

**Blowup Phenomena for Nonlinear Dissipative Wave Equations**

By

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Abstract

We study the initial-boundary value problem for the nonlinear wave equations with nonlinear dissipative terms: $\Box u + |u'|^{\beta}u' = |u|^\alpha u$ with $u(0) = u_0$, $u'(0) = u_1$, and $u|_{\partial \Omega} = 0$. When the initial energy $E(u_0, u_1) < 0$ and the inner product $(u_0, u_1) > 0$, the solution blows up at some finite time $T$ which is estimated from above. On the other hand, when $0 \leq E(u_0, u_1) < 1$ and $u_0 \in W_*$, the solution exists globally in time and has the energy decay $E(u(t), u'(t)) \leq c(1 + t)^{-2/\beta}$ for $t \geq 0$.

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1. Introduction

In this paper we mainly investigate on the blowup phenomena to the initial-boundary value problem for the following nonlinear wave equations with nonlinear dissipative terms:

\[
\begin{cases}
u'' + Au + \delta_1 u' + \delta_2 |u'|^\beta u' + \delta_3 Au' = |u|^\alpha u & \text{in } \Omega \times [0, +\infty) \\
u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), \quad \text{and } u(x, t)|_{\partial \Omega} = 0,
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $' = \partial_t \equiv \partial/\partial t$, $A = -\Delta \equiv \sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\delta_1 \geq 0, \delta_2 \geq 0, \delta_3 \geq 0, \beta > 0$, and $\alpha > 0$ are constants. Let $H$ be the usual real separable Hilbert space $L^2(\Omega)$ with norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$. We denote $L^p(\Omega)$-norm by $\| \cdot \|_p$ ($\| \cdot \| = \| \cdot \|_2$).
We define the energy associated with Eq. (0.1) by

\[(0.2)\quad E(u, u') \equiv \|u'\|^2 + J(u),\]

where we put

\[(0.3)\quad J(u) \equiv \|A^{1/2}u\|^2 - \frac{2}{\alpha + 2}\|u\|^{\alpha+2},\]

and following Nakao and Ono [17], we introduce the K-positive set and the K-negative set:

\[(0.4)\quad \mathcal{W}_* \equiv \{ u \in \mathcal{D}(A) : K(u) > 0 \} \cup \{0\}\]

and

\[(0.5)\quad \mathcal{V}_* \equiv \{ u \in \mathcal{D}(A) : K(u) < 0 \},\]

respectively, where we set

\[(0.6)\quad K(u) \equiv \|A^{1/2}u\|^2 - \|u\|^{\alpha+2}\]

(cf. see [9, 20, 26]).

In the non-dissipative case \((\delta_1 = \delta_2 = \delta_3 = 0)\), many authors have already studied on blowup solutions for the problem (0.1), see for example the works of [1–5, 9, 24, 26]. In particular, when \(\alpha \leq 4/(N-2)\) (\(\alpha < +\infty\) if \(N = 1, 2\)), we observe that the solution of (0.1) with \(\delta_1 = \delta_2 = \delta_3 = 0\) can not be extended globally in time under the assumptions which \(u_0 \in \mathcal{V}_*\) and \(E(u_0, u_1) < d\) (i.e. \(E(u_0, u_1) \ll 1\)), where \(d\) is the so-called potential well depth, and that the solution can be extended globally in time under the assumptions which \(u_0 \in \mathcal{W}_*\) and \(0 \leq E(u_0, u_1) < d\) (e.g. see [20]).

In the case of \(\delta_2 = 0\) in (0.1), Levine [10–12] has given an upper estimate of the blowup time \(T\) under \(E(u_0, u_1) < 0\) by using the so-called concavity methods. We note that \(u \in \mathcal{V}_*\) if \(E(u, u') < 0\). Recently, when \(\alpha \leq 4/(N-2)\) (\(\alpha < +\infty\) if \(N = 1, 2\)), in the case of \(\delta_1 > 0\) and \(\delta_2 = \delta_3 = 0\) in (0.1), Ohta [18] has proved that the solution can not be extended globally in time under the assumptions which \(u_0 \in \mathcal{V}_*\) and \(E(u_0, u_1) < d\). On the other hand, we shall prove that the problem (0.1) admits a unique global solution, and that the solution has some decay properties under the assumptions which \(u_0 \in \mathcal{W}_*\) and \(0 \leq E(u_0, u_1) \ll 1\) without the assumption \(\alpha \leq 4/(N-2)\) in Section 5.

In the case of \(\delta_2 > 0\) and \(\delta_1 = \delta_3 = 0\) in (0.1), Georgiev and Todorova [6] have proved that the solution can not be extended globally in time under the assumptions which the initial energy is sufficiently negative \((E(u_0, u_1) \ll -1)\) and \(\beta < \alpha \leq 2/(N-2)\) without any estimates of the blowup time \(T\). Our purpose of the present paper is mainly to relax the assumption connected with the initial energy to \(E(u_0, u_1) < 0\), and to derive an upper estimate of the blowup time \(T\), where we treat the case of \(\delta_1 \geq 0, \delta_2 > 0, \) and \(\delta_3 \geq 0\) in Section 2. Moreover, we also give the arranged proof for blowup results in the case of \(\delta_2 = 0\) in Sections 3 and 4.
On the other hand, in the case of \( \delta_1 + \delta_2 + \delta_3 > 0 \) in (0.1), we show that the problem (0.1) admits a unique global solution, and that the solution and its energy have some decay properties under the assumptions which \( \mu_0 \in \mathcal{W}_\ast \) and \( 0 \leq E(u_0, u_1) \ll 1 \). In particular, when \( \delta_2 > 0 \) in (0.1), the energy \( E(u(t), u'(t)) \) has some decay rate polynomially. When \( \delta_1 + \delta_3 > 0 \) in (0.1), the energy \( E(u(t), u'(t)) \) has some decay rate exponentially (see Section 5). Nakao [16] has studied the existence and decay properties of a unique global solution for the problem (0.1) with \(-|u|^\alpha u \) (monotone) instead of \(|u|^\alpha u \) in the case of \( \delta_2 > 0 \) and \( \delta_1 = \delta_3 = 0 \), but his results cannot apply our problem immediately. Our results of the global in time solvability can apply to the problem (0.1) with \(|u|^\alpha u \) replaced by the nonlinear function \( f(u) \) such that \( |f(u)| \leq k_1|u|^\alpha + 1 \) and \( |f'(u)| \leq k_2|u|^\alpha \) with positive constants \( k_1 \) and \( k_2 \).

We denote \([a]^+ = \max\{0, a\}\) where \(1/[a]^+ = +\infty \) if \([a]^+ = 0\).

1. Preliminaries

We give the local in time existence theorem for the problem (0.1) applying the Banach contraction mapping theorem.

**Theorem 1.** (Local Existence) Let the initial data \( \{u_0, u_1\} \) belong to \( \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \). Suppose that

\[ \alpha \leq 2/(N - 4) \quad (\alpha < +\infty \text{ if } N \leq 4). \]

Then the problem (0.1) admits a unique local solution \( u \) belonging to

\[ C_\mathcal{W}^\alpha([0, T); \mathcal{D}(A)) \cap C_\mathcal{W}^1([0, T); \mathcal{D}(A^{1/2})) \cap C^0([0, T); \mathcal{D}(A^{1/2})) \cap C^1([0, T); H) \]

for some \( T = T(\|A u_0\|, \|A^{1/2} u_1\|) > 0 \), and \( u \) satisfies

\begin{align*}
(1.1) & \quad u' \in L^{p+2}((0, T) \times \Omega) \quad \text{if} \quad \delta_2 > 0, \\
(1.2) & \quad u' \in L^2(0, T; \mathcal{D}(A^{1/2})) \quad \text{if} \quad \delta_3 > 0.
\end{align*}

Moreover, if \( \delta_2 = 0 \), then

\[ u \in C^0([0, T); \mathcal{D}(A)) \cap C^1([0, T); \mathcal{D}(A^{1/2})) \cap C^2([0, T); H). \]

Furthermore, at least one of the following statements is valid:

(i) \( T = +\infty \)

(ii) \( \|A u(t)\|^2 + \|A^{1/2} u'(t)\|^2 \to +\infty \quad \text{as} \quad t \to T^- \).

**Proof.** For \( T > 0 \) and \( R > 0 \), we define the two-parameter space \( X_{T, R} \) of the solutions by

\[ X_{T, R} \equiv \{ u(t) \in C_\mathcal{W}^\alpha([0, T); \mathcal{D}(A)) \cap C_\mathcal{W}^1([0, T); \mathcal{D}(A^{1/2})) \cap C^0([0, T); \mathcal{D}(A^{1/2})) \cap C^1([0, T); H) : \|A^{1/2} u'(t)\|^2 + \|A u(t)\|^2 \leq R^2 \text{ on } [0, T] \}. \]
It is easy to see that $X_{T,R}$ can be organized as a complete metric space with the distance:

$$d(u,v) \equiv \sup_{0 \leq t \leq T} \left\{ \|u'(t) - v'(t)\|^2 + \|A^{1/2}(u(t) - v(t))\|^2 \right\}.$$ 

We define a nonlinear mapping $S$ as follows. For $v \in X_{T,R}$, $u = Sv$ is the unique solution of the following equations:

$$\begin{cases}
  u'' + Au' + \delta_1 u' + \delta_2 |u'|^\beta u' + \delta_3 Au' = |v|^\alpha v & \text{in } \Omega \times [0,T] \\
  u(0) = u_0, \quad u'(0) = u_1, \quad \text{and } u|_{\partial \Omega} = 0.
\end{cases}$$

Using the fact that $(|u_1'|^\beta u_1' - |u_2'|^\beta u_2', u_1' - u_2') \geq 0$, we can prove that there exist $T > 0$ and $R > 0$ such that $S$ maps $X_{T,R}$ into itself; $S$ is a contraction mapping with respect to the metric $d(\cdot, \cdot)$ (e.g. see Theorem 3.1 in [13], Theorem 2.1 in [6]). By applying the Banach contraction mapping theorem, we obtain a unique solution $u$ belonging to $X_{T,R}$ of (0.1). Moreover, noting that $(|u'|^\beta u', u') = \|u'\|_{\beta+2}^{\beta+2}$ and $(Au', u') = \|A^{1/2}u'\|^2$, we get (1.1) and (1.2), respectively (see [23]). When $\delta_2 = 0$, by the continuity argument for wave equations (e.g. see [13, 22, 25]), we see that the solution $u$ belongs to (1.3). We omit the detail here. \end{proof}

In what follows, we put $E(t) = E(u(t), u'(t))$, $f(u) = |u|^\alpha u$, and $g(u') = |u'|^\beta u'$ for simplicity.

