Vol. 31 (1997), 29–61

Theory of Fourier Microfunctions of Several Types (III)

By

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(Received September 12, 1997)

Abstract

In this paper, we define the concept of partial, partial-modified and partial-mixed Fourier microfunctions and investigate their structures. Thereby we obtain the decomposition of singularity of partial, partial-modified and partial-mixed Fourier hyperfunctions. Then we can deduce the qualitative and quantitative property of partial, partial-modified and partial-mixed Fourier hyperfunctions by examining only their singularity spectrums. We also investigate their vector-valued versions.

1991 Mathematics Subject Classification. Primary 32A45;Secondary 46F15

Introduction

This paper is the third part of the series of papers on the theory of Fourier microfunctions of several types.

In this paper, we define the concept of partial, partial-modified and partial-mixed Fourier microfunctions and their vector-valued versions, and study their fundamental properties. We can investigate these in a similar way to S. K. K. [21] and Ito[4], [4 bis], [5].

Sheaf homomorphisms $\mathscr{A}^{\flat} \longrightarrow \mathscr{B}^{\flat}$, $\mathscr{A}^{\natural} \longrightarrow \mathscr{B}^{\natural}$ and $\mathscr{A}^{*} \longrightarrow \mathscr{B}^{*}$ are defined and they become injections. Thereby, the concept of partial, partial-modified and partial-mixed Fourier hyperfunctions can be considered as a generalization of the concept of slowly increasing and real-analytic functions. One purpose of this paper is to analyse the structure of the quotient sheaves $\mathscr{B}^{\flat}/\mathscr{A}^{\flat}$, $\mathscr{B}^{\natural}/\mathscr{A}^{\natural}$ and $\mathscr{B}^{*}/\mathscr{A}^{*}$. We can analyse this structure by a similar way to the theory of Sato and Fourier microfunctions. For Sato and Fourier microfunctions, we refer the reader to Kaneko[9], [10], [11], Kashiwara-Kawai-Kimura[12], Morimoto[14], [15], Sato[17], [18], [19], [20], Sato-Kawai-

Kashiwara[21] and Ito[4], [4bis], [5]. The first target is to show that we can define the sheaves $\mathscr{C}^{\flat}$, $\mathscr{C}^{\natural}$ and $\mathscr{C}^{*}$ of partial, partial-modified and partial-mixed Fourier microfunctions over S^*M, respectively, which is the cosphere bundle over M, and we can have the fundamental exact sequences

$$0 \longrightarrow \mathscr{A}^{\flat} \longrightarrow \mathscr{B}^{\flat} \longrightarrow \pi_* \mathscr{C}^{\flat} \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{A}^{\natural} \longrightarrow \mathscr{B}^{\natural} \longrightarrow \pi_* \mathscr{C}^{\natural} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathscr{A}^{*} \longrightarrow \mathscr{B}^{*} \longrightarrow \pi_* \mathscr{C}^{*} \longrightarrow 0,$$

where $\pi : S^*M \longrightarrow M$ is the projection and $\pi_* \mathscr{C}^{\flat}$, $\pi_* \mathscr{C}^{\natural}$ and $\pi_* \mathscr{C}^{*}$ denote the direct images of $\mathscr{C}^{\flat}$, $\mathscr{C}^{\natural}$ and $\mathscr{C}^{*}$ with respect to π, respectively.

Further we investigate more precise structures of partial, partial-modified and partial-mixed Fourier microfunctions. Until now, the flabbiness of the sheaves $\mathscr{C}^{\flat}$, $\mathscr{C}^{\natural}$ and $\mathscr{C}^{*}$ is not yet known.

Next, we consider a similar construction of the theory of vector-valued versions.

At last we note that partial, partial-modified and partial-mixed Fourier microfunctions and their vector-valued versions on an open set in $S^*\boldsymbol{R}^{|n|}$ are nothing else but Sato microfunctions and vector-valued Sato microfunctions, respectively, where $S^*\boldsymbol{R}^{|n|}$ is the cosphere bundle over $\boldsymbol{R}^{|n|}$.

In section 8, we construct the theory of partial Fourier microfunctions.

In section 9, we construct the theory of vector-valued and partial Fourier microfunctions.

