of $\pi(x)$ in M,*
2) *x is not contained in the closure of $Z - M \subset {}^{M}\tilde{N}^{*}$.*

Proposition 1.5. *Let $\mathscr{F}$ be a sheaf on N. Then, for every proper, open and convex subset U in $S_{M}^{*}N$ (i.e. for every point $x \in M$, $\pi^{-1}(x) \cap U$ is convex and $\neq \pi^{-1}(x)$), we have*

$$H^{k}(U, \boldsymbol{R}\Gamma_{S_{M}^{*}N}(\pi^{-1}\mathscr{F}) \cong \lim\ \mathrm{ind}_{Z}\, H^{k}_{Z}(N, \mathscr{F}),$$

where Z runs over a family of locally closed subsets in N such that

1) *Z contains $\pi(U)$,*
2) *the closure of $Z-M$ in ${}^{M}\tilde{N}^*$ is disjoint from U.*

Proposition 1.6. *Suppose that $\mathscr{F}$ is a sheaf on N. Then we have a triangle*

$$\begin{array}{ccc} & \mathscr{F}|_M & \\ \swarrow & & \nwarrow^{+1} \\ \boldsymbol{R}\Gamma_M(\mathscr{F})[d] & \longrightarrow & \boldsymbol{R}\pi_*\boldsymbol{R}\Gamma_{S_M^*N}(\pi^{-1}\mathscr{F})[d], \end{array}$$

where d is the codimension of M in N.

1.2. Sheaves on sphere bundles and on cosphere bundles. In this subsection we recall some general results concerning sheaves on sphere bundles and on cosphere bundles following S.K.K. [19].

We consider the following situation.

Let X be a topological space, V a (real) vector bundle of dimension n, and V^* a dual bundle of V. We denote by S and S^* the sphere bundle corresponding to V and V^* respectively, that is, $S=(V-X)/\boldsymbol{R}^+$, $S^*=(V^*-X)/\boldsymbol{R}^+$.

We set $D=\{(\bar{\xi},\bar{\eta})\in S\underset{X}{\times}S^*;\ \langle\xi,\eta\rangle\geqq 0\}$, where $(\bar{\xi},\bar{\eta})$ is the equivalent class of $(\xi,\eta)\in(V-X)\underset{X}{\times}(V^*-X)$, $I=\{(\bar{\xi},\bar{\eta})\in S\underset{X}{\times}S^*;\ \langle\xi,\eta\rangle>0\}$ and $E=\{(\bar{\xi},\bar{\eta})\in S\underset{X}{\times}S^*;\ \langle\xi,\eta\rangle=0\}$. We have the following diagrams:

$$(1.4)\qquad \begin{array}{ccc} & D & \\ {}^{\pi}\swarrow & & \searrow^{\tau} \\ S & & S^*, \\ {}^{\tau}\searrow & & \swarrow^{\pi} \\ & X & \end{array} \qquad \begin{array}{ccc} & I & \\ {}^{\pi'}\swarrow & & \searrow^{\tau'} \\ S & & S^*. \\ {}^{\tau}\searrow & & \swarrow^{\pi} \\ & X & \end{array}$$

Proposition 1.7. *The derived category of abelian sheaves $\mathscr{F}$ on S and $\mathscr{G}$ on S^* are equivalent under the following correspondences:*

$$\mathscr{G}=\boldsymbol{R}\tau_*\pi^{-1}\mathscr{F}[n-1],$$
$$\mathscr{F}=\boldsymbol{R}\pi'_!\tau'^{-1}\mathscr{G}.$$

Remark. Let X and Y be two topological spaces, $f\colon X\to Y$ a continuous map, and $\mathscr{F}$ a sheaf on X. We set

$$\Gamma_{f-\mathrm{pr}}(X,\mathscr{F})=\{s\in\Gamma(X,\mathscr{F});\ \mathrm{supp}(s)\longrightarrow Y \text{ is proper}\}.$$

$f_!(\mathscr{F})$ is defined to be the sheaf

$$Y\supset U\longrightarrow\Gamma_{(f|U)-\mathrm{pr}}(f^{-1}(U),\mathscr{F}|_{f^{-1}(U)}).$$

$R^k f_!(\mathscr{F})$ is its k-th derived functor.

The following lemma is frequently used in this paper.

Lemma 1.8. *Let $f: X \to Y$ be a continuous map of topological spaces satisfying*

1) *f is separated, and*
2) *f is locally proper, that is, for every point $x \in X$, there exist a (not necessarily open) neighborhood U of x and a neighborhood V of $f(x)$ such that $U \cap f^{-1}(V) \to V$ is proper.*

Let $g: Y' \to Y$ be a continuous map of topological spaces. Set $X' = X \underset{Y}{\times} Y'$, $f' = f \underset{Y}{\times} Y': X' \to Y'$ and $g' = g \underset{Y}{\times} X: X' \to X$. Then, for every sheaf $\mathscr{F}$ on X, the homomorphism

$$g^{-1} R^k f_!(\mathscr{F}) \longrightarrow R^k f'_!(g'^{-1}\mathscr{F})$$

is an isomorphism.

For the proof of Proposition 1.7, we need the following lemma.

Lemma 1.9. *Let $\mathscr{F}$ be a complex of abelian sheaves on $S \underset{X}{\times} S$, $\pi': D \underset{S^*}{\times} I \to S \underset{X}{\times} S$ the canonical projection defined by the projections $\pi: D \to S$ and $\pi': I \to S$. Then we have*

$$\boldsymbol{R}\pi'_! \pi'^{-1} \mathscr{F} \cong \mathscr{F}|_{S \underset{S}{\times} S}[1-n].$$

We denote by a the involutive automorphism of S (or S^*) deduced from $V \ni \xi \to -\xi \in V$.

Proposition 1.10. *Let $\mathscr{F}$ be a sheaf on S. Set*

$$\mathscr{G} = \boldsymbol{R}\tau_* \pi^{-1} \mathscr{F}[1-n],$$
$$\mathscr{E} = \boldsymbol{R}\tau_* \mathscr{F}[n-1] = \boldsymbol{R}\pi_* \mathscr{G}.$$

We have then the canonical triangle

$$\begin{array}{ccc} & \mathscr{F}^a & \\ \swarrow & & \nwarrow {\scriptstyle +1} \\ \tau^{-1}\mathscr{E} & \longrightarrow & \boldsymbol{R}\pi_* \tau^{-1}\mathscr{G}, \end{array}$$

where $\mathscr{F}^a$ is the inverse image of $\mathscr{F}$ by a.