Multiplying Eq.(0.1) by $2u'$ (or $u$) and integrating it over $\Omega$, we have immediately the following differential equalities associated with Eq.(0.1).

**Lemma 1.1.** Let $u$ be a solution of (0.1). Then

$$\partial_t E(t) + 2\{\delta_1 \|u'(t)\|^2 + \delta_2 \|u'(t)\|_{\beta+2}^{\beta+2} + \delta_3 \|A^{1/2}u'(t)\|^2\} = 0$$

where $E(t) = E(u(t), u'(t))$, and

$$K(u(t)) = \|u'(t)\|^2 - \partial_t (u(t), u'(t)) - (\delta_1 u'(t) + \delta_2 g(u'(t)) + \delta_3 Au'(t), u(t))$$

where $K(u) = \|A^{1/2}u\|^2 - \|u\|_{\alpha+2}^{\alpha+2}$ and $g(u') = |u'|^\beta u'$.

We see from (1.4) that

$$E(t) + 2 \int_0^t \{\delta_1 \|u'(s)\|^2 + \delta_2 \|u'(s)\|_{\beta+2}^{\beta+2} + \delta_3 \|A^{1/2}u'(s)\|^2\} ds = E(0)$$

where $E(0) = E(u_0, u_1)$.

To pull out blowup properties of solutions, we apply the concavity methods (see Levine [10–12]). We define the nonnegative function $P$ by

$$P(t) \equiv \|u(t)\|^2 + \int_0^t (\delta_1 \|u(s)\|^2 + \delta_3 \|A^{1/2}u(s)\|^2) ds$$

$$+ (T_0 - t)(\delta_1 \|u_0\|^2 + \delta_3 \|A^{1/2}u_0\|^2) + r(t + \tau)^2$$
for a solution \( u(t), t \in [0, T_0] \), where \( T_0 > 0, r \geq 0, \) and \( \tau > 0 \) are some constants which are specified later on, then we observe the following properties.

**Proposition 1.2.** The function \( P(t) \) satisfies

\[
P''(t) = 2\{r - E(t) + 2\|u(t)\|^2 + \frac{\alpha}{\alpha + 2}\|u(t)\|_{\alpha/2}^{\alpha + 2} - \delta_2(g(u'(t)), u(t))\}
\]

with \( g(u') = |u'|^{\beta}u' \), and

\[
P(t)P''(t) - (\alpha/4 + 1)P'(t)^2 \geq P(t)Q(t),
\]

where

\[
Q(t) = - (\alpha + 2)(r + E(0)) + \alpha\{\|A^{1/2}u(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds\} + 2\delta_2((\alpha + 2)\int_0^t \|u'(s)\|_{\beta/2}^{\beta+2} ds - (g(u'(t)), u(t))).
\]

**Proof.** Differentiating (1.7) with respect to \( t \), we have

\[
P'(t) = 2(u(t), u'(t)) + (\delta_1\|u(t)\|^2 + \delta_3\|A^{1/2}u(t)\|^2) - (\delta_1\|u_0\|^2 + \delta_3\|A^{1/2}u_0\|^2) + 2r(t + \tau)
\]

\[
= 2\{(u(t), u'(t)) + \int_0^t (\delta_1(u(s), u'(s)) + \delta_3(A^{1/2}u(s), A^{1/2}u'(s)))\, ds + r(t + \tau)\}
\]

and

\[
P''(t) = 2\{(u(t), u''(t) + \delta_1u'(t) + \delta_2Au(t)) + \|u'(t)\|^2 + r\}
\]

\[
= 2\{r + \|u'(t)\|^2 - K(u(t)) - \delta_2(g(u'(t)), u(t))\} + \frac{\alpha}{\alpha + 2}\|u(t)\|_{\alpha/2}^{\alpha + 2} - \delta_2(g(u'(t)), u(t))),
\]

which implies the desired equality (1.8). Next, we set

\[
R(t) \equiv \{\|u(t)\|^2 + \int_0^t (\delta_1\|u(s)\|^2 + \delta_3\|A^{1/2}u(s)\|^2)\, ds + r(t + \tau)^2\} \times
\]

\[
\times \{\|u'(s)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\, ds + r\}
\]

\[
- \{(u(t), u'(t)) + \int_0^t (\delta_1(u(s), u'(s)) + \delta_3(A^{1/2}u(s), A^{1/2}u'(s)))\, ds + r(t + \tau)^2\},
\]
then we see $R(t) \geq 0$ and
\[
R(t) = \{P(t) - (T_0 - t)(\delta_1 \|u_0\|^2 + \delta_3 \|A^{1/2}u_0\|^2)\} \\
\times \{|u'(t)|^2 + \int_0^t (\delta_1 \|u'(s)\|^2 + \delta_3 \|A^{1/2}u'(s)\|^2) \, ds + r\} - (1/4)P'(t)^2
\]
or
\[
(1.13) \\
P'(t) = 4\{P(t) - (T_0 - t)(\delta_1 \|u_0\|^2 + \delta_3 \|A^{1/2}u_0\|^2)\} \\
\times \{|u'(t)|^2 + \int_0^t (\delta_1 \|u'(s)\|^2 + \delta_3 \|A^{1/2}u'(s)\|^2) \, ds + r\} - R(t).
\]
Thus it follows from (1.13) that
\[
P(t)P''(t) - (\alpha/4 + 1)P'(t)^2 \\
\geq P(t)[P''(t) - (\alpha + 4)\{|u'(t)|^2 \\
+ \int_0^t (\delta_1 \|u'(s)\|^2 + \delta_3 \|A^{1/2}u'(s)\|^2) \, ds + r\}].
\]
To derive (1.9), we shall show that the above $[\cdots]$ is equal to $Q(t)$.
\[
[\cdots] = 2\{r + \|u'(t)\|^2 - K(u(t)) - \delta_2(g(u'(t)), u(t))\} \\
- (\alpha + 4)\{r + \|u'(t)\|^2 + \int_0^t (\delta_1 \|u'(s)\|^2 + \delta_3 \|A^{1/2}u'(s)\|^2) \, ds\} \\
= -(\alpha + 2)(r + \|u'(t)\|^2) + 2\{-K(u(t)) - \delta_2(g(u'(t)), u(t))\} \\
- (\alpha + 4)\int_0^t (\delta_1 \|u(s)\|^2 + \delta_3 \|A^{1/2}u(s)\|^2) \, ds \\
= -(\alpha + 2)(r + E(t)) + 2\int_0^t (\delta_1 \|u'(s)\|^2 + \delta_2 \|u'(s)\|^{\beta+2} \\
+ \delta_3 \|A^{1/2}u'(s)\|^2) \, ds \}
+ \alpha\{\|A^{1/2}u(t)\|^2 + \int_0^t (\delta_1 \|u'(s)\|^2 + \delta_3 \|A^{1/2}u'(s)\|^2) \, ds\} \\
+ 2\delta_2((\alpha + 2)\int_0^t \|u'(s)\|^{\beta+2} \, ds - (g(u'(t)), u(t)))
\]
and noting (1.6), it is equal to $Q(t)$. The proof of Proposition 1.2 is now completed. \qed

2. Blow Up \textbf{I} ($\delta_2 > 0$)

When $\delta_2 > 0$, $\delta_1 \geq 0$, $\delta_3 \geq 0$ in Eq. (0.1), we shall show that the solution $u(t)$ blows up at some finite time under the assumptions which $E(0) < 0$ and $(u_0, u_1) > 0$ and $\alpha > \beta$. We denote by $|\Omega|$ the measure of $\Omega$, and we assume $|\Omega| \geq 1$ for simplicity.
Our main result is as follows.

**Theorem 2.** \((\delta_2 > 0)\) Let \(\delta_2 > 0\) in (0.1). Suppose that \(\alpha > \beta\) and \(E(0) < 0\) and \(G(0) \equiv -(E(0))^{\omega} + 2\omega m_0^{-1}(u_0, u_1) > 0\).