In section 10, we construct the theory of partial and modified Fourier microfunctions.

In section 11, we construct the theory of vector-valued, partial and modi fied Fourier microfunctions.

In section 12, we construct the theory of partial and mixed Fourier microfunctions.

In section 13, we construct the theory of vector-valued, partial and mixed Fourier microfunctions.

8. Partial Fourier microfunctions

8.1. Partial Fourier hyperfunctions. In this subsection we recall the notion of partial Fourier hyperfunctions following Ito[2].

Let $n = (n_1, n_2)$ be a pair of nonnegative integers with $|n| = n_1 + n_2 \neq 0$.

We denote the product spaces $\boldsymbol{C}^{n_1} \times \widetilde{\boldsymbol{C}}^{n_2}$ and $\boldsymbol{R}^{n_1} \times \widetilde{\boldsymbol{R}}^{n_2}$ by $\boldsymbol{C}^{\flat \cdot n}$ and $\boldsymbol{R}^{\flat \cdot n}$ respectively. Also put $\boldsymbol{C}^{|n|} = \boldsymbol{C}^{n_1} \times \boldsymbol{C}^{n_2}$, $X = \boldsymbol{C}^{\flat \cdot n}$ and $M = \boldsymbol{R}^{\flat \cdot n}$. Then M is the closure of $\boldsymbol{R}^{|n|} = \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$ in X. We denote $z = (z', z'') \in \boldsymbol{C}^{|n|}$ so that $z' = (z_1, \cdots, z_{n_1})$, $z'' = (z_{n_1+1}, \cdots, z_{|n|})$.

Let $\mathcal{O}^{\flat}$ be the sheaf of slowly increasing and holomorphic functions on X following Ito [2], Definition 7. 1. 1, p. 43. Put $\mathscr{A}^{\flat}=\mathcal{O}^{\flat}|_{M}$. Then $\mathscr{A}^{\flat}$ is the sheaf of slowly increasing and real-analytic functions on M. Then we have $\mathscr{A}^{\flat}=\iota^{-1}\mathcal{O}^{\flat}$, where $\iota : M \hookrightarrow X$ is the canonical injection.

As in Ito [2], we define the sheaf of partial Fourier hyperfunctions on M :

Definition 8.1. The sheaf $\mathscr{B}^{\flat}$ is, by definition,

$$\mathscr{B}^{\flat}=\mathscr{H}^{|n|}_{M}(\mathcal{O}^{\flat})=\mathrm{Dist}^{|n|}(M,\mathcal{O}^{\flat}),$$

where the notation in the right hand side of the above equality is due to Sato [16], p. 405. A section of $\mathscr{B}^{\flat}$ is called a partial Fourier hyperfunction.

As stated in Ito [2], $\mathscr{H}^{k}_{M}(\mathcal{O}^{\flat})=0$ for $k \neq |n|$ and $\mathscr{B}^{\flat}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1. 1 of Ito [4] to this case where $\mathscr{F}$, X and Y correspond to $\mathcal{O}^{\flat}$, X and M respectively. Then we obtain the sheaf homomorphism

$$\mathscr{A}^{\flat}\longrightarrow\mathscr{B}^{\flat}$$

which will be proved to be injective later. This injection allows us to consider partial Fourier hyperfunctions as a generalization of slowly increasing and real-analytic functions. The one purpose of this section is to analyse the structure of the quotient sheaf $\mathscr{B}^{\flat}/\mathscr{A}^{\flat}$ by a similar way to S.K.K. [21].

8.2. Definition of partial Fourier microfunctions. Suppose that $M=\boldsymbol{R}^{\flat\cdot n}$ and $X=\boldsymbol{C}^{\flat\cdot n}$. We denote by $\mathcal{O}^{\flat}$ the sheaf of partially slowly increasing and holomorphic functions defined on X. The (co-)sphere bundle iSM (resp. $iS^{*}M$) are defined similarly to subsection 2. 2 of Ito [4]. We also use a similar notation to subsection 2. 2 of Ito [4]. Then we have the following diagram :

$$\begin{array}{ccccccccc}
 & & & {}^{M}\widetilde{X}^{+} & \hookleftarrow & DM & & & \\
 & \pi & \swarrow & \tau & \times & \pi & \searrow & \tau & \\
{}^{M}\widetilde{X} & \hookleftarrow & iSM & & & & {}^{M}\widetilde{X}^{*} & \hookleftarrow & iS^{*}M \\
 & \tau & \searrow & \tau & \times & \pi & \swarrow & \pi & \\
 & & & X & \hookleftarrow & M & & &
\end{array}$$