2. Fourier microfunctions

2.1. Fourier hyperfunctions. In the sequel of this paper we apply the general theory of section 1 to certain special situations and construct the theory of Fourier microfunctions.

In this subsection we recall the notion of Fourier hyperfunctions following Kawai [11], Kaneko [7] and Ito [2].

Let $\boldsymbol{R}^n$ be an n-dimensional Euclidean space and $\boldsymbol{R}_n$ be its dual space. Let $\boldsymbol{D}^n = \boldsymbol{R}^n \sqcup S_\infty^{n-1}$ be the radial compactification of $\boldsymbol{R}^n$ in the sense of Kawai [11], Definition 1.1.1, p. 468. We denote this $\boldsymbol{D}^n$ by $\tilde{\boldsymbol{R}}^n$ and put $M = \tilde{\boldsymbol{R}}^n$ and $X = \tilde{\boldsymbol{C}}^n = M \times i\boldsymbol{R}^n$ endowed with the natural topology. Here we denote $i = \sqrt{-1}$.

Let $\tilde{\mathcal{O}}$ be the sheaf of slowly increasing and holomorphic functions on X following Kawai [11], Definition 1.1.2, p. 468, and put $\tilde{\mathscr{A}} = \tilde{\mathcal{O}}|_M$. Then $\tilde{\mathscr{A}}$ is the sheaf of slowly increasing and real-analytic functions on M. Then we have $\tilde{\mathscr{A}} = \iota^{-1}\tilde{\mathcal{O}}$, where $\iota\colon M \to X$ is the canonical injection.

As in Kawai [11], we define the sheaf of Fourier hyperfunctions on M.

Definition 2.1. The sheaf $\tilde{\mathscr{B}}$ is, by definition,

$$\tilde{\mathscr{B}} = \mathscr{H}_M^n(\tilde{\mathcal{O}}) = \mathrm{Dist}^n(M, \tilde{\mathcal{O}}),$$

where the notation in the right hand side of the above equality is due to Sato [14], p. 405.

A section of $\tilde{\mathscr{B}}$ is called a Fourier hyperfunction.

As stated in Kawai [11], Kaneko [7] and Ito [2], we have $\mathscr{H}_M^k(\tilde{\mathcal{O}}) = 0$ for $k \neq n$ and $\tilde{\mathscr{B}}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 to this case where $\mathscr{F}$, X and Y correspond to $\tilde{\mathcal{O}}$, X and M respectively. Then we obtain the sheaf homomorphism

$$\tilde{\mathscr{A}} \longrightarrow \tilde{\mathscr{B}},$$

which will be proved to be injective later. This injection allows us to consider Fourier hyperfunctions as a generalization of slowly increasing and real-analytic functions. The purpose of this section is to analyse the structure of the quotient sheaf $\tilde{\mathscr{B}}/\tilde{\mathscr{A}}$ by a similar way to S.K.K. [19].

2.2. Definition of Fourier microfunctions. Suppose that $M = \tilde{\boldsymbol{R}}^n$ and $X = \tilde{\boldsymbol{C}}^n$. Then we have the following isomorphisms

$$T(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n} \cong T\boldsymbol{R}^n \oplus iT\boldsymbol{R}^n, \quad T\boldsymbol{R}^n \cong \boldsymbol{R}^n \times \boldsymbol{R}^n,$$

$$T^*(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n} \cong T^*\boldsymbol{R}^n \oplus iT^*\boldsymbol{R}^n, \; T^*\boldsymbol{R}^n \cong \boldsymbol{R}^n \times \boldsymbol{R}_n$$

by the complex structure of $X \cap \boldsymbol{C}^n = \boldsymbol{C}^n$. Here $\boldsymbol{R}_n$ is the dual space of

$\boldsymbol{R}^n$. Hence we have the isomorphisms

$$T_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong T\boldsymbol{R}^n,\ S_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong S\boldsymbol{R}^n \cong \boldsymbol{R}^n \times S^{n-1},$$
$$T^*_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong T^*\boldsymbol{R}^n,\ S^*_{\boldsymbol{R}^n}(X \cap \boldsymbol{C}^n) \cong S^*\boldsymbol{R}^n \cong \boldsymbol{R}^n \times S_{n-1}$$

by the identification $i\xi \leftrightarrow \xi$. Since X is the radial compactification of $X \cap \boldsymbol{C}^n$ along the real subspace, we can take the radial compactification of $T(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n}$ and $T^*(X \cap \boldsymbol{C}^n)|_{\boldsymbol{R}^n}$ along the base space. Hence we obtain the isomorphisms

$$TX|_M \cong TM \oplus iTM,\ TM \cong M \times \boldsymbol{R}^n,$$
$$T^*X|_M \cong T^*M \oplus iT^*M,\ T^*M \cong M \times \boldsymbol{R}_n.$$

Hence we have the isomorphisms

$$T_M X \cong TM,\ S_M X \cong SM \cong M \times S^{n-1},$$
$$T^*_M X \cong T^*M,\ S^*_M X \cong S^*M \cong M \times S_{n-1}.$$

Taking account of this fact, we denote $S_M X$ and $S^*_M X$ by iSM and iS^*M, respectively. The point of iSM (resp. iS^*M) is frequently denoted by $x + i\xi 0$ (resp. $(x, i\langle \eta, dx \rangle \infty) = (x, i\eta\infty)$), where $\xi \in S^{n-1}$ (resp. $\eta \in S_{n-1}$).

We use the general discussions of subsection 1.1 to this special case. We denote

$$DM = \{(x + i\xi 0, (x, i\eta\infty)) \in iSM \underset{M}{\times} iS^*M;\ \langle i\xi, i\eta \rangle = -\langle \xi, \eta \rangle \geqq 0\},$$
$$IM = \{(x + i\xi 0, (x, i\eta\infty)) \in iSM \underset{M}{\times} iS^*M;\ \langle i\xi, i\eta \rangle = -\langle \xi, \eta \rangle > 0\}.$$

We have the following diagram:

$$\begin{array}{ccccccc}
 & & {}^M\tilde{X}^+ & \hookleftarrow & DM & & \\
 & \overset{\pi}{\swarrow} & & \overset{\tau}{\searrow}\ \overset{\pi}{\swarrow} & & \overset{\tau}{\searrow} & \\
{}^M\tilde{X} & \hookleftarrow & iSM & & {}^M\tilde{X}^* & \hookleftarrow & iS^*M \\
 & \overset{\tau}{\searrow} & & \overset{\tau}{\searrow}\ \overset{\pi}{\swarrow} & & \overset{\pi}{\swarrow} & \\
 & & X & \hookleftarrow & M & & .
\end{array}$$