Then there exists a \(T\) such that

\[
0 < T \leq m_0 m_1 \omega(1 - \omega)^{-1} G(0)^{-1/(\omega)} \omega
\]

and the local solution \(u(t)\) in the sense of Theorem 1 blows up at the finite time \(T\), where \(\omega, m_0,\) and \(m_1\) are positive constants such that

\[
\omega = 1 - \left(\frac{1}{\beta + 2} - \frac{1}{\alpha + 2}\right) \quad (1/2 < \omega < 1),
\]

\[
m_0 = (2\delta_2(1 + 2/\alpha)\Omega^{\frac{\alpha}{\alpha + 2}} (-E(0))^{1/(\delta + 1)}),
\]

\[
m_1 = 2 \max \left\{1, (1 + 2/\alpha)(2\Omega^{\frac{\alpha}{\alpha + 2}} m_0^{-1})^{\frac{1}{\alpha + 2}} (-E(0))^{1/(\omega + 1)} \right\}.
\]

**Proof.** We put \(r = 0\) in (1.7) and we shall estimate \(P''(t)\) given by (1.8) with \(r = 0\). We have that for \(g(u') = |u'|^\beta u'\)

\[
|\delta_2 g(u'), u)| \leq \delta_2 |u'|^{\beta + 1} \|u\|_{\beta + 2} \leq \delta_2 B_1 |u'|^{\beta + 1} \|u\|_{\alpha + 2}
\]

(2.1)

\[
= \delta_2 B_1 |u'|^{\beta + 1} \|u\|_{\alpha + 2}
\]

with \(B_1 = |\Omega|^{\frac{\beta}{\alpha + 2}} (\delta + 3)\). Since we see from (0.2), (0.3), and (1.6) that

\[
\|u(t)\|_{\alpha + 2} \geq (-E(t))^{1/(\alpha + 2)} \geq (-E(0))^{1/(\alpha + 2)} > 0
\]

and from the Young inequality that

\[
\delta_2 B_1 |u'|^{\beta + 1} \|u\|_{\alpha + 2}^{\beta + 2} \leq \frac{\beta + 1}{\beta + 2} (\epsilon^{-1} \delta_2 B_1)^{\frac{\beta + 1}{\beta + 2}} |u'|^{\beta + 2} \|u\|_{\alpha + 2}^{\alpha + 2}
\]

for any \(\epsilon > 0\), we observe from (2.1) that

\[
\delta_2 |(g(u'(t)), u(t))| \leq (\epsilon^{-1} \delta_2 B_1)^{\frac{\beta + 1}{\beta + 2}} (-E(0))^{1/(\beta + 1)} |u'(t)|^{\beta + 2} + \epsilon^{\beta + 2} (-E(0))^{1/(\beta + 1)} \|u(t)\|_{\alpha + 2}^{\alpha + 2}
\]

if \(\alpha > \beta\), and we obtain from (1.8) with \(r = 0\) that

\[
P''(t) \geq 2(-E(t)) + 2|u'(t)|^2 + \frac{\alpha}{\alpha + 2} |u(t)|_{\alpha + 2}^2
\]

\[
- (\epsilon^{-1} \delta_2 B_1)^{\frac{\beta + 1}{\beta + 2}} (-E(t))^{1/(\beta + 1)} |u'(t)|^{\beta + 2}
\]

\[
- \epsilon^{\beta + 2} (-E(0))^{1/(\beta + 1)} |u(t)|_{\alpha + 2}^{\alpha + 2}
\]

\[
\geq 2(-E(t)) + 2|u'(t)|^2 + \frac{\alpha}{2(\alpha + 2)} |u(t)|_{\alpha + 2}^2
\]

\[
- \delta_2 m_0 (-E(t))^{1/(\beta + 1)} |u'(t)|^{\beta + 2},
\]

(2.3)
where we put $\varepsilon^{\beta+2} = (\alpha/2)(\alpha + 2)^{-1}(-E(0))^{(\beta+1)/2} > 0$ if $E(0) < 0$ and $m_0 = (2\delta_2(1 + 2/\alpha)B_1^{\beta+2}(-E(0))^{-1/2} > 0)$.

We introduce the function $G(t)$ as

$$G(t) \equiv (-E(t))^\omega + \omega m_0^{-1} P'(t)$$

with $\omega = 1 - (1/\beta+2 - 1/\alpha+2) (1/2 < \omega < 1)$, then we observe the following.

**Claim A.** If $E(0) < 0$ and $\alpha > \beta$, then

$$G'(t) \geq m_0^{-1} H(t), \quad (2.4)$$

where

$$H(t) \equiv (-E(t)) + 2\|u'(t)\|^2 + \frac{\alpha}{2(\alpha + 2)} \|u(t)\|^{\alpha+2} > 0. \quad (2.5)$$

Indeed, we see from (1.4) and (2.3) that

$$G'(t) = \omega(-E(t))^{-(1-\omega)}(-E'(t)) + \omega m_0^{-1} P''(t)$$

$$\geq 2\omega(-E(t))^{-(1-\omega)}(\delta_1 \|u'(t)\|^2 + \delta_2 \|u'(t)\|^{\beta+2} + \delta_3 \|A^{1/2} u'(t)\|^2)$$

$$+ 2\omega m_0^{-1} \{(-E(t)) + 2\|u'(t)\|^2 + \frac{\alpha}{2(\alpha + 2)} \|u(t)\|^{\alpha+2}\}$$

$$- \delta_2 m_0(-E(t))^{-(1-\omega)} \|u'(t)\|^{\beta+2}$$

$$\geq 2\omega m_0^{-1} \{(-E(t)) + 2\|u'(t)\|^2 + \frac{\alpha}{2(\alpha + 2)} \|u(t)\|^{\alpha+2}\} \geq m_0^{-1} H(t),$$

where we used the fact $1/2 < \omega < 1$, which implies (2.4).

Moreover, we observe the following.

**Claim B.** If $E(0) < 0$ and $\alpha > \beta$, then

$$G(t)^{1/\omega} \leq m_1 H(t), \quad (2.6)$$

with $m_1 = 2\max\{1, (1 + 2/\alpha)(2B_2 m_0^{-1})^{\frac{2}{2+\alpha}} (-E(0))^{-(1-(\omega-1/2)/\alpha+2)}\}$ and $B_2 = \|u\|^{\alpha+2}$. Indeed, Since $\|(u', u)\| \leq B_2 \|u'\| \|u\|^{\alpha+2}$, we have that

$$G(t)^{1/\omega} \leq 2\{(-E(t)) + (m_0^{-1} P'(t))^{1/\omega}\}$$

$$\leq 2\{(-E(t)) + (2B_2 m_0^{-1} \|u'(t)\| \|u(t)\|^{\alpha+2})^{1/\omega}\}$$

$$\leq 2\{(-E(t)) + 2\|u'(t)\|^2 + (1/2)(2B_2 m_0^{-1} \|u(t)\|^{\alpha+2})^{2/(2\omega-1)}\},$$
where we used the Young inequality. Moreover, since \(2/(2\omega - 1) < \alpha + 2\) and

\[
(E(0))^{-1/(\alpha+2)}\|u(t)\|_{\alpha+2} \geq 1 \quad \text{if} \quad E(0) < 0
\]

(see (2.2)), we observe that

\[
G(t)^{1/\omega} \leq 2((-E(t)) + 2\|u'(t)\|^2 + (1/2)(2B_2m_0^{-1})^{2-\frac{2}{\omega}}(-E(0))^{-(1-(2-2(\alpha+2))))^2\|u(t)\|_{\alpha+2}^2}
\]

and hence, we obtain (2.6).

Therefore it follows from Claim A and Claim B that

\[
\partial_t \{G(t)^{1-1/\omega}\} = -\frac{1-\omega}{\omega} G(t)^{-1/\omega}G'(t) \leq -\frac{1-\omega}{\omega} (m_0m_1)^{-1},
\]

and hence,

\[
G(t) \geq (G(0)^{(1-1/\omega)} - \frac{1-\omega}{\omega} (m_0m_1)^{-1})^{1-1/\omega}
\]

for some \(t > 0\) if \(G(0) > 0\). Here, we put \(T_0 \equiv m_0m_1(1 - \omega)^{-1}G(0)^{-1(1-1/\omega)}\). Then there exists a \(T\) such that \(0 < T \leq T_0\) and \(\lim_{t \to T} G(t) = +\infty\).

Since it follows from (0.2) and (0.3) that \((-E(t)) + \|u'(t)\|^2 \leq 2(\alpha + 2)^{-1}\|u(t)\|_{\omega+2}^2\), we have from (2.5) and (2.6) that \(G(t)^{1/\omega} \leq \text{Const.}\|u(t)\|_{\omega+2}^{2\omega}\). Thus, we see that \(\lim_{t \to T} \|u(t)\|_{\omega+2}^2 = \lim_{t \to T} \{\|Au(t)\|^2 + \|A^{1/2}u'(t)\|^2\} = +\infty\), and hence, the local solution \(u(t)\) can not be continued to the finite time \(T\). The proof of Theorem 2 is now completed. \(\square\)

**Remark 2.1.** Since \(G'(t) \geq m_0^{-1}H(t) \geq m_0^{-1}(-E(0)) > 0\), there exists a \(t_0 > 0\) such that \(G(t) > 0\) for \(t \geq t_0\), and hence, we see that if \(\alpha > \beta\) and \(E(0) < 0\) (without \(G(0) > 0\)), the local solution blows up at some finite time.

3. Blow Up II \((\delta_2 = 0)\)

When \(\delta_2 = 0\) \((\delta_1 \geq 0, \delta_3 \geq 0)\) in Eq. (0.1), we shall show that the solution blows up at some finite time under the assumptions which \(E(0) < 0\), or \(E(0) = 0\) and \((u_0, u_1) > 0\) (see [10–12]).

Our results are as follows.