Theorem 8.2. *We have* $\mathscr{H}^{k}_{iSM}(\tau^{-1}\mathcal{O}^{\flat})=0$ *for* $k\neq 1$, *where* $\tau : {}^{M}\widetilde{X}\longrightarrow X$ *is the canonical projection.*

The following theorem is the most essential one in the theory of partial Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 8.3. *We have* $\mathscr{H}^k_{iS^*M}(\pi^{-1}\mathscr{O}^b)=0$ *for* $k\neq|n|$, *where* $\pi: {}^M\widetilde{X}^*\longrightarrow X$ *is the canonical projection.*

In the proof of the above theorem, the following theorem is essential.

Theorem 8.4(the "Edge of the Wedge" Theorem). *Put* $G=\{z=x+iy\in \boldsymbol{C}^{|n|};\ y_j\geqq 0\ (1\leqq j\leqq|n|)\}^{cl}$. *Then we have, for each* $x\in M$,

$$\mathscr{H}^k_G(\mathscr{O}^b)_x=0 \quad \textit{for } k\neq|n|.$$

Definition 8.5. We define the sheaf $\mathscr{C}^b$ on iS^*M by the relation

$$\mathscr{C}^b=\mathscr{H}^{|n|}_{iS^*M}(\pi^{-1}\mathscr{O}^b)^a,$$

where we denote by a the antipodal map $iS^*M\longrightarrow iS^*M$, and by $\mathscr{F}^a$ the inverse image under a of a sheaf $\mathscr{F}$ on iS^*M. A section of $\mathscr{C}^b$ is called a partial Fourier microfunction.

Now we define the sheaves $\mathscr{Q}^b$, $\mathscr{O}^{b,\beta}$ and $\mathscr{A}^{b,\beta}$ by

$$\mathscr{Q}^b=\mathscr{H}^1_{iSM}(\tau^{-1}\mathscr{O}^b),$$
$$\mathscr{O}^{b,\beta}=j_*(\mathscr{O}^b|_{X-M}),$$
$$\mathscr{A}^{b,\beta}=\mathscr{O}^{b,\beta}|_{iSM},$$

where $j:X-M\hookrightarrow {}^M\widetilde{X}$, $\pi: {}^M\widetilde{X}^*\longrightarrow X$ and $\tau: {}^M\widetilde{X}\longrightarrow X$ are canonical maps.

By Proposition 1.3 of Ito[4] and Theorems 8.2 and 8.3, we have the following.

Proposition 8.6. *We have*

$$R^k\tau_*\pi^{-1}\mathscr{Q}^b=\begin{cases}(\mathscr{C}^b)^a, & (k=|n|-1),\\ 0, & (k\neq|n|-1).\end{cases}$$

Theorem 8.7. *We have*

$$R^k\pi_*\mathscr{C}^b=R^{k+|n|-1}\tau_*\mathscr{Q}^b=0 \quad \textit{for } k\neq 0,$$

and we have the exact sequence

$$0\longrightarrow\mathscr{A}^b\longrightarrow\mathscr{B}^b\longrightarrow\pi_*\mathscr{C}^b\longrightarrow 0.$$

This is the required decomposition of singularity of partial Fourier hyperfunctions.

Corollary 8.8. *We have the exact sequence*

$$0\longrightarrow\mathscr{A}^b(M)\xrightarrow{\lambda}\mathscr{B}^b(M)\xrightarrow{\mathrm{sp}}\mathscr{C}^b(iS^*M)\longrightarrow 0.$$

Definition 8.9. Let $u \in \mathscr{B}^b(M)$. We call $\mathrm{sp}(u) \in \mathscr{C}^b(iS^*M)$ a spectrum of u. We denote by S.S.u the support supp sp(u) of sp(u) and call it a singularity spectrum of u. π (S.S.u) is evidently the subset where u is not partially slowly increasing nor real-analytic and is called the singular support of u.