Theorem 2.2 *We have* $\mathscr{H}^k_{iSM}(\tau^{-1}\tilde{\mathscr{O}}) = 0$ *for* $k \neq 1$, *where* $\tau\colon {}^M\tilde{X} \to X$ *is the canonical projection.*

Proof. Let $x = x_0 + i\xi 0$ be a point of iSM. Then we have

$$\mathscr{H}^k_{iSM}(\tau^{-1}\tilde{\mathscr{O}})_x \cong \lim\operatorname{ind}_{\tilde{U} \ni x} H^{k-1}(\tilde{U} - iSM, \tilde{\mathscr{O}}), \qquad \text{for } k > 1,$$

and we have the exact sequence

$$0 \longrightarrow \mathscr{H}^0_{iSM}(\tau^{-1}\tilde{\mathcal{O}})_x \longrightarrow \tilde{\mathcal{O}}_{x_0} \xrightarrow{\alpha} \lim\operatorname{ind}_{\tilde{U}\ni ix} H^0(\tilde{U} - iSM, \tilde{\mathcal{O}}),$$

where $\tilde{U}$ runs over the neighborhoods of x. Since $\tilde{U} - iSM \neq \emptyset$, α is injective by the property of unique continuation of holomorphic functions. Therefore we have $\mathscr{H}^0_{iSM}(\tau^{-1}\tilde{\mathcal{O}})_x = 0$. On the other hand, there is a fundamental system of neighborhoods $\{\tilde{U}\}$ of x such that $\tilde{U} - iSM$ is an $\tilde{\mathcal{O}}$-pseudoconvex open set. It follows from the Oka-Cartan-Kawai Theorem B that we have

$$\mathscr{H}^k_{iSM}(\tau^{-1}\tilde{\mathcal{O}})_x = 0 \qquad \text{for } k > 1. \qquad \text{(Q.E.D.)}$$

The following theorem is the most essential one in the theory of Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 2.3. *We have $\mathscr{H}^k_{iS^*M}(\pi^{-1}\tilde{\mathcal{O}}) = 0$ for $k \neq n$, where $\pi: {}^M\tilde{X}^* \to X$ is the canonical projection.*

Proof. Let $x = (x_0, i\eta\infty) \in iS^*M$. Then, by Proposition 1.4, we have

$$\mathscr{H}^k_{iS^*M}(\pi^{-1}\tilde{\mathcal{O}})_x \cong \lim\operatorname{ind}_Z H^k_Z(X, \tilde{\mathcal{O}}),$$

where Z runs over the family of

$$Z = \{z = x + iy \in U;\ \langle y, \eta_j\rangle \leqq 0\ (1 \leqq j \leqq n)\}$$

where U is a neighborhood of $\pi(x) = x_0$ in X and $\eta_1, \cdots, \eta_n \in S_{n-1}$ are chosen so that the convex hull of $\eta_0(=\eta), \eta_1, \cdots, \eta_n$ contains a certain neighborhood of the origin. Moreover, we have

$$\lim\operatorname{ind}_Z H^k_Z(X, \tilde{\mathcal{O}}) = \lim\operatorname{ind}_G \mathscr{H}^k_G(\tilde{\mathcal{O}})_{x_0}$$

where G runs all over the family of

$$G = \{z = x + iy \in X;\ \langle y, \eta_j\rangle \leqq 0\ (1 \leqq j \leqq n)\},$$

for $\eta_1, \cdots, \eta_n$ varying in a neighborhood of $-\eta$. By the "Edge of the Wedge" Theorem (see the following theorem 2.4), we have

$$\mathscr{H}^k_G(\tilde{\mathcal{O}})_{x_0} = 0 \qquad \text{for } k \neq n.$$

Therefore, we have

$$\mathscr{H}^k_{iS^*M}(\pi^{-1}\tilde{\mathcal{O}}) = 0 \qquad \text{for } k \neq n. \qquad \text{(Q.E.D.)}$$

Theorem 2.4 (the "Edge of the Wedge" Theorem). *Put $G = \{z = x + iy \in X;$ $y_j \geqq 0\ (1 \leqq j \leqq n)\}$. Then we have, for each $x \in M$,*

$$\mathscr{H}^k_G(\tilde{\mathcal{O}})_x = 0 \qquad \text{for } k \neq n.$$

In the proof of Theorem 2.3, we have only use the linear transformation of

G in Theorem 2.4.

The proof of Theorem 2.4 goes in a similar way to Kashiwara-Kawai-Kimura [10], Theorem 2.2.2, p. 60.

In order to prove Theorem 2.4, we prepare the following Lemma.

Lemma 2.5. *Let K_1, K_2 $(K_2 \subset K_1)$ be two compact subsets of X. Assume the following* (i) *and* (ii):

(i) *K_1 and K_2 satisfy the conditions of Martineau-Harvey Theorem* (Ito [2], (I), Theorem 1.5.1, p. 70).

(ii) *For sufficiently small $a > 0$, $|\mathrm{Im}\, z| < a (z \in K_1 \cap \boldsymbol{C}^n)$ holds.*

Then we have

$$H^k_{K_1 \setminus K_2}(V, \tilde{\mathcal{O}}) = 0, \qquad (k \neq n).$$

Here V is an open neighborhood of K_1.

Proof. We note that we have the long exact sequence of relative cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0_{K_2}(V, \tilde{\mathcal{O}}) \longrightarrow H^0_{K_1}(V, \tilde{\mathcal{O}}) \longrightarrow H^0_{K_1 \setminus K_2}(V, \tilde{\mathcal{O}}) \\ &\longrightarrow H^1_{K_2}(V, \tilde{\mathcal{O}}) \longrightarrow \cdots \\ &\longrightarrow H^k_{K_2}(V, \tilde{\mathcal{O}}) \longrightarrow H^k_{K_1}(V, \tilde{\mathcal{O}}) \longrightarrow H^k_{K_1 \setminus K_2}(V, \tilde{\mathcal{O}}) \longrightarrow \cdots. \end{aligned}$$

By Martineau-Harvey Theorem, we have

$$H^k_{K_1}(V, \tilde{\mathcal{O}}) = H^k_{K_2}(V, \tilde{\mathcal{O}}) = 0, \qquad (k \neq n).$$

Hence we have

$$H^k_{K_1 \setminus K_2}(V, \tilde{\mathcal{O}}) = 0, \qquad (k \neq n-1,\ n).$$

Since we have the isomorphisms

$$H^n_{K_1}(V, \tilde{\mathcal{O}}) \cong \underset{\sim}{Q}(K_1)', \qquad H^n_{K_2}(V, \tilde{\mathcal{O}}) \cong \underset{\sim}{Q}(K_2)'$$

and the canonical mapping

$$\underset{\sim}{Q}(K_2)' \longrightarrow \underset{\sim}{Q}(K_1)'$$

is injective by the new theorem of Runge type, we have

$$H^{n-1}_{K_1 \setminus K_2}(V, \tilde{\mathcal{O}}) = 0.$$

This completes the proof. (Q.E.D.)