**Theorem 3.** \((\delta_2 = 0)\) Let \(\delta_2 = 0\) in (0.1). Suppose that \(E(0) < 0\).

Then there exists a \(T\) such that

\[
0 < T \leq \alpha^{-2}(-E(0))^{-1}\left\{[(2\delta_1\|u_0\|^2+2\delta_3\|A^{1/2}u_0\|^2-\alpha(u_0,u_1))^2
+\alpha(-E(0))\|u_0\|^2)^{1/2}+2\delta_1\|u_0\|^2+2\delta_3\|A^{1/2}u_0\|^2-\alpha(u_0,u_1)\right\}
\]
and the local solution $u(t)$ in the sense of Theorem 1 blows up at the finite time $T$.

**Theorem 4.** ($\delta_2 = 0$) Let $\delta_2 = 0$ in (0.1). Suppose that

$$E(0) = 0 \quad \text{and} \quad (u_0, u_1) > 0.$$ 

Then there exists a $T$ such that

$$0 < T \leq 2\alpha^{-1}(u_0, u_1)^{-1}\|u_0\|^2$$

and the local solution $u(t)$ in the sense of Theorem 1 blows up at the finite time $T$.

Here, we denote the Sobolev-Poincaré constant by

$$(3.1) \quad c_{s, p} \equiv \sup \{ \|v\|_p\|A^{1/2}v\|^{-1} : v \in \mathcal{D}(A^{1/2}), \ v \neq 0 \}$$

for $2 \leq p \leq 4/[N - 2]^{+}$ ($2 \leq p < +\infty$ if $N = 2$).

**Theorem 5.** ($\delta_1 = \delta_2 = \delta_3 = 0$) Let $\delta_1 = \delta_2 = \delta_3 = 0$ in (0.1). Suppose that

$$(3.2) \quad E(0) \leq \alpha(\alpha + 2)^{-1}c_{s, 2}^{-2}\|u_0\|^2 \quad \text{and} \quad (u_0, u_1) > 0.$$ 

Then there exists a $T$ such that

$$(3.3) \quad 0 < T \leq 2\alpha^{-1}(u_0, u_1)^{-1}\|u_0\|^2$$

and the local solution $u(t)$ in the sense of Theorem 1 blows up at the finite time $T$.

**Proof of Theorem 3 and 4.** We put $r = -E(0) \geq 0$ and $\delta_2 = 0$ in (1.7), then we see from (1.10) that

$$Q(t) = \alpha\{\|A^{1/2}u(t)\|^2 + \int_0^t (\delta_1\|u'(s)\|^2 + \delta_3\|A^{1/2}u'(s)\|^2)\,ds\} \geq 0$$

and from (1.9) that

$$(P(t)^{-(\alpha/4)})'' = -(\alpha/4)P(t)^{-(\alpha/4+2)}\{P(t)P''(t) - (\alpha/4 + 1)P'(t)^2\} \leq 0,$$

and hence,

$$(3.4) \quad P(t) \geq \left\{\frac{P(0)^{\alpha/4+1}}{4P(0) - \alpha P'(0)t}\right\}^{\alpha/4}$$

for some $t > 0$ if $P(0) > 0$.

**Case I.** When $E(0) < 0$, we choose $\tau > 0$ such that

$$P'(0) = 2\{((u_0, u_1) + (-E(0))\tau) > 0,$$
and we take

\[ T_0 = \frac{4P(0)}{(\alpha P'(0))} \quad (> 0). \]

Then we see that

\[ T_0 = T(\tau) \equiv \frac{2\{\|u_0\|^2 + (-E(0))\gamma\}}{\alpha\{(u_0, u_1) + (-E(0))\gamma\} - 2(\delta_1\|u_0\|^2 + \delta_3\|A^{1/2}u_0\|^2)}, \]

and we find that \( T(\tau) \) takes a minimum at

\[ \tau = \tau_0 \equiv \alpha^{-2}(-E(0))^{-1}[[((2\delta_1\|u_0\|^2 + 2\delta_3\|A^{1/2}u_0\|^2 - \alpha(u_0, u_1))^2
\quad + \alpha^2(-E(0))\|u_0\|^2)^{1/2} + 2\delta_1\|u_0\|^2 + 2\delta_3\|A^{1/2}u_0\|^2 - \alpha(u_0, u_1)].\]

Here, we put

\[ T_0 = \min_{\tau > 0} T(\tau) = T(\tau_0). \]

Then, we see from (3.4) that there exists a \( T \) such that \( 0 < T \leq T_0 \) and

\[ \lim_{t \to T} \{\|u(t)\|^2 + \int_0^t(\delta_1\|u(s)\|^2 + \delta_3\|A^{1/2}u(s)\|^2) ds\} = +\infty, \]

that is, \( \lim_{t \to T} \|A^{1/2}u(t)\| = +\infty \) if \( \delta_3 > 0 \) and \( \lim_{t \to T} \|u(t)\| = +\infty \) if \( \delta_3 = 0 \), and hence, the local solution \( u(t) \) can not be continued to the finite time \( T \). The proof of Theorem 3 is now completed.

**Case II.** When \( E(0) = 0 \) and \( (u_0, u_1) > 0 \), we see that

\[ P(0) > 0 \quad \text{and} \quad P'(0) > 0. \]

Putting \( T_0 = \frac{4P(0)}{(\alpha P'(0))} \quad (> 0) \), we see from (3.4) that (3.5) holds for some \( 0 < T \leq T_0 \). The proof of Theorem 4 is now completed. \( \Box \)

**Proof of Theorem 5** We put \( r = 0 \) and \( \delta_1 = \delta_2 = \delta_3 = 0 \) in (1.7) i.e. \( P(t) = \|u(t)\|^2 \), then we see from (1.10) that

\[ Q(t) = -(\alpha + 2)E(0) + \alpha\|A^{1/2}u(t)\|^2. \]

We assume that \( (u_0, u_1) > 0 \), then

\[ P'(t) = 2(u(t), u'(t)) > 0 \]

for near \( t = 0 \), that is, \( P(t) \) is a increasing function and

\[ 0 < \|u_0\|^2 = P(0) \leq P(t) = \|u(t)\|^2 \leq c_{*,2}^2\|A^{1/2}u(t)\|^2 \]

for near \( t = 0 \). Thus we obtain that

\[ Q(t) \geq -(\alpha + 2)E(0) + \alpha c_{*,2}^{-2}\|u_0\|^2 \geq 0 \]
if \( E(0) \leq \alpha(\alpha + 2)^{-1}c_{\ast, \alpha}^{-2}\|u_0\|^2 \). Then it follows from (1.9) that
\[
(P(t)^{-\alpha/4})'' = -(\alpha/4)P(t)^{-\alpha/4+2}\{P(t)P''(t) - (\alpha/4 + 1)P'(t)^2\} \leq 0
\]
and
\[
(P(t)^{-\alpha/4})' = -(\alpha/4)P(t)^{-\alpha/4+1}P'(t) < 0
\]
for near \( t = 0 \). Thus we have that
\[
\partial_t\{(P(t)^{-\alpha/4})'\}^2 = 2(P(t)^{-\alpha/4})''(P(t)^{-\alpha/4})' \geq 0,
\]
and hence,
\[
\{(P(t)^{-\alpha/4})'\}^2 \geq \{(P(0)^{-\alpha/4})'\}^2 = -(\alpha/4)P(0)^{-\alpha/4+1}P'(0))^2 > 0
\]
for near \( t = 0 \). Therefore, we conclude that \( (P(t)^{-\alpha/4})' \) can not be change sign for \( t \geq 0 \), and we see that
\[
P(t) > 0, \quad P'(t) > 0, \quad \text{and} \quad (P(t)^{-\alpha/4})'' \leq 0
\]
for \( t \geq 0 \). Putting \( T_0 = 4P(0)/(\alpha P'(0))( > 0) \), we see from (3.4) that (3.5) with \( \delta_1 = \delta_3 = 0 \) holds for some \( 0 < T \leq T_0 \). The proof of Theorem 5 is now completed. \( \square \)

4. Blow Up III \( (\delta_2 = \delta_3 = 0 \& \alpha \leq 4/(N - 2)) \)

In this section, even if initial energy \( E(0) \) is positive, we shall show that the solution for the problem (0.1) with \( \delta_2 = \delta_3 = 0 \) \( (\delta_1 \geq 0) \) can not be continued globally under the assumptions which \( u_0 \in \mathcal{V}_\ast \) and \( E(0) \ll 1 \) and \( \alpha \leq 4/(N - 2) \) \( (\alpha < +\infty \text{ if } N = 1, 2) \).

We observe the following useful results connected with the \( K \)-negative set \( \mathcal{V}_\ast \).

**Proposition 4.1.** Let \( u \) be a solution of Eq.(0.1). Suppose that
\[
\alpha \leq 4/(N - 2) \quad (\alpha < +\infty \text{ if } N = 1, 2),
\]
\[
u_0 \in \mathcal{V}_\ast \equiv \{u \in D(A) : K(u) < 0\},
\]
and
\[
E(0) < \alpha(\alpha + 2)^{-1}c_{\ast, \alpha + 2}^{-2(\alpha + 2)/\alpha} \quad (\equiv D_\ast)
\]
with a positive constant \( c_{\ast, \alpha + 2} \) given by (3.1). Then
\[
K(u(t)) \equiv \|A^{1/2}u(t)\|^2 - \|u(t)\|_{\alpha + 2}^{\alpha + 2} < 0
\]
and
\[
E(t) < D_\ast \leq \alpha(\alpha + 2)^{-1}\|A^{1/2}u(t)\|^2
\]
for \( t \geq 0 \) (cf. (3.2)).