Corollary 8.10. Let $u \in \mathscr{B}^b(M)$. *Then u is a partially slowly increasing and real-analytic function on M if and only if* S.S.$u = \emptyset$.

Since $\mathscr{A} = \mathscr{A}^b|_{R^{|n|}}$, $\mathscr{B} = \mathscr{B}^b|_{R^{|n|}}$ and $\mathscr{C} = \mathscr{C}^b|_{iS^*R^{|n|}}$ hold in the notation of S.K.K. [21], we have the following Corollary by restricting the exact sequence in Theorem 8.7.

Corollary 8.11. *Let $\pi : iS^*\boldsymbol{R}^{|n|} \longrightarrow \boldsymbol{R}^{|n|}$. Then we have the exact sequence*

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_* \mathscr{C} \longrightarrow 0 .$$

8.3. Fundamental diagram on $\mathscr{C}^b$. We apply the arguments in the subsection 1.2 of Ito [4] to a special case. At first we apply Proposition 1.10 of Ito [4] to the situation $\mathscr{F} = (\mathscr{Q}^b)^a$, $X = M$, $S = iSM$. Then we have $\mathscr{G} = \mathscr{C}^b$ and $\mathscr{E} = \pi_* \mathscr{C}^b$. We obtain the following.

Proposition 8.12. *We have*

$$R^k \pi_* \tau^{-1} \mathscr{C}^b = 0 \quad \textit{for } k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow \mathscr{Q}^b \longrightarrow \tau^{-1} \pi_* \mathscr{C}^b \longrightarrow \pi_* \tau^{-1} \mathscr{C}^b \longrightarrow 0 .$$

Now we apply the same proposition to the case where $\mathscr{F} = \mathscr{A}^{b,\beta}$. Thus we obtain a homomorphism

(8.1)
$$\mathscr{A}^{b,\beta} \longrightarrow \tau^{-1} R^{|n|-1} \tau_* \mathscr{A}^{b,\beta},$$
$$\mathscr{A}^{b,\beta} = \boldsymbol{R} j_* (\mathscr{O}^b|_{X-M})|_{iSM},$$

where $j : X - M \hookrightarrow {}^M\widetilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1} \tau_* \mathscr{A}^{b,\beta} = R^{|n|-1} (\tau \circ j)_* (\mathscr{O}^b|_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1} \tau_* \mathscr{A}^{b,\beta} \longrightarrow \mathscr{B}^b .$$

It yields, together with (8.1), a homomorphism $\mathscr{A}^{b,\beta} \longrightarrow \tau^{-1} \mathscr{B}^b$. Summing up, we have obtained the following.

Theorem 8.13. *We have the following diagram of exact sequences of sheaves on iSM:*

$$
(8.2)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{-1}\mathscr{A}^{b} & \longrightarrow & \mathscr{A}^{b,\beta} & \longrightarrow & \mathscr{Q}^{b} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \\
0 & \longrightarrow & \tau^{-1}\mathscr{A}^{b} & \longrightarrow & \tau^{-1}\mathscr{B}^{b} & \longrightarrow & \tau^{-1}\pi_{*}\mathscr{C}^{b} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \pi_{*}\tau^{-1}\mathscr{C}^{b} & = & \pi_{*}\tau^{-1}\mathscr{C}^{b} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}
$$

Let us transform the diagram (8.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \longrightarrow iS^*M$ and $IM \longrightarrow iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1.7 of Ito[4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}\mathscr{C}^b \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}\mathscr{C}^b \cong \mathscr{C}^b[1-|n|].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (8.2), we obtain

$$\boldsymbol{R}^k\tau'_!\pi'^{-1}\mathscr{Q}^b = 0 \quad \text{for } k \neq |n|-1,$$
$$\boldsymbol{R}^k\tau'_!\pi'^{-1}\mathscr{A}^{b,\beta} = 0 \quad \text{for } k \neq |n|-1.$$

We define the sheaves $\mathscr{A}^{b,\vee}$ and $\mathscr{Q}^{b,\vee}$ on iS^*M by

$$\mathscr{A}^{b,\vee} = \boldsymbol{R}^{|n|-1}\tau'_!\pi'^{-1}\mathscr{A}^{b,\beta},$$
$$\mathscr{Q}^{b,\vee} = \boldsymbol{R}^{|n|-1}\tau'_!\pi'^{-1}\mathscr{Q}^{b}.$$

Then, in this way, we obtain the following theorem.