Now we prove Theorem 2.4.

Proof of Theorem 2.4. By Kashiwara-Kawai-Kimura [10], Theorem 2.2.2, p. 60, we have, for $x \in \boldsymbol{R}^n$,

$$\mathscr{H}_G^k(\tilde{\mathscr{O}})_x = \mathscr{H}_{G\cap C^n}^k(\mathscr{O})_x = 0, \qquad (k \neq n).$$

For $x \in M \setminus \boldsymbol{R}^n$, we can prove the Theorem by using the Lemma 2.5.

In order to fix the argument, put $x = (1, 0, \cdots, 0)\infty$. Then it is sufficient to prove that, for a sufficiently small neighborhood Ω of x in X, we have

$$H_{G\cap\Omega}^k(\Omega, \tilde{\mathscr{O}}) = 0, \qquad (k \neq n).$$

Assume $2n - 1 > a > 0$. Further assume that a is sufficiently small. Put

$$\varphi = -a^2(x_1 - 1/a)^2 + (x_2^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2),$$

$$\psi = \frac{1}{\varphi^2} + \frac{a^2}{(y_1^2 - a)^2} + \cdots + \frac{a^2}{(y_n^2 - a)^2} - n - 1.$$

If we put

$$K_1 = G \cap \{z = x + iy \in \boldsymbol{C}^n;\ x_1 \geqq 0,\ y_j^2 \leqq a\ (1 \leqq j \leqq n),\ \varphi(z) \leqq 0\}^{cl},$$

$$K_2 = K_1 \cap \{z \in \boldsymbol{C}^n;\ \psi \geqq 0\}^{cl},$$

where, for a subset F of X, we denote by F^{cl} the closure of F in X, then K_1 and K_2 are $\tilde{\mathscr{O}}$-pseudoconvex compact sets and satisfy the assumptions of Lemma 2.5. Therefore we have

$$H_{K_1\setminus K_2}^k(X, \tilde{\mathscr{O}}) = 0, \qquad (k \neq n).$$

Put

$$\Omega = \operatorname{int}(\{z = x + iy \in \boldsymbol{C}^n;\ \psi < 0,\ x_1 \geqq 0,\ \varphi(z) \leqq 0,\ y_j^2 \leqq a,\ (1 \leqq j \leqq n)\}^{cl}).$$

Then we have

$$K_1 \setminus K_2 = G \cap \Omega.$$

Letting $a \to 0$, then the family of all the corresponding Ω becomes a fundamental system of neighborhoods of $x = (1, 0, \cdots, 0)\infty$. Therefore, by Lemma 2.5, we have

$$H_{G\cap\Omega}^k(\Omega, \tilde{\mathscr{O}}) = H_{K_1\setminus K_2}^k(X, \tilde{\mathscr{O}}) = 0, \qquad (k \neq n).$$

Letting $a \to 0$ and taking the inductive limit with respect to the corresponding Ω, we have, after all,

$$\mathscr{H}_G^k(\tilde{\mathscr{O}})_x = 0, \qquad (k \neq n). \qquad \text{(Q.E.D.)}$$

Definition 2.6. We define the sheaf $\tilde{\mathscr{C}}$ on iS^*M by

$$\tilde{\mathscr{C}} = \mathscr{H}_{iS^*M}^n(\pi^{-1}\tilde{\mathscr{O}})^a,$$

where we denote by a the antipodal map $iS^*M \to iS^*M$, and by $\mathscr{F}^a$ the inverse image under a of a sheaf $\mathscr{F}$ on iS^*M. A section of $\tilde{\mathscr{C}}$ is called a Fourier microfunction.

Now we define the sheaves $\tilde{\mathscr{Q}}$, $\tilde{\mathscr{O}}^{\beta}$ and $\tilde{\mathscr{A}}^{\beta}$ by

$$\tilde{\mathscr{Q}} = \mathscr{H}^1_{iSM}(\tau^{-1}\tilde{\mathscr{O}}),$$
$$\tilde{\mathscr{O}}^{\beta} = j_*(\tilde{\mathscr{O}}|_{X-M}),$$
$$\tilde{\mathscr{A}}^{\beta} = \tilde{\mathscr{O}}^{\beta}|_{iSM},$$

where $j: X - X \hookrightarrow {}^M\tilde{X}$, $\pi: {}^M\tilde{X}^* \to X$ and $\tau: {}^M\tilde{X} \to X$ are canonical maps.

By Proposition 1.3 and Theorems 2.2 and 2.3, we have the following.

Proposition 2.7. *We have*

$$R^k\tau_*\pi^{-1}\tilde{\mathscr{Q}} = \begin{cases} \tilde{\mathscr{C}}^a, & (k = n-1) \\ 0, & (k \neq n-1) \end{cases}.$$

Theorem 2.8. *We have*

$$R^k\pi_*\tilde{\mathscr{C}} = R^{k+n-1}\tau_*\tilde{\mathscr{Q}} = 0 \qquad \textit{for } k \neq 0,$$

and we have the exact sequence

$$0 \longrightarrow \tilde{\mathscr{A}} \longrightarrow \tilde{\mathscr{B}} \longrightarrow \pi_*\tilde{\mathscr{C}} \longrightarrow 0.$$

Proof. $R^k\pi_*\tilde{\mathscr{C}} = R^{k+n-1}\tau_*\tilde{\mathscr{Q}}$ is the trivial corollary of the preceding proposition. The triangle obtained in Proposition 1.6 implies immediately $R^k\pi_*\tilde{\mathscr{C}} = 0$ for $k \neq 0$ and yields the exact sequence in the theorem. (Q.E.D.)

This is the required decomposition of singularity of Fourier hyperfunctions.

Corollary 2.9. *We have the exact sequence*

$$0 \longrightarrow \tilde{\mathscr{A}}(M) \xrightarrow{\lambda} \tilde{\mathscr{B}}(M) \xrightarrow{\mathrm{sp}} \tilde{\mathscr{C}}(iS^*M) \longrightarrow 0.$$

Definition 2.10. Let $u \in \tilde{\mathscr{B}}(M)$. We call $\mathrm{sp}(u) \in \tilde{\mathscr{C}}(iS^*M)$ a spectrum of u. We denote by S.S. u the support $\mathrm{supp}\,\mathrm{sp}(u)$ of $\mathrm{sp}(u)$ and call it a singularity spectrum of u. $\pi(\text{S.S. } u)$ is evidently the subset where u is not slowly increasing nor real-analytic and is called the singular support of u.