**Proof.** Since \( E(t) \leq E(0) \) (see (1.6)), we get from (4.2) immediately that

\[
E(t) < D_*.
\]

Let

\[
T \equiv \text{sup}\{t \in [0, +\infty) : K(u(s)) < 0 \text{ for } 0 \leq s < t\},
\]

then we see \( T > 0 \) by (4.1) and \( K(u(t)) < 0 \) and \( u(t) \neq 0 \) for \( 0 \leq t < T \). If \( T = +\infty \), then \( K(u(T)) = 0 \), and hence,

\[
J(u(T)) = \frac{\alpha}{\alpha + 2}\|A^{1/2}u(T)\|^2.
\]

Now, when \( K(u) < 0 \) and \( u \neq 0 \), we see from (3.1) that

\[
\|A^{1/2}u\|^2 < \|u\|_{\alpha + 2}^{\alpha + 2} \leq c_{*,\alpha+2}^{-1}\|A^{1/2}u\|^{\alpha + 2}
\]

for \( \alpha \leq 4/(N-2) \) (\( \alpha < +\infty \) if \( N = 1, 2 \)), and hence,

\[
\|A^{1/2}u\|^2 > c_{*,\alpha+2}^{-2(\alpha+2)/\alpha} \quad (> 0).
\]

Thus, we have from (4.7) and the continuity that

\[
\|A^{1/2}u(T)\|^2 \geq c_{*,\alpha+2}^{-2(\alpha+2)/\alpha}.
\]

Thus we get from (0.2), (4.6), and (4.8) that

\[
E(T) \geq J(T) \geq \alpha(\alpha + 2)^{-1}\|A^{1/2}u(T)\|^2 \geq D_*,
\]

which contradicts (4.5), and hence, we see \( T = +\infty \). Moreover, from (4.5) and (4.7) we obtain (4.4).

When \( \delta_1 = \delta_2 = \delta_3 = 0 \) in (0.1) (non-dissipative case), we obtain the following result.

**Theorem 6.** \((\delta_1 = \delta_2 = \delta_3 = 0)\) Let \( \delta_1 = \delta_2 = \delta_3 = 0 \) in (0.1). Under the assumption of proposition 4.1, the local solution blows up at some finite time.

**Remark 4.2.** If we assume that \((u_0, u_1) > 0\), then the conclusion of Theorem 3 holds true, that is, the local solution blows up at the finite time \( T \) given by (3.3).

**Proof.** We put \( r = 0 \) and \( \delta_1 = \delta_2 = \delta_3 = 0 \) in (1.7) i.e. \( P(t) = \|u(t)\|^2 \), then we see from (1.12) that

\[
P''(t) = 2\|u'(t)\|^2 - K(u(t))
\]

\[
= (\alpha + 4)\|u'(t)\|^2 + \{\alpha\|A^{1/2}u(t)\|^2 - (\alpha + 2)E(t)\}
\]

\[
\geq (\alpha + 4)\|u'(t)\|^2,
\]
where we used (4.4) at the last inequality. Thus we have that
\[ P''(t)P(t) - (\alpha/4 + 1)P'(t)^2 \geq (\alpha + 4)\left\| u'(t) \right\|^2 \left\| u(t) \right\|^2 - (u(t), u'(t))^2 \geq 0 \]
(4.11)
for \( t \geq 0 \).

On the other hand, we see from (4.10), (4.7), and (1.6) with \( \delta_1 = \delta_2 = \delta_3 = 0 \) that
\[ P''(t) \geq \alpha \| A^{1/2} u(t) \|^2 - (\alpha + 2)E(t) \geq (\alpha + 2)\{ D_* - E(0) \} \equiv n_0 > 0, \]
where we used the assumption (4.2). Then we obtain that
\[ P'(t) \geq P'(0) + n_0 t, \]
and hence, there exists \( t_0 \) such that
\[ P'(t) = 2(u(t), u'(t)) > 0 \]
(4.12)
for \( t \geq t_0 \). Thus, from (4.11) and (4.12) we arrived at our conclusion by the argument as in Section 2. \( \Box \)

**Theorem 7**. (\( \delta_1 > 0, \delta_2 = \delta_3 = 0 \)) Let \( \delta_1 > 0 \) and \( \delta_2 = \delta_3 = 0 \) in (0.1). Under the assumption of Proposition 4.1, the local solution blows up at some finite time.

**Proof.** Following Ohta [18], we shall prove the theorem. We put
\[ \tilde{P}(t) \equiv \| u(t) \|^2, \]
then we see from (1.5) (cf. (4.9)) that
\[ \tilde{P}''(t) + \delta_1 \tilde{P}'(t) = 2(\| u'(t) \|^2 - K(u(t))) \]
\[ = (\alpha + 4)\| u'(t) \|^2 + \{ \alpha \| A^{1/2} u(t) \|^2 - (\alpha + 2)E(t) \} \]
\[ \geq (\alpha + 4)\| u'(t) \|^2 + (\alpha + 2)\{ D_* - E(t) \}, \]
(4.13)
where we used (4.7). Next, we put
\[ H(t) \equiv \delta_1 \tilde{P}'(t) - (\alpha/2 + 2)\{ D_* - E(t) \}, \]
(4.14)
then we see from (1.4) with \( \delta_2 = \delta_3 = 0 \) and (4.13) that
\[ H'(t) = \delta_1 \tilde{P}''(t) + (\alpha/2 + 2)E'(t) \]
\[ = \delta_1 \tilde{P}''(t) - (\alpha + 4)\delta_1 \| u'(t) \|^2 \]
\[ \geq -\delta_1^2 \tilde{P}'(t) + \delta_1 (\alpha + 2)\{ D_* - E(t) \} \]
\[ \geq -\delta_1 H(t) + \delta_1 (\alpha/2)\{ D_* - E(0) \}, \]
where we used the fact $E(t) \leq E(0)$ (see (1.6)). Thus we get

$$H(t) \geq e^{-\delta t}(H(0) - n_1) + n_1,$$

where $n_1 = (\alpha/2)(D_\ast - E(0)) > 0$ by (4.2), and hence, there exists a $t_1$ such that

$$H(t) > 0 \quad \text{for} \quad t \geq t_1.$$

Therefore, it follows from (4.14) and (4.4) that

$$\delta_1 \bar{P}'(t) > (\alpha/2 + 2)(D_\ast - E(t)) > 0,$$

that is,

$$P(t) > 0 \quad \text{and} \quad P'(t) > 0 \quad \text{for} \quad t \geq t_1.$$

On the other hand, we observe from (4.15) and (1.4) with $\delta_2 = \delta_3 = 0$ that

$$\begin{align*}
\delta_1 [(D_\ast - E(t)) \bar{P}(t)^{-\alpha/4+1}] & = -E'(t)\bar{P}(t)^{-\alpha/4+1} - (\alpha/4 + 1)(D_\ast - E(t))\bar{P}'(t)\bar{P}(t)^{-\alpha/4+2} \\
& \geq -\{E'(t)\bar{P}(t) + (\delta_1/2)\bar{P}'(t)^2\}\bar{P}(t)^{-\alpha/4+2} \\
& = 2\delta_1 \{\|u(t)\|^2 \|u(t)\|^2 - (u(t), u'(t))^2\}\bar{P}(t)^{-\alpha/4+2} \geq 0,
\end{align*}$$

and hence,

$$\begin{align*}
\|D_\ast - E(t)\| \geq n_2 \bar{P}(t)^{\alpha/4+1}
\end{align*}$$

for $t \geq t_1$, where $n_2 = \{D_\ast - E(t_1)\} \bar{P}(t_1)^{-(\alpha/4+1)} > 0$ by (4.4)). Thus we have from (4.13) and (4.16) that

$$\bar{P}''(t) + \delta_1 \bar{P}'(t) \geq n_2 \bar{P}(t)^{\alpha/4+1}$$

with $\bar{P}(t) > 0$ and $\bar{P}'(t) > 0$ for $t \geq t_1$, and hence, we conclude from Lemma 4.3 below that $\bar{P}(t) = \|u(t)\|^2$ blows up at some finite time. The proof of Theorem 7 is now completed. □

**Lemma 4.3.** (see [14, 21]) Let the function $P(t)$ satisfy

$$P''(t) + \delta P'(t) \geq c_0 P(t)^{1+r}$$

for $t \geq 0$ with $\delta \geq 0, c_0 > 0, r > 0$, and $P(0) > 0$ and $P'(0) > 0$. Then $P(t)$ blows up at some finite time.

**Proof.** We consider that the differential equation $Q'(t) = \varepsilon Q(t)^{1+r/2}$ for $Q(t) \in C^2([0, +\infty))$ and $0 < \varepsilon \ll 1$ with $Q(0) = P(0) > 0$. Then we see that $Q(t) = \{Q(0)^{-r/2} - (r/2)\varepsilon t\}^{-2/r} \leq \frac{1}{3} \varepsilon t$ for some $t > 0$, and that $Q(t)$ blows up at some finite time $T_0$. Since

$$\varepsilon Q(0)^{1+r/2} = \varepsilon Q'(0) < P'(0)$$
for some small $\varepsilon > 0$, we have that

$$Q''(t) = \varepsilon(1 + r/2)Q(t)^{r/2}Q'(t) = \varepsilon^2(1 + r/2)Q(t)^{1+r},$$

and hence, from $Q(t) \geq Q(0)$,

$$Q''(t) + \delta Q'(t) = \varepsilon^2(1 + r/2)Q(t)^{1+r} + \varepsilon\delta Q(t)^{1+r/2}$$

$$(4.18) \quad \leq \{\varepsilon^2(1 + r/2) + \varepsilon\delta Q(0)^{-r/2}\}Q(t)^{1+r} \leq c_0 Q(t)^{1+r}$$

for small $\varepsilon > 0$. Since $Q'(0) < P'(0)$, we see that $Q'(t) < P'(t)$ for near $t = 0$. Let $T \equiv \sup\{t \in [0, +\infty) : Q'(s) < P'(s) \text{ for } 0 \leq s < t\}$, then we see $T > 0$ and $Q'(t) < P'(t)$ for $0 \leq t < T$ and $Q(t) < P(t)$ for $0 < t < T$. If $T < T_0$, then we observe that

$$Q'(T) = P'(T), \quad Q''(T) \geq P''(T), \quad \text{and} \quad Q(T) < P(T).$$

On the other hand, it follows from (4.17) and (4.18) that

$$(Q''(T) - P''(T)) + \delta(Q'(T) - P'(T)) \leq c_0(Q(T)^{1+r} - P(T)^{1+r}),$$

which is a contradiction, and hence, we see that $T \geq T_0$ and

$$Q(t) \leq P(t) \quad \text{for} \quad 0 \leq t \leq T_0.$$

Thus, $P(t)$ blows up at some finite time. □

5. Global Existence and Decay

In this section we shall study on the global in time existence and energy decay properties of the solution for Eq. (0.1) with $\delta_1 + \delta_2 + \delta_3 > 0$ under the assumptions that $0 \leq E(0) \equiv E(u_0, u_1) \ll 1$ and

$$u_0 \in W_* \equiv \{u \in D(A) : K(u) > 0\} \cup \{0\}.$$

We observe the following useful results connected with the $K$-positive set $W_*$. 