Theorem 8.14. *We have the diagram of exact sequences of sheaves on iS^*M:*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}\mathscr{A}^{b} \arrow[r] \arrow[d, equal] & \mathscr{A}^{b,\vee} \arrow[r] \arrow[d] & \mathscr{Q}^{b,\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}\mathscr{A}^{b} \arrow[r] & \pi^{-1}\mathscr{B}^{b} \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_{*}\mathscr{C}^{b} \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{C}^{b} \arrow[r, equal] \arrow[d] & \mathscr{C}^{b} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{8.3}
$$

and the diagram (8. 2) *and the diagram* (8. 3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_{!}\pi'^{-1}[|n|-1]$ *and* $\boldsymbol{R}\pi_{*}\tau^{-1}$.

We give a direct application of Theorem 8. 13, which gives a relation between singularity spectrum and the domain of the defining function of a partial Fourier hyperfunction.

By using similar notions as in subsection 2. 3 of Ito [4], we can state the following proposition.

Proposition 8.15. *Let* U *be an open subset of* iSM *with convex fiber, and* V *a convex hull of* U. *Then we have*

1) *If* $\varphi\in\Gamma(U,\mathscr{A}^{b,\beta})$, *then* $\mathrm{S.S.}(\lambda(\varphi))\subset U^{\circ}$. *Conversely, if* $f(x)\in\Gamma(\tau U,\mathscr{B}^{b})$ *satisfies* $\mathrm{S.S.}(f)\subset U^{\circ}$, *then there exists a unique* $\varphi\in\Gamma(U,\mathscr{A}^{b,\beta})$ *such that* $f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$0\longrightarrow\mathscr{A}^{b,\beta}(U)\xrightarrow{\ \lambda\ }\mathscr{B}^{b}(\tau U)\xrightarrow{\ \mathrm{sp}\ }\mathscr{C}^{b}(iS^{*}M-U^{\circ}).$$

2) $\Gamma(V,\mathscr{A}^{b,\beta})\longrightarrow\Gamma(U,\mathscr{A}^{b,\beta})$ *is an isomorphism.*

Definition 8.16. We say $u\in\mathscr{B}^{b}(\Omega)$ to be micro-analytic at $(x, i\eta\infty)$ in $iS^{*}M$ if $(x, i\eta\infty)\notin\mathrm{S.S.}u$. This is equivalent to being represented as

$$u=\sum_{j}\lambda(\varphi_{j}),\ \varphi_{j}\in\mathscr{A}^{b,\beta}(U_{j}),\ (x, i\eta\infty)\notin U_{j}^{\circ}.$$

9. Vector-valued and partial Fourier microfunctions

9.1. Vector-valued and partial Fourier hyperfunctions. In this subsection we recall the notion of vector-valued and partial Fourier hyperfunctions following Ito[2].

We use a similar notation to subsection 8. 1. Let E be a Fréchet space over the complex number field.

Let ${}^{E}\mathcal{O}^{b}$ be the sheaf of E-valued, partially slowly increasing, and holomorphic functions on X following Ito[2], Definition 8.1.1, p. 55, and put ${}^{E}\mathcal{A}^{b} = {}^{E}\mathcal{O}^{b}|_{M}$. Then ${}^{E}\mathcal{A}^{b}$ is the sheaf of E-valued, partially slowly increasing, and real-analytic functions on M. Then we have ${}^{E}\mathcal{A}^{b} = \iota^{-1}{}^{E}\mathcal{O}^{b}$, where $\iota: M \hookrightarrow X$ is the canonical injection.