Corollary 2.11. *Let $u \in \tilde{\mathscr{B}}(M)$. Then u is a slowly increasing and real-analytic function on M if and only if* S.S. $u = \emptyset$.

Since $\mathscr{A} = \tilde{\mathscr{A}}|_{\boldsymbol{R}^n}$, $\mathscr{B} = \tilde{\mathscr{B}}|_{\boldsymbol{R}^n}$ and $\mathscr{C} = \tilde{\mathscr{C}}|_{iS^*\boldsymbol{R}^n}$ hold in the notation of S.K.K. [19], we have the following Corollary by restricting the exact sequence in Theorem 2.8.

Corollary 2.12. *Let $\pi: iS^*\boldsymbol{R}^n \to \boldsymbol{R}^n$ be the canonical projection. Then we have the exact sequence*

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_* \mathscr{C} \longrightarrow 0.$$

2.3. Fundamental diagram on $\tilde{\mathscr{C}}$. We apply the arguments in the subsection 1.2 to a special case. At first we apply Proposition 1.10 to the situation $\mathscr{F} = \tilde{\mathscr{Q}}^a$, $X = M$, $S = iSM$. Then $\mathscr{G} = \tilde{\mathscr{C}}$, $\mathscr{E} = \pi_* \tilde{\mathscr{C}}$. We obtain the following.

Proposition 2.13. *We have*

$$R^k \pi_* \tau^{-1} \tilde{\mathscr{C}} = 0 \qquad \textit{for } k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow \tilde{\mathscr{Q}} \longrightarrow \tau^{-1} \pi_* \tilde{\mathscr{C}} \longrightarrow \pi_* \tau^{-1} \tilde{\mathscr{C}} \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F} = \tilde{\mathscr{A}}^{\beta}$. Thus we obtain a homomorphism

$$\tilde{\mathscr{A}}^{\beta} \longrightarrow \tau^{-1} R^{n-1} \tau_* \tilde{\mathscr{A}}^{\beta}, \tag{2.1}$$

$$\tilde{\mathscr{A}}^{\beta} = Rj_*(\tilde{\mathscr{O}}|_{X-M})|_{iSM},$$

where $j\colon X - M \hookrightarrow {}^M\tilde{X}$ is the canonical injection, which implies that

$$R^{n-1} \tau_* \tilde{\mathscr{A}}^{\beta} = R^{n-1}(\tau \circ j)_* (\tilde{\mathscr{O}}|_{X-M}).$$

Hence we can define the canonical map

$$R^{n-1} \tau_* \tilde{\mathscr{A}}^{\beta} \longrightarrow \tilde{\mathscr{B}}.$$

It yields, together with (2.1), a homomorphism $\tilde{\mathscr{A}}^{\beta} \to \tau^{-1} \tilde{\mathscr{B}}$. Summing up, we have obtained the following.

Theorem 2.14. *We have the following diagram of exact sequences of sheaves on iSM:*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{-1}\tilde{\mathscr{A}} & \longrightarrow & \tilde{\mathscr{A}}^{\beta} & \longrightarrow & \tilde{\mathscr{Q}} & \longrightarrow & 0 \\
 & & \| & & \lambda\downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{-1}\tilde{\mathscr{A}} & \longrightarrow & \tau^{-1}\tilde{\mathscr{B}} & \longrightarrow & \tau^{-1}\pi_*\tilde{\mathscr{C}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \pi_*\tau^{-1}\tilde{\mathscr{C}} & = & \pi_*\tau^{-1}\tilde{\mathscr{C}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.2}$$

Proof. It has already been proved that the rows are exact. The right column is exact by Proposition 2.13. Hence it follows that the middle column is exact. (Q.E.D.)

Let us transform the diagram (2.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \to iS^*M$ and $IM \to iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-n].$$

By Proposition 1.7,

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}\tilde{\mathscr{C}} \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}\tilde{\mathscr{C}} \cong \tilde{\mathscr{C}}[1-n].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (2.2), we obtain

$$R^k\tau'_!\pi'^{-1}\tilde{\mathscr{Q}} = 0 \qquad \text{for } k \neq n-1,$$

$$R^k\tau'_!\pi'^{-1}\tilde{\mathscr{A}}^{\beta} = 0 \qquad \text{for } k \neq n-1.$$

We define the sheaves $\tilde{\mathscr{A}}^{\vee}$ and $\mathscr{Q}^{\vee}$ on iS^*M by

$$\tilde{\mathscr{A}}^{\vee} = R^{n-1}\tau'_!\pi'^{-1}\tilde{\mathscr{A}}^{\beta},$$

$$\tilde{\mathscr{Q}}^{\vee} = R^{n-1}\tau'_!\pi'^{-1}\tilde{\mathscr{Q}}.$$

Then, in this way, we obtain the following theorem.

Theorem 2.15. *We have the diagram of exact sequences of sheaves on iS^*M:*

$$(2.3) \qquad \begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^{-1}\mathscr{A} & \longrightarrow & \tilde{\mathscr{A}}^{\vee} & \longrightarrow & \tilde{\mathscr{Q}}^{\vee} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^{-1}\mathscr{A} & \longrightarrow & \pi^{-1}\tilde{\mathscr{B}} & \longrightarrow & \pi^{-1}\pi_*\tilde{\mathscr{C}} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle sp} & & \downarrow & & \\
 & & & & \tilde{\mathscr{C}} & = & \tilde{\mathscr{C}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

and the diagram (2.2) *and the diagram* (2.3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_!\pi'^{-1}[n-1]$ *and* $\boldsymbol{R}\pi_*\tau^{-1}$.

We give a direct application of Theorem 2.14, which gives a relation between the singularity spectrum and the domain of the defining function of a Fourier hyperfunction.

A subset Z of iS^*M is said to be convex if each fiber $Z_x = Z \cap \tau^{-1}(x)$ is convex. It means, by definition, that every arc joining two points in Z_x is contained in Z_x. An arc joining two antipodal points is understood to be $\tau^{-1}(x)$. For every subset Z in iSM, we call the smallest convex subset containing Z the convex hull of Z. The polar Z° is the subset of iS^*M defined by $\{(x, i\eta\infty) \in iS^*M; \langle \xi, \eta \rangle \geqq 0$ for every $x + i\xi 0 \in Z\}$. By using these notions we can state the following proposition.