**Proposition 5.1.** (i) If $\alpha < 4/[N - 4]^+$, then

$$(5.1) \quad W_* \text{ is a neighborhood of } 0 \text{ in } D(A^{1/2}) = H_0^1(\Omega) \text{ and an open set.}$$

(ii) If $u \in \overline{W}_*$, then

$$(5.2) \quad d_*^{-1}\|A^{1/2}u\|^2 \leq J(u) \quad (\leq E(u, u'))$$

where $d_* = (1 + 2\alpha^{-1}) (\geq 1)$. 
PROOF. We see from Lemma 5.2 below that
\begin{equation}
\|u\|^\alpha+2 \leq c_*^{\alpha+2} \|A^{1/2}u\|^\alpha-(\alpha+2)^2 \|Au\|^{(\alpha+2)^2} \|A^{1/2}u\|^2,
\end{equation}
where \(\theta_1 = \left[(N - 2)\alpha - 4 \right]^{+} / (2(\alpha + 2))\) and \(\alpha - (\alpha + 2)\theta_1 > 0\) if \(\alpha < 4/\left[N - 4\right]^{+}\), and hence, \(K(u) > 0\) if \(D(A^{1/2})\)-norm of \(u\) is sufficiently small and \(u \neq 0\), which implies (5.1). From the definitions of \(W_*\) and \(J(u)\), (5.2) follows immediately. \(\square\)

We use well-known lemma without the proof.

**Lemma 5.2.** (Gagliardo-Nirenberg) Let \(1 \leq r < p \leq +\infty\) and \(p \geq 2\). Then, the inequality
\[\|v\|_p \leq c_* \|A^{m/2}v\|^\theta \|v\|_r^{1-\theta}\] for \(v \in D(A^{m/2}) \cap L^r(\Omega)\)
holds with some constant \(c_*\) and
\[\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} + \frac{m}{N} - \frac{1}{2}\right)^{-1}\]
provided that \(0 < \theta \leq 1\) (\(0 < \theta < 1\) if \(m - N/2\) is a nonnegative integer).

(Sobolev-Poincaré) Let \(1 \leq p \leq 2N/\left[N - 2m\right]^{+}\) \((1 \leq p < +\infty\) if \(N = 2m\)). Then, the inequality
\[\|v\|_p \leq c_* \|A^{m/2}v\|\] for \(v \in D(A^{m/2})\)
holds with some constant \(c_*\).

Moreover, we use the inequality \(\|u\| \leq c_* \|u\|_p\) for \(u \in L^p(\Omega), p \geq 2\), with some constant \(c_*\). In what follows, we assume \(c_* \geq 1\) for simplicity.

To state our results we define the second energy associated with Eq.(0.1) by
\[E_2(u, u') \equiv \|A^{1/2}u\|^2 + \|Au\|^2\]
Then, multiplying Eq.(0.1) by \(2Au'\) and integrating it over \(\Omega\), we have
\begin{equation}
\partial_t E_2(t) + 2\delta_1 \|A^{1/2}u'(t)\|^2 + \delta_2 (\beta + 1) \int_{\Omega} |u'(t)|^\beta |A^{1/2}u'(t)|^2 dx
\end{equation}
\begin{equation}
+ \delta_3 \|Au(t)\|^2 = 2(f(u(t)), Au(t))\]
where we put \(E_2(t) \equiv E_2(u(t), u'(t))\) \((E_2(0) \equiv E_2(u_0, u_1))\) for simplicity.

In what follows, we denote by \(c_{ij}, j = 1, 2, \ldots\), constants independent of the initial data and depending only on \(\alpha, \beta, N, c_*\), \(\delta_1, \delta_2\), and \(\delta_3\).

Our results are as follows:

**Theorem 8.** (\(\delta_2 > 0\)) Let \(\delta_2 > 0\) and \(\delta_1 = \delta_3 = 0\) in (0.1), and let the initial data \(\{u_0, u_1\}\) belong to \(W_* \subset D(A) \times D(A^{1/2})\). Suppose that
\[
\alpha < 2/\left[N - 4\right]^{+}, \quad \beta \leq 4/(N - 2) \quad (\beta < +\infty \text{ if } N = 1, 2),
\]
\[
\beta < \alpha - \left[(N/2 - 1)\alpha - 1\right]^{+},
\]
and that the initial energy $E(0)$ is small ($0 \leq E(0) \ll 1$ but $E_2(0) \geq 1$) such that

(i) when $\alpha \leq 4/(N-2)$ ($\alpha < +\infty$ if $N \leq 2$),

$$
(5.5) \quad 0 \leq c_1 E(0)^{\alpha/2} < 1 \quad \text{and} \quad \omega_1 c_3 E(0)^{\omega_3} E_2(0)^{\omega_1} < 1,
$$

(ii) when $4/(N-2) < \alpha < 2/(N-4)^+ \quad (N \geq 3),

$$
(5.6) \quad 0 \leq \{\omega_1 c_3 E(0)^{\omega_3} + c_4 E(0)^{\omega_3}\} E_2(0)^{\omega_1} < 1,
$$

where $\omega_1 = [(N-2)\alpha^{+}/4 \geq 0], \omega_2 = (\alpha - \beta)/2 - \omega_1 \ (> 0), \text{and} \ \omega_3 = \omega_1 (4 - (N - 4)\alpha)/((N - 2)\alpha - 4) \ (> 0)$. Then, the problem (0.1) admits a unique global solution $u \in \mathcal{W}_* \text{ satisfying}

$$
(5.7) \quad \|u'(t)\|^2 + \|A^{1/2} u(t)\|^2 \leq d_* E(t) \leq c(1 + t)^{-2/\beta}
$$

for $t \geq 0$ with a constant $c$.

**Theorem 9.** ($\delta_1 + \delta_3 > 0$). Let $\delta_1 + \delta_3 > 0$ and $\delta_2 \geq 0$ in (0.1), and let the initial data $\{u_0, u_1\}$ belong to $\mathcal{W}_* \times \mathcal{D}(A^{1/2})$. Suppose that

$$
\alpha \leq 2/[N-4]^+ \quad \text{and} \quad \beta \leq 4/N-2 \quad (\beta < +\infty \text{ if } N = 1,2),
$$

and that the initial energy $E(0)$ is small ($0 \leq E(0) \ll 1$ but $E_2(0) \geq 1$) such that

(i) when $\alpha \leq 4/(N-2)$ ($\alpha < +\infty$ if $N = 1,2$),

$$
(5.8) \quad 0 \leq c_1 E(0)^{\alpha/2} < 1 \quad \text{and} \quad \omega_1 c_3 E(0)^{\omega_3} E_2(0)^{\omega_1} < 1,
$$

(ii) when $4/(N-2) < \alpha < 2/[N-4]^+ \quad (N \geq 3),

$$
(5.9) \quad 0 \leq \{\omega_1 c_3 E(0)^{\omega_3} + c_4 E(0)^{\omega_3}\} E_2(0)^{\omega_1} < 1.
$$

where $\omega_1 = [(N-2)\alpha^{+}/4 \geq 0], \tilde{\omega}_2 = \alpha/2 - \omega_1 \ (> 0), \text{and} \ \omega_3 = \omega_1 (4 - (N - 4)\alpha)/((N - 2)\alpha - 4) \ (> 0)$. Then, the problem (0.1) admits a unique global solution $u \in \mathcal{W}_* \text{ satisfying}

$$
(5.10) \quad \|u'(t)\|^2 + \|A^{1/2} u(t)\|^2 \leq d_* E(t) \leq ce^{-kt}
$$

for $t \geq 0$ with constants $c$ and $k > 0$.

**Remark 5.3.** When we consider the problem (0.1) with $|u|^{\alpha} u$ replaced by the non-linear function $f(u)$ such that

$$
|f(u)| \leq k_1 |u|^\alpha + 1 \quad \text{and} \quad |f'(u)| \leq k_1 |u|^\alpha
$$

with positive constants $k_1$ and $k_2$, we can get the similar results as Theorem 8 and Theorem 9. Then we need to redefine (0.3) and (0.6) by

$$
J(u) \equiv \|A^{1/2} u\|^2 - 2 \int_{\Omega} F(u) \, dx
$$
with \( F(u) = \int_0^u f(\eta) \, d\eta \) and

\[
K(u) \equiv \|A^{1/2}u\|^2 - k_1 \|u\|_{\alpha+2}^{\alpha+2},
\]

respectively.