As in Ito[2], we define the sheaf of E-valued and partial Fourier hyperfunctions on M :

Definition 9.1. The sheaf ${}^{E}\mathcal{B}^{b}$ is, by definition,

$$ {}^{E}\mathcal{B}^{b} = \mathcal{H}_{M}^{|n|}({}^{E}\mathcal{O}^{b}) = \mathrm{Dist}^{|n|}(M, {}^{E}\mathcal{O}^{b}). $$

A section of ${}^{E}\mathcal{B}^{b}$ is called an E-valued and partial Fourier hyperfunction.

As stated in Ito[2], $\mathcal{H}_{M}^{k}({}^{E}\mathcal{O}^{b}) = 0$ for $k \neq |n|$ and ${}^{E}\mathcal{B}^{b}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito[4] to this case where $\mathcal{F}$, X and Y correspond to ${}^{E}\mathcal{O}^{b}$, X and M respectively. Then we obtain the sheaf homomorphism

$$ {}^{E}\mathcal{A}^{b} \longrightarrow {}^{E}\mathcal{B}^{b}, $$

which will be proved to be injective later. This injection allows us to consider E-valued and partial Fourier hyperfunctions as a generalization of E-valued, partially slowly increasing, and real-analytic functions. The one purpose of this chapter is to analyse the structure of the quotient sheaf ${}^{E}\mathcal{B}^{b}/{}^{E}\mathcal{A}^{b}$ by a similar way to S.K.K. [21].

9.2. Definition of vector-valued and partial Fourier microfunctions. We use a similar notation to subsection 8.2. Let E be a Fréchet space over the complex number field. We denote by ${}^{E}\mathcal{O}^{b}$ the sheaf of E-valued, partially slowly increasing, and holomorphic functions defined on X. We have the following.

Theorem 9.2. *We have $\mathcal{H}_{iSM}^{k}(\tau^{-1}{}^{E}\mathcal{O}^{b}) = 0$ for $k \neq 1$, where $\tau : {}^{M}\widetilde{X} \longrightarrow X$ is the canonical projection.*

The following theorem is the most essential one in the theory of E-valued and partial Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 9.3. *We have $\mathcal{H}_{iS^{*}M}^{k}(\pi^{-1}{}^{E}\mathcal{O}^{b}) = 0$ for $k \neq |n|$, where $\pi : {}^{M}\widetilde{X}^{*} \longrightarrow X$ is the canonical projection.*

In the proof of the above theorem, the following theorem is essential.

Theorem 9.4 (the "Edge of the Wedge" Theorem). *Put $G = \{z = x + iy \in \boldsymbol{C}^{|n|};\ y_j \geq 0 (1 \leq j \leq |n|)\}^{\mathrm{cl}}$. Then we have, for each $x \in M$,*

$$\mathscr{H}^k_G({}^E\mathcal{O}^b)_x = 0 \ for \ k \neq |n|.$$

Definition 9.5. We define the sheaf ${}^E\mathscr{C}^b$ in iS^*M by the relation

$$^E\mathscr{C}^b = \mathscr{H}^{|n|}_{iS^*M}(\pi^{-1}\,{}^E\mathcal{O}^b)^a.$$

A section of ${}^E\mathscr{C}^b$ is called an E-valued and partial Fourier microfunctions.

Now we define the sheaves ${}^E\mathscr{Q}^b$, ${}^E\mathcal{O}^{b,\beta}$, ${}^E\mathscr{A}^{b,\beta}$ by the relations

$$\begin{aligned} &{}^E\mathscr{Q}^b = \mathscr{H}^1_{iSM}(\tau^{-1}\,{}^E\mathcal{O}^b), \\ &{}^E\mathcal{O}^{b,\beta} = j_*({}^E\mathcal{O}^b|_{X-M}), \\ &{}^E\mathscr{A}^{b,\beta} = {}^E\mathcal{O}^{b,\beta}|_{iSM}, \end{aligned}$$

where $j: X-M \hookrightarrow {}^M\widetilde{X}$, $\pi: {}^M\widetilde{X}^* \longrightarrow X$ and $\tau: {}^M\widetilde{X} \longrightarrow X$ are canonical maps.

By Proposition 1.3 of Ito [4] and Theorems 9.2 and 9.3, we have the following.