Proposition 2.16. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

(1) *If $\varphi \in \Gamma(U, \tilde{\mathscr{A}}^\beta)$, then* S.S. $(\lambda(\varphi)) \subset U^\circ$. *Conversely, if $f(x) \in \Gamma(\tau U, \tilde{\mathscr{B}})$ satisfies* S.S. $(f) \subset U^\circ$, *then there exists a unique $\varphi \in \Gamma(U, \tilde{\mathscr{A}}^\beta)$ such that $f = \lambda(\varphi)$. Namely, we have the exact sequence*

$$0 \longrightarrow \tilde{\mathscr{A}}^\beta(U) \xrightarrow{\lambda} \tilde{\mathscr{B}}(\tau U) \xrightarrow{\mathrm{sp}} \tilde{\mathscr{C}}(iS^*M - U^\circ).$$

(2) *$\Gamma(V, \tilde{\mathscr{A}}^\beta) \longrightarrow \Gamma(U, \tilde{\mathscr{A}}^\beta)$ is an isomorphism.*

Proof. Consider the exact sequence

$$0 \longrightarrow \tilde{\mathscr{A}}^\beta \longrightarrow \tau^{-1}\tilde{\mathscr{B}} \longrightarrow \pi_* \tau^{-1}\tilde{\mathscr{C}} \longrightarrow 0.$$

From this, we have the following diagram

$$(2.4)\qquad \begin{array}{ccccccc} 0 \longrightarrow & \Gamma(V, \tilde{\mathscr{A}}^\beta) & \longrightarrow & \Gamma(V, \tau^{-1}\tilde{\mathscr{B}}) & \longrightarrow & \Gamma(V, \pi_*\tau^{-1}\tilde{\mathscr{C}}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \Gamma(U, \tilde{\mathscr{A}}^\beta) & \longrightarrow & \Gamma(U, \tau^{-1}\tilde{\mathscr{B}}) & \longrightarrow & \Gamma(U, \pi_*\tau^{-1}\tilde{\mathscr{C}}) \end{array}$$

with exact rows. Since $V \to \tau V = \tau U$ and $U \to \tau U$ are open mappings with convex fibers, we have

$$\Gamma(V, \tau^{-1}\tilde{\mathscr{B}}) = \Gamma(U, \tau^{-1}\tilde{\mathscr{B}}) = \Gamma(\tau U, \tilde{\mathscr{B}}).$$

Since $\pi^{-1}V \to \tau\pi^{-1}V = iS^*M - V^\circ = iS^*M - U^\circ$ is an open mapping with connected fiber, we have

$$\begin{aligned} \Gamma(V, \pi_*\tau^{-1}\tilde{\mathscr{C}}) &= \Gamma(\pi^{-1}V, \tau^{-1}\tilde{\mathscr{C}}) \cong \Gamma(\tau\pi^{-1}V, \tilde{\mathscr{C}}) \\ &= \Gamma(iS^*M - U^\circ, \tilde{\mathscr{C}}). \end{aligned}$$

On the other hand,

$$\Gamma(iS^*M - U^\circ, \tilde{\mathscr{C}}) \longrightarrow \Gamma(\pi^{-1}U, \tau^{-1}\tilde{\mathscr{C}}^a) = \mathscr{C}(U, \pi_*\tau^{-1}\tilde{\mathscr{C}}^a)$$

is injective. Summing up, the middle arrow in the diagram (2.4) is an isomorphism and the right one is injective. Hence it follows that the left one is isomorphic. Moreover,

$$0 \longrightarrow \Gamma(U, \tilde{\mathscr{A}}^{\beta}) \longrightarrow \Gamma(\tau U, \tilde{\mathscr{B}}) \longrightarrow \Gamma(iS^*M - U^\circ, \tilde{\mathscr{C}})$$

is exact, which completes the proof. (Q.E.D.)

Definition 2.17. We say $u \in \tilde{\mathscr{B}}(\Omega)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty) \notin \text{S.S. } u$. This is equivalent to being represented as

$$u = \Sigma_j \lambda(\varphi_j), \quad \varphi_j \in \tilde{\mathscr{A}}^{\beta}(U_j), \quad (x, i\eta\infty) \notin U_j^\circ.$$

3. Vector-valued Fourier microfunctions

3.1. Vector-valued Fourier hyperfunctions. In this section we recall the notion of vector-valued Fourier hyperfunctions following Ito-Nagamachi [4], Junker [5], [6] and Ito [2].

We use the similar notation to subsection 2.1. Let E be a Fréchet space over the complex number field.

Let ${}^E\tilde{\mathcal{O}}$ be the sheaf of E-valued slowly increasing and holomorphic functions on X following Ito [2], Definition 2.1.1, p. 75, and put ${}^E\tilde{\mathscr{A}} = {}^E\tilde{\mathcal{O}}|_M$. Then ${}^E\tilde{\mathscr{A}}$ is the sheaf of E-valued, slowly increasing and real-analytic functions on M. Then we have ${}^E\tilde{\mathscr{A}} = \iota^{-1}\,{}^E\tilde{\mathcal{O}}$, where $\iota: M \hookrightarrow X$ is the canonical injection.

As in Junker [6] and Ito [2], we define the sheaf of E-valued Fourier hyperfunctions on M.

Definition 3.1. The sheaf ${}^E\tilde{\mathscr{B}}$ is, by definition,

$${}^E\tilde{\mathscr{B}} = \mathscr{H}^n_M({}^E\tilde{\mathcal{O}}) = \text{Dist}^n(M, {}^E\tilde{\mathcal{O}}).$$

A section of ${}^E\tilde{\mathscr{B}}$ is called an E-valued Fourier hyperfunction.

As stated in Junker [6] and Ito [2], we have $\mathscr{H}^k_M({}^E\tilde{\mathcal{O}}) = 0$ for $k \neq n$ and ${}^E\tilde{\mathscr{B}}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 to this case where $\mathscr{F}$, X and Y correspond to ${}^E\tilde{\mathcal{O}}$, X and M respectively. Then we obtain the sheaf homomorphism

$${}^E\tilde{\mathscr{A}} \longrightarrow {}^E\tilde{\mathscr{B}},$$

which will be proved to be injective later. This injection allows us to consider E-valued Fourier hyperfunctions as a generalzation of E-valued, slowly increasing and real-analytic functions. The purpose of this chapter is to analyse the structure of the quotient sheaf ${}^E\tilde{\mathscr{B}}/{}^E\tilde{\mathscr{A}}$ by a similar way to S.K.K. [19].