First, we shall prepare for those proof. We put

\[
T_1 \equiv \sup \{ t \in [0, +\infty) : u(s) \in \mathcal{W}_* \text{ for } 0 \leq s < t \},
\]

then we see \( T_1 > 0 \) and \( u(t) \in \mathcal{W}_* \) for \( 0 \leq t < T_1 \) because \( u_0 \in \mathcal{W}_* \) being an open set (see \((5.1))\). If \( T_1 < +\infty \), then \( u(T_1) \in \partial \mathcal{W}_* \), that is,

\[
K(u(T_1)) = 0 \quad \text{and} \quad u(T_1) \neq 0.
\]

We see from \((1.6), (5.2), \) and \((5.3)\) that

\[
\|u(t)\|_{\alpha+2}^{\alpha+2} \leq (1/2)B(t)\|A^{1/2}u(t)\|^2
\]

for \( 0 \leq t \leq T_1 \) where

\[
B(t) \equiv c_1 E(0)^{(\alpha-(\alpha+2)\delta_1)/2}\|Au(t)\|^{(\alpha+2)\delta_1}
\]

with \( c_1 = 2c_2^{\alpha+2}d_1^{\alpha-(\alpha+2)\delta_1}/2 \).

We put

\[
T_2 \equiv \sup \{ t \in [0, +\infty) : B(s) < 1 \text{ for } 0 \leq s < t \},
\]

then we see \( T_2 > 0 \) and \( B(t) < 1 \) for \( 0 \leq t < T_2 \) because \( B(0) < 1 \) by \((5.5), (5.6), (5.8), \) or \((5.9)\). If \( T_2 < T_1 (< +\infty)\), then

\[
B(T_2) = 1,
\]

and

\[
K(u(t)) \geq \|A^{1/2}u(t)\|^2 - (1/2)B(t)\|A^{1/2}u(t)\|^2 \geq (1/2)\|A^{1/2}u(t)\|^2
\]

for \( 0 \leq t \leq T_2 \).

**Proof of Theorem 8.** Following Nakao [16], we shall derive the decay property of the energy \( E(t) \equiv E(u(t), u'(t)) \) associated with Eq.\((0.1)\) with \( \delta_2 > 0 \) and \( \delta_1 = \delta_3 = 0 \). In what follows, we put \( \delta_2 = 1 \) without loss of generality.

For a moment, we assume that \( T_2 > 1 \). Integrating \((1.4)\) with \( \delta_2 = 1 \) and \( \delta_1 = \delta_3 = 0 \) over \([t, t+1], 0 < t < T_2 - 1\), we have

\[
2 \int_t^{t+1} \|u'(s)\|^{\gamma+2}_{\beta_+^*} \, ds = E(t) - E(t + 1) \quad (\equiv 2D(t)^{\gamma+2})
\]
and

\( (5.17) \quad \int_t^{t+1} \|u'(s)\|^2 ds \leq c_s^2 \int_t^{t+1} \|u'(s)\|_{\beta+2}^2 ds \leq c_s^2 D(t)^2. \)

Then there exist \( t_1 \in [t, t + 1/4] \) and \( t_2 \in [t + 3/4, t + 1] \) such that

\( (5.18) \quad \|u'(t_i)\| \leq 2c_s D(t) \quad i = 1, 2. \)

Since \( |(g(u'), u)| \leq \|u\|^{\beta+1}_{\beta+2} \|u\|_{\beta+2} \), we see from (1.5) and (5.15) that

\[
\frac{1}{2} \int_t^{t_1} \|A^{1/2} u(s)\|^2 ds \leq \int_t^{t_1} K(u(s)) ds
\]

\[
\leq \int_t^{t+1} \|u'(s)\|^2 ds + \sum_{i=1}^2 \|u'(t_i)\| \|u(t_i)\| + \int_t^{t+1} \|u'(s)\|_{\beta+2} \|u(s)\|_{\beta+2} ds
\]

\[
\leq \int_t^{t+1} \|u'(s)\|^2 ds + \|u'(t_i)\| + \sum_{i=1}^2 \|u'(t_i)\|
\]

\[
\left[ \int_t^{t+1} \|u'(s)\|_{\beta+2} ds \right] \sup_{t \leq s \leq t+1} \|A^{1/2} u(s)\|,
\]

where we used the fact that \( \|u\|_{\beta+2} \leq c_s \|A^{1/2} u\| \) for \( \beta \leq 4/(N - 2) \). Integrating (1.4) over \([t, t_2]\), we have from (5.19) that

\[
E(t) = E(t_2) + 2 \int_t^{t_2} \|u'(s)\|_{\beta+2}^2 ds
\]

\[
\leq 2 \int_t^{t_2} E(s) ds + 2 \int_t^{t+1} \|u'(s)\|_{\beta+2}^2 ds
\]

\[
\leq 2 \int_t^{t+1} \{\|u'(s)\|^2 + \|u'(s)\|_{\beta+2}^2\} ds + 2 \int_t^{t_2} \|A^{1/2} u(s)\|^2 ds
\]

\[
\leq 2 \int_t^{t+1} \left\{ 3 \|u'(s)\|^2 + \|u'(s)\|_{\beta+2}^2 \right\} ds
\]

\[
+ 4c_s \left\{ \sum_{i=1}^2 \|u'(t_i)\| + \left( \int_t^{t+1} \|u'(s)\|_{\beta+2} ds \right)^{\beta+1} \right\} \sup_{t \leq s \leq t+1} \|A^{1/2} u(s)\|,
\]

and from (5.16), (5.17), and (5.18) that

\[
E(t) \leq 2 \{3c_s^2 D(t)^2 + D(t)^{\beta+2}\} + 4c_s \{4c_s D(t) + D(t)^{\beta+1}\} (d_s E(t))^{1/2}.
\]

Since \( 2D(t)^{\beta+2} \leq E(t) \leq E(0) \leq 1 \), we see

\[
E(t) \leq 2^\beta c_s^4 d_s D(t)^2 + (1/2) E(t),
\]
and hence,
\[
E(t)^{1+\beta/2} \leq (2^\theta c_* d_*)^{(\beta+2)/2} D(t)^{\beta+2} \\
\leq 2^{-1}(2^\theta c_* d_*)^{(\beta+2)/2} \{E(t) - E(t+1)\}.
\]

Thus, noting the fact \(E(t) \leq E(0)\) and applying Lemma 5.4 below, we obtain the following energy decay estimate:
\[
E(t) \leq \{E(0)^{-\beta/2} + d_0^{-1}[t-1]^+\}^{-2/\beta}
\]
for \(0 \leq t \leq T_2\) with \(d_0 = \beta^{-1}/(2^\theta c_* d_*)^{(\beta+2)/2} \geq 1\).

Next, using the energy decay (5.20), we shall estimate the second energy \(E_2(t) \equiv E_2(u(t), u'(t))\). It follows from (5.4) and Lemma 5.2 that
\[
\partial_t E_2(t) \leq 2(f(u(t)), Au'(t)) \leq 2c_* (\alpha + 1)\|u(t)\|_{\mathbb{H}_\alpha}^\alpha \|Au(t)\|_{\mathbb{A}^{1/2}} \|A^{1/2} u'(t)\|
\leq 2c_*^{\alpha+1}(\alpha + 1)\|A^{1/2}(u(t))\|^\alpha \|A^{1/2} u'(t)\|^\beta
\leq c_2 E(t)^{\alpha(1-\theta_2)/2} E_2(t)^{\omega_2+1},
\]
where \(c_2 = 2c_*^{\alpha+1}(\alpha + 1)d_0^{(1-\theta_2)/2}, \theta_2 = ([N-2]\alpha - 2)\alpha/(2\alpha), \) and \(\omega_1 = \alpha \theta_2/2\). We observe from (5.20) that if \(\alpha(1-\theta_2) > \beta\),
\[
\int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds = \int_0^1 c_2 E(0)^{\omega_1},
\]
with \(c_3 = c_2 d_0(\alpha(1-\theta_2))/((\alpha(1-\theta_2) - \beta)\omega_2 = (\alpha(1-\theta_2) - \beta)/2\).

When \(\alpha \leq 2/(N-2)\) (i.e. \(\omega_1 = 0\)), we have from (5.21) and (5.22) that
\[
E_2(t) \leq E_2(0) \exp\{\int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds\}
\leq E_2(0) \exp\{c_3 E(0)^{\omega_2}\} \leq +\infty.
\]

On the other hand, when \(\alpha > [N-2]^+\) (i.e. \(\omega_1 > 0\)), we have
\[
E_2(t) \leq \{E_2(0)^{-\omega_1} - \omega_1 \int_0^t c_2 E(s)^{\alpha(1-\theta_2)/2} ds\}^{-1/\omega_1}
\leq \{E_2(0)^{-\omega_1} - \omega_1 c_3 E(0)^{\omega_2}\}^{-1/\omega_1} \leq +\infty
\]
if \(\omega_1 c_3 E(0)^{\omega_2} E_2(0)^{\omega_2} < 1\).