Proposition 9.6. *We have*

$$R^k\tau_*\pi^{-1}\,{}^E\mathscr{Q}^b = \begin{cases} ({}^E\mathscr{C}^b)^a, & (k=|n|-1), \\ 0, & (k\neq|n|-1). \end{cases}$$

Theorem 9.7. *We have*

$$R^k\pi_*\,{}^E\mathscr{C}^b = R^{k+|n|-1}\tau_*\,{}^E\mathscr{Q}^b = 0 \quad for \ k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow {}^E\mathscr{A}^b \longrightarrow {}^E\mathscr{B}^b \longrightarrow \pi_*\,{}^E\mathscr{C}^b \longrightarrow 0.$$

This is the required decomposition of singularity of E-valued and partial Fourier hyperfunctions.

Corollary 9.8. *We have the exact sequence*

$$0 \longrightarrow \mathscr{A}^b(M;E) \xrightarrow{\lambda} \mathscr{B}^b(M;E) \xrightarrow{\mathrm{sp}} \mathscr{C}^b(iS^*M;E) \longrightarrow 0.$$

Definition 9.9. Let $u \in \mathscr{B}^b(M;E)$. We call $\mathrm{sp}(u) \in \mathscr{C}^b(iS^*M;E)$ a spectrum of u. We denote by S.S.u the support supp sp(u) of sp(u) and call it a singularity spectrum of u. π (S.S.u) is evidently the subset where u is not slowly increasing nor real-analytic, and it is called the singular support of u.

Corollary 9.10. *Let* $u \in \mathscr{B}^b(M;E)$. *Then u is an E-valued, partially slowly increasing, and real-analytic function on M if and only if* S.S.$u = \phi$.

Put ${}^E\mathscr{A} = {}^E\mathscr{A}^b|_{R^{|n|}}$, ${}^E\mathscr{A} = {}^E\mathscr{A}^b|_{R^{|n|}}$ and ${}^E\mathscr{C} = {}^E\mathscr{C}^b|_{iS^*R^{|n|}}$. Then we have the following Corollary by restricting the exact sequence in Theorem 9.7.

Corollary 9.11. *Let* $\pi: iS^*\boldsymbol{R}^{|n|} \longrightarrow \boldsymbol{R}^{|n|}$ *be the canonical projection. Then we have the exact sequence*

$$0 \longrightarrow {}^E\mathscr{A} \longrightarrow {}^E\mathscr{B} \longrightarrow \pi_* {}^E\mathscr{C} \longrightarrow 0.$$

9.3. Fundamental diagram on ${}^E\mathscr{C}^b$. We apply the arguments in the subsection 1.2 of Ito[4] to a special case. At first we apply Proposition 1.10 of Ito[4] to the situation $\mathscr{F}=({}^E\mathscr{Q}^b)^a$, $X=M$, $S=iSM$. Then we have $\mathscr{G}={}^E\mathscr{C}^b$ and $\mathscr{E}=\pi_*{}^E\mathscr{C}^b$. We obtain the following.

Proposition 9.12. *We have*

$$R^k\pi_*\tau^{-1}{}^E\mathscr{C}^b=0 \text{ for } k\neq 0$$

and we have the exact sequence

$$0 \longrightarrow {}^E\mathscr{Q}^b \longrightarrow \tau^{-1}\pi_*{}^E\mathscr{C}^b \longrightarrow \pi_*\tau^{-1}{}^E\mathscr{C}^b \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F}={}^E\mathscr{A}^{b,\beta}$. Thus we obtain a homomorphism

$$(9.1) \qquad \begin{gathered} {}^E\mathscr{A}^{b,\beta} \longrightarrow \tau^{-1}R^{|n|-1}\tau_*{}^E\mathscr{A}^{b,\beta}, \\ {}^E\mathscr{A}^{b,\beta}=\boldsymbol{R}j_*({}^E\mathscr{O}^b|_{X-M})|_{iSM}, \end{gathered}$$

where $j:X-M \hookrightarrow {}^M\widetilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1}\tau_*{}^E\mathscr{A}^{b,\beta}=R^{|n|-1}(\tau\circ j)_*({}^E\mathscr{O}^b|_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1}\tau_*{}^E\mathscr{A}^{b,\beta} \longrightarrow {}^E\mathscr{B}^b.$$

It yields, together with (9.1), a homomorphism ${}^E\mathscr{A}^{b,\beta} \longrightarrow \tau^{-1}{}^E\mathscr{B}^b$. Summing up, we have obtained the following.