3.2. Definition of vector-valued Fourier microfunctions. We use the similar notation to subsection 2.2. Let E be a Fréchet space over the complex number field. We denote by ${}^{E}\tilde{\mathcal{O}}$ the sheaf of E-valued, slowly increasing and holomorphic functions defined on X. We have the following.

Theorem 3.2. *We have* $\mathscr{H}^k_{iSM}(\tau^{-1}\,{}^{E}\tilde{\mathcal{O}}) = 0$ *for* $k \neq 1$, *where* τ: ${}^{M}\tilde{X} \to X$ *is the canonical projection.*

Proof. It goes in a similar way to Theorem 2.2. (Q.E.D.)

The following theorem is the most essential one in the theory of E-valued Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 3.3. *We have* $\mathscr{H}^k_{iS^*M}(\pi^{-1}\,{}^{E}\tilde{\mathcal{O}}) = 0$ *for* $k \neq n$, *where* π: ${}^{M}\tilde{X}^* \to X$ *is the canonical projection.*

Proof. It goes in a similar way to Theorem 2.3. (Q.E.D.)

In the above proof, the following theorem is essential.

Theorem 3.4 (the "Edge of the Wedge" Theorem). *Put* $G = \{z = x + iy \in X;\ y_j \geqq 0\ (1 \leqq j \leqq n)\}$. *Then we have, for each* $x \in M$,

$$\mathscr{H}^k_G({}^{E}\tilde{\mathcal{O}})_x = 0 \qquad \text{for } k \neq n.$$

Definition 3.5. We define the sheaf ${}^{E}\tilde{\mathscr{C}}$ in iS^*M by

$${}^{E}\tilde{\mathscr{C}} = \mathscr{H}^n_{iS^*M}(\pi^{-1}\,{}^{E}\tilde{\mathcal{O}})^a.$$

A section of ${}^{E}\tilde{\mathscr{C}}$ is called an E-valued Fourier microfunction.

Now we define the sheaves ${}^{E}\tilde{\mathscr{Q}}$, ${}^{E}\tilde{\mathcal{O}}^{\beta}$, ${}^{E}\tilde{\mathscr{A}}^{\beta}$ by

$${}^{E}\tilde{\mathscr{Q}} = \mathscr{H}^1_{iSM}(\tau^{-1}\,{}^{E}\tilde{\mathcal{O}}),$$
$${}^{E}\tilde{\mathcal{O}}^{\beta} = j_*({}^{E}\tilde{\mathcal{O}}|_{X-M}),$$
$${}^{E}\tilde{\mathscr{A}}^{\beta} = {}^{E}\tilde{\mathcal{O}}^{\beta}|_{iSM},$$

where j: $X - M \hookrightarrow {}^{M}\tilde{X}$, π: ${}^{M}\tilde{X}^* \to X$ and τ: ${}^{M}\tilde{X} \to X$ are canonical maps.

By proposition 1.3 and Theorems 3.2 and 3.3, we have the following.

Proposition 3.6. *We have*

$$R^k\tau_*\pi^{-1}\,{}^{E}\tilde{\mathscr{Q}} = \begin{cases} {}^{E}\tilde{\mathscr{C}}^a, & (\text{for } k = n-1), \\ 0, & (\text{for } k \neq n-1). \end{cases}$$

Theorem 3.7. *We have*

$$R^k\pi_*\,{}^{E}\tilde{\mathscr{C}} = R^{k+n-1}\tau_*\,{}^{E}\tilde{\mathscr{Q}} = 0 \qquad \text{for } k \neq 0,$$

and we have the exact sequence

$$0 \longrightarrow {}^{E}\tilde{\mathscr{A}} \longrightarrow {}^{E}\tilde{\mathscr{B}} \longrightarrow \pi_* {}^{E}\tilde{\mathscr{C}} \longrightarrow 0.$$

Proof. It goes in a similar way to Theorem 2.7. (Q.E.D.)

This is the required decomposition of singularity of E-valued Fourier hyperfunctions.

Corollary 3.8. *We have the exact sequence*

$$0 \longrightarrow \tilde{\mathscr{A}}(M; E) \xrightarrow{\lambda} \tilde{\mathscr{B}}(M; E) \xrightarrow{\mathrm{sp}} \tilde{\mathscr{C}}(iS^*M; E) \longrightarrow 0.$$

Definition 3.9. Let $u \in \tilde{\mathscr{B}}(M; E)$. We call $\mathrm{sp}(u) \in \tilde{\mathscr{C}}(iS^*M; E)$ a spectrum of u. We denote by S.S. u the support supp sp(u) of sp(u) and call it a singularity spectrum of u. π(S.S. u) is evidently the subset where u is not slowly increasing nor real-analytic and is called the singular support of u.

Corollary 3.10. *Let $u \in \tilde{\mathscr{B}}(M; E)$. Then u is an E-valued, slowly increasing and real-analytic function on M if and only if* S.S. $u = \emptyset$.

Put ${}^{E}\mathscr{A} = {}^{E}\tilde{\mathscr{A}}|_{\boldsymbol{R}^n}$, ${}^{E}\mathscr{B} = {}^{E}\tilde{\mathscr{B}}|_{\boldsymbol{R}^n}$ and ${}^{E}\mathscr{C} = {}^{E}\tilde{\mathscr{C}}|_{iS^*\boldsymbol{R}^n}$. Then we have the following Corollary by restricting the exact sequence in Theorem 3.7.

Corollary 3.11. *Let $\pi: iS^*\boldsymbol{R}^n \to \boldsymbol{R}^n$ be the canonical projection. Then we have the exact sequence*

$$0 \longrightarrow {}^{E}\mathscr{A} \longrightarrow {}^{E}\mathscr{B} \longrightarrow \pi_* {}^{E}\mathscr{C} \longrightarrow 0.$$

Here a section of ${}^{E}\mathscr{C}$ is called a E-valued Sato microfunction. Thus this shows the decomposition of singularity of E-valued Sato hyperfunctions.

3.3. Fundamental diagram on ${}^{E}\tilde{\mathscr{C}}$. We apply the arguments in the subsection 1.2 to a special case. At first we apply Proposition 1.10 to the situation $\mathscr{F} = {}^{E}\tilde{\mathscr{Q}}^a$, $X = M$, $S = iSM$. Then $\mathscr{G} = {}^{E}\tilde{\mathscr{C}}$, $\mathscr{E} = \pi_* {}^{E}\tilde{\mathscr{C}}$. We obtain the following.