When \(\alpha \leq 4/(N-2)\) (i.e. \(\theta_1 = 0\)), we have from (5.5) and (5.13) that
\[
B(t) = c_1 E(0)^{\alpha/2} < 1.
\]

On the other hand, when \(\alpha > 4/[N-2]^+\) (i.e. \(\theta_1 > 0\)), we have from (5.13) and (5.24) that
\[
B(t) \leq c_1 E(0)^{1-(N-4)\alpha/4} E_2(t)^{(N-2)\alpha/4-1}
\leq c_1 E(0)^{1-(N-4)\alpha/4} \{E_2(0)^{-\omega_1} - \omega_1 c_3 E(0)^{\omega_2}\}^{-1}(N-2)^{\alpha/4-1} \leq 1.
\]
if we assume (5.6), that is,
\[ \{\omega_1 c_3 E(0)^{\omega_3} + c_4 E(0)^{\omega_3}\} E_2(0)^{\omega_4} < 1 \]
with \( c_4 = c_1^{\omega_1} / ((N - 2)\alpha - 4) \) and \( \omega_3 = \omega_1 (4 - (N - 4)\alpha) / ((N - 2)\alpha - 4) \). Thus we conclude that (5.25) and (5.26) contradict (5.14), and hence, we see that \( T_2 \geq T_1 \). Moreover, we observe from (5.11) and (5.15) that
\[ 0 = K(u(T_1)) \geq (1/2)\|A^{1/2}u(T_1)\|^2 > 0, \]
which is a contradiction, and hence, we see that \( T_1 = +\infty \), that is, (5.20), (5.23), and (5.24) hold true for all \( t \geq 0 \). The proof of Theorem 8 is now completed. \( \Box \)

We used the following useful lemma in the proof of Theorem 8. (We omit the proof here, see [15, 17].)

**Lemma 5.4.** (Nakao [15]) Let \( \phi \) be a bounded and nonnegative function on \([0, +\infty)\) satisfying
\[ \sup_{t \leq s \leq t + 1} \phi(s)^{1 + r} \leq k \{\phi(t) - \phi(t + 1)\} \]
for \( t > 0 \) and \( k > 0 \). Then
\[ \phi(t) \leq \{\phi(0)^{-r} + rk^{-1}[t - 1]^+\}^{-1/r} \text{ for } t \geq 0. \]

**Proof of Theorem 9.** From (1.5), we have
\[
\partial_t \{2(u(t), u'(t)) + \delta_1 \|u(t)\|^2 + \delta_3 \|A^{1/2}u(t)\|^2\} = 2\|u'(t)\|^2 - 2K(u(t)) - 2(g(u'(t)), u(t)),
\]
and hence, from this and (1.4), we have
\[
\partial_t E^*(t) = -2\{(\delta_1 - \varepsilon)\|u'(t)\|^2 + \delta_2 \|u'(t)\|^{\delta_2 + 2} + \delta_3 \|A^{1/2}u'(t)\|^2\}
- 2\varepsilon K(u(t)) - 2\varepsilon \delta_2 (g(u'(t)), u(t)) \tag{5.27}
\]
for \( \varepsilon < 1 \), where we set
\[
E^*(t) \equiv E(t) + \varepsilon \{2(u(t), u'(t)) + \delta_1 \|u(t)\|^2 + \delta_3 \|A^{1/2}u(t)\|^2\}. \tag{5.28}
\]
Then we see that for
\[
(2d_*)^{-1}(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2) \leq E^*(t) \leq 2(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2). \tag{5.29}
\]
if \( \varepsilon \leq (2d_*(c_* + c_*^2 \delta_1 + \delta_3))^{-1} \). Indeed, since
\[
d_*^{-1}(\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2) \leq E(t) \leq \|u'(t)\|^2 + \|A^{1/2}u(t)\|^2 \tag{5.30}
\]
by (5.2) and
\[
|2(u, u') + \delta_1 \|u\|^2 + \delta_3 \|A^{1/2} u\|^2| \\
\leq 2c_* \|A^{1/2} u\| |u'| + c_*^2 \delta_1 \|A^{1/2} u\|^2 + \delta_3 \|A^{1/2} u\|^2 \\
\leq (c_* + c_*^2 \delta_1 + \delta_3)(\|u'\|^2 + \|A^{1/2} u\|^2),
\]
we see (5.29) immediately.

To proceed the estimation of (5.27), we observe from (1.6) and (5.2) that
\[
|\delta_2(g(u'), u)| \leq \delta_2 \|u\|_{\beta+2} \|u'\|_{\beta+2}^{\beta+1} \\
\leq \delta_2 c_* \|A^{1/2} u\| |u'|_{\beta+2}^{\beta+1}, \quad \beta \leq 4/(N-2) \\
= \delta_2 c_* \|A^{1/2} u\|_{\beta+2}^{\beta+1} \|A^{1/2} u\|_{\beta+2} \|u'|_{\beta+2}^{\beta+1} \\
\leq \delta_2 c_* (d_* E(0))^{\beta+1} \|u'|_{\beta+2}^{\beta+1} \|A^{1/2} u\|_{\beta+2} \\
\leq \frac{\beta+1}{\beta+2} (\delta_2 c_* (d_* E(0))^{\beta+1}) \|u'|_{\beta+2}^{\beta+1} + \frac{1}{\beta+2} \|A^{1/2} u\|^2 \\
\leq (\delta_2 c_* d_*)^{\frac{\beta+1}{\beta+2}} \|u'|_{\beta+2}^{\beta+2} + (1/2) \|A^{1/2} u\|^2,
\]
and hence,
\[
\partial_t E^*(t) \leq -2(\delta_1 + c_*^2 \delta_3 - \epsilon) \|u'(t)\|^2 - \epsilon \|A^{1/2} u(t)\|^2 \\
- 2(\delta_2 - \epsilon (\delta_2 c_* d_*)^{\frac{\beta+1}{\beta+2}}) \|u'(t)\|_{\beta+2}^{\beta+2} \\
\leq -2\epsilon (\|u'(t)\|^2 + \|A^{1/2} u(t)\|^2),
\]
where we used (5.15) and we put
\[
\epsilon = \min\{(\delta_1 + c_*^2 \delta_3)/2, \delta_2 (\delta_2 c_* d_*)^{-\frac{\beta+1}{\beta+2}}, (2d_*(c_* + c_*^2 \delta_1 + \delta_3))^{-1}\}
\]
(We note that \(\epsilon > 0\) by \(\delta_1 + \delta_3 > 0\).) Thus we obtain from (5.29), (5.30), and (5.31) that
\[
E^*(t) \leq E^*(0) e^{-\epsilon t}
\]
or
\[
E(t) \leq \|u'(t)\|^2 + \|A^{1/2} u(t)\|^2 \leq (2d_*)^2 E(0) e^{-\epsilon t}
\]
for \(0 \leq t \leq T_2\).

Next, using the decay (5.32), we shall estimate the second energy \(E_2(t)\). It follows from (5.4) and (5.21) that
\[
\partial_t E_2(t) \leq c_2 E(t)^{\alpha(1-\theta_1)/2} E_2(t)^{\alpha+1}.
\]
We observe from (5.32) that if \( \alpha(1 - \theta_2) > \beta \),
\[
(5.34) \quad \int_0^t c_2 E(s)^{\alpha(1 - \theta_2)/2} ds \leq c_5 E(0)^{\tilde{\omega}_2} \]
with \( c_5 = c_2 (2d_*)^{2\tilde{\omega}_2}/\tilde{\omega}_2 \) and \( \tilde{\omega}_2 = (1 - \theta_2)/2 \ (> 0) \).

When \( \alpha \leq 2/(N - 2) \) (i.e. \( \omega_1 = 0 \)), we have from (5.33) and (5.34) that
\[
(5.35) \quad E_2(t) \leq E_2(0) \exp \{ c_5 E(0)^{\alpha/2} \} \quad (< +\infty) .
\]

On the other hand, when \( \alpha > 2/[N - 2]^+ \), we have that
\[
E_2(t) \leq \{ E_2(0)^{-\omega_1} - \omega_1 \int_0^t c_2 E(s)^{\alpha(1 - \theta_2)/2} ds \}^{-1/\omega_1} \leq \{ E_2(0)^{-\omega_1} - \omega_1 c_5 E(0)^{\tilde{\omega}_2} \}^{-1/\omega_1} \quad (< +\infty) \]
if \( \omega_1 c_5 E(0)^{\tilde{\omega}_2} E_2(0)^{\omega_1} < 1 \).

When \( \alpha \leq 4/(N - 2) \) (i.e. \( \theta_1 = 0 \)), we have from (5.8) and (5.13) that
\[
(5.37) \quad B(t) = c_1 E(0)^{\alpha/2} < 1 .
\]

On the other hand, when \( \alpha > 4/[N - 2]^+ \) (i.e. \( \theta_1 > 0 \)), we have (5.13) and (5.36) that
\[
B(t) \leq c_1 E(0)^{1 - (N - 4)/4} E_2(t)^{(N - 2)/4 - 1} \leq c_1 E(0)^{1 - (N - 4)/4} \{ E_2(0)^{-\omega_1} - \omega_1 c_5 E(0)^{\tilde{\omega}_2} \}^{-(N - 2)/4 \omega_1} < 1 .
\]
if we assume (5.9), that is,
\[
\{ \omega_1 c_5 E(0)^{\tilde{\omega}_2} + c_4 E(0)^{\omega_1} \} E_2(0)^{\omega_1} < 1 .
\]

Thus we conclude that (5.37) and (5.38) contradict (5.14), and hence, we see \( T_2 \geq T_1 \).
Moreover, we observe from (5.11) and (5.15) that
\[
0 = K(u(T_1)) \geq (1/2)\| A^{1/2} u(T_1) \|^2 > 0 ,
\]
which is a contradiction, and hence, we see \( T_1 = +\infty \), that is, (5.32), (5.35), and (5.36) hold true for all \( t \geq 0 \). The proof of Theorem 9 is now completed. \( \square \)

References

Blowup Phenomena for Nonlinear Dissipative Wave Equations