Proposition 3.12. *We have*

$$R^k \pi_* \tau^{-1} {}^{E}\tilde{\mathscr{C}} = 0 \qquad \textit{for} \;\; k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow {}^{E}\tilde{\mathscr{Q}} \longrightarrow \tau^{-1} \pi_* {}^{E}\tilde{\mathscr{C}} \longrightarrow \pi_* \tau^{-1} {}^{E}\tilde{\mathscr{C}} \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F} = {}^{E}\tilde{\mathscr{A}}^{\beta}$. Thus we obtain a homomorphism

(3.1) $${}^{E}\tilde{\mathscr{A}}^{\beta} \longrightarrow \tau^{-1} R^{n-1} \tau_* {}^{E}\tilde{\mathscr{A}}^{\beta},$$

$${}^{E}\tilde{\mathscr{A}}^{\beta} = \boldsymbol{R}j_* ({}^{E}\tilde{\mathscr{O}}|_{X-M})|_{iSM},$$

where $j: X - M \hookrightarrow {}^{M}\tilde{X}$ is the canonical injection, which implies that

$$R^{n-1}\tau_* {}^E\tilde{\mathscr{A}}^\beta = R^{n-1}(\tau \circ j)_*({}^E\tilde{\mathcal{O}}|_{X-M}).$$

Hence we can define the canonical map

$$R^{n-1}\tau_* {}^E\tilde{\mathscr{A}}^\beta \longrightarrow {}^E\tilde{\mathscr{B}}.$$

It yields, together with (3.1), a homomorphism ${}^E\tilde{\mathscr{A}}^\beta \to \tau^{-1}{}^E\tilde{\mathscr{B}}$. Summing up, we have obtained the following.

Theorem 3.13. *We hae the following diagram of exact sequences of sheaves on iSM:*

$$(3.2)\qquad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tau^{-1}{}^E\tilde{\mathscr{A}} & \longrightarrow & {}^E\tilde{\mathscr{A}}^\beta & \longrightarrow & {}^E\tilde{\mathscr{Q}} & \longrightarrow & 0 \\ & & \| & & \downarrow\lambda & & \downarrow & & \\ 0 & \longrightarrow & \tau^{-1}{}^E\tilde{\mathscr{A}} & \longrightarrow & \tau^{-1}{}^E\tilde{\mathscr{B}} & \longrightarrow & \tau^{-1}\pi_*{}^E\tilde{\mathscr{C}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \pi_*\tau^{-1}{}^E\tilde{\mathscr{C}} & = & \pi_*\tau^{-1}{}^E\tilde{\mathscr{C}} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Proof. It has already been proved that the rows are exact. The right column is exact by Proposition 3.12. Hence it follows that the middle column is exact. (Q.E.D.)

Let us transform the diagram (3.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \to iS^*M$ and $IM \to iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-n].$$

By Proposition 1.7,

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}{}^E\tilde{\mathscr{C}} \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}{}^E\tilde{\mathscr{C}} \cong {}^E\tilde{\mathscr{C}}[1-n].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (3.2), we obtain

$$R^k\tau'_!\pi'^{-1}{}^E\tilde{\mathscr{Q}} = 0 \qquad \text{for } k \neq n-1,$$

$$R^k\tau'_!\pi'^{-1}{}^E\tilde{\mathscr{A}}^\beta = 0 \qquad \text{for } k \neq n-1.$$

We define the sheaves ${}^E\tilde{\mathscr{A}}^\vee$ and ${}^E\tilde{\mathscr{Q}}^\vee$ on iS^*M by

$$ {}^E\tilde{\mathscr{A}}^{\vee} = R^{n-1}\tau'_!\pi'^{-1}\,{}^E\tilde{\mathscr{A}}^{\beta}, $$
$$ {}^E\tilde{\mathscr{Q}}^{\vee} = R^{n-1}\tau'_!\pi'^{-1}\,{}^E\tilde{\mathscr{Q}}. $$

Then, in this way, we obtain the following theorem.

Theorem 3.14. *We have the diagram of exact sequences of sheaves on iS*M:*

$$
(3.3)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^{-1}\,{}^E\tilde{\mathscr{A}} & \longrightarrow & {}^E\tilde{\mathscr{A}}^{\vee} & \longrightarrow & {}^E\tilde{\mathscr{Q}}^{\vee} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^{-1}\,{}^E\tilde{\mathscr{A}} & \longrightarrow & \pi^{-1}\,{}^E\tilde{\mathscr{B}} & \longrightarrow & \pi^{-1}\pi_*{}^E\tilde{\mathscr{C}} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow & & \\
 & & & & {}^E\tilde{\mathscr{C}} & = & {}^E\tilde{\mathscr{C}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}
$$

and the diagram (3.2) *and the diagram* (3.3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_!\pi'^{-1}[n-1]$ *and* $\boldsymbol{R}\pi_*\tau^{-1}$.

We give a direct application of Theorem 3.13, which gives a relation between the singularity spectrum and the domain of the defining function of an E-valued Fourier hyperfunction.

By using the similar notation to Proposition 2.16, we can state the following proposition.

Proposition 3.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

(1) *If* $\varphi\in\Gamma(U, {}^E\tilde{\mathscr{A}}^{\beta})$, *then* $\mathrm{S.S.}(\lambda(\varphi))\subset U^{\circ}$. *Conversely, if* $f(x)\in\Gamma(\tau U, {}^E\tilde{\mathscr{B}})$ *satisfies* $\mathrm{S.S.}(f)\subset U^{\circ}$, *then there exists a unique* $\varphi\in\Gamma(U, {}^E\tilde{\mathscr{A}}^{\beta})$ *such that* $f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$ 0 \longrightarrow \tilde{\mathscr{A}}^{\beta}(U;E) \xrightarrow{\ \lambda\ } \tilde{\mathscr{B}}(\tau U;E) \xrightarrow{\ \mathrm{sp}\ } \tilde{\mathscr{C}}(iS^*M-U^{\circ};E). $$

(2) $\Gamma(V, {}^E\tilde{\mathscr{A}}^{\beta}) \longrightarrow \Gamma(U, {}^E\tilde{\mathscr{A}}^{\beta})$ *is an isomorphism.*

Proof. It goes in a similar way to Proposition 2.16. (Q.E.D.)

Definition 3.16. We say $u\in\tilde{\mathscr{B}}(\Omega;E)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty)\notin \mathrm{S.S.}\,u$. This is equivalent to being represented as

$$ u=\Sigma_j\lambda(\varphi_j),\quad \varphi_j\in\tilde{\mathscr{A}}^{\beta}(U_j;E),\quad (x, i\eta\infty)\notin U_j^{\circ}. $$

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