Theorem 9.13. *We have the following diagram of exact sequences of sheaves on iSM :*

$$(9.2) \qquad \begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \tau^{-1}{}^E\mathscr{A}^b \arrow[r] \arrow[d, equal] & {}^E\mathscr{A}^{b,\beta} \arrow[r] \arrow[d] & {}^E\mathscr{Q}^b \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tau^{-1}{}^E\mathscr{A}^b \arrow[r] & \tau^{-1}{}^E\mathscr{B}^b \arrow[r] \arrow[d] & \tau^{-1}\pi_*{}^E\mathscr{C}^b \arrow[r] \arrow[d] & 0 \\ & & \pi_*\tau^{-1}{}^E\mathscr{C}^b \arrow[r, equal] \arrow[d] & \pi_*\tau^{-1}{}^E\mathscr{C}^b \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

.

Let us transform the diagram (9. 2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \longrightarrow iS^*M$ and $IM \longrightarrow iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1. 7 of Ito[4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}{}^E\mathscr{C}^b \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}{}^E\mathscr{C}^b \cong {}^E\mathscr{C}^b[1-|n|].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (9. 2), we obtain

$$\boldsymbol{R}^k\tau'_!\pi'^{-1}{}^E\mathscr{Q}^b = 0 \quad \text{for } k \neq |n|-1,$$
$$\boldsymbol{R}^k\tau'_!\pi'^{-1}{}^E\mathscr{A}^{b,\beta} = 0 \quad \text{for } k \neq |n|-1.$$

We define the sheaves ${}^E\mathscr{A}^{b,\vee}$ and ${}^E\mathscr{Q}^{b,\vee}$ on iS^*M by the relations

$${}^E\mathscr{A}^{b,\vee} = R^{|n|-1}\tau'_!\pi'^{-1}{}^E\mathscr{A}^{b,\beta},$$
$${}^E\mathscr{Q}^{b,\vee} = R^{|n|-1}\tau'_!\pi'^{-1}{}^E\mathscr{Q}^{b,\beta}.$$

Then, in this way, we obtain the following theorem.

Theorem 9.14. *We have the diagram of exact sequences of sheaves on iS^*M:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}{}^E\mathscr{A}^b \arrow[r] \arrow[d, equal] & {}^E\mathscr{A}^{b,\vee} \arrow[r] \arrow[d] & {}^E\mathscr{Q}^{b,\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}{}^E\mathscr{A}^b \arrow[r] & \pi^{-1}{}^E\mathscr{B}^b \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_*{}^E\mathscr{C}^b \arrow[r] \arrow[d] & 0 \\
 & & {}^E\mathscr{C}^b \arrow[r, equal] \arrow[d] & {}^E\mathscr{C}^b \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{9. 3}$$

and the diagram (9. 2) *and the diagram* (9. 3) *are mutually transformed by the functors $\boldsymbol{R}\tau'_!\pi'^{-1}[|n|-1]$ and $\boldsymbol{R}\pi_*\tau^{-1}$.*

We give a direct application of Theorem 9. 13, which gives a relation between singularity spectrum and the domain of the defining function of an E-valued and partial Fourier hyperfunction.

Proposition 9.15. *Let U be an open subset of iSM with convex fiber, and*

V a convex hull of U. Then we have

1) *If* $\varphi\in\Gamma(U, {}^E\mathscr{A}^{\flat,\beta})$, *then* S.S.$(\lambda(\varphi))\subset U^\circ$. *Conversely, if* $f(x)\in\Gamma(\tau U, {}^E\mathscr{B}^\flat)$ *satisfies* S.S.$(f)\subset U^\circ$, *then there exists a unique* $\varphi\in\Gamma(U, {}^E\mathscr{A}^{\flat,\beta})$ *such that* $f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$0\longrightarrow\mathscr{A}^{\flat,\beta}(U;E)\xrightarrow{\lambda}\mathscr{B}^\flat(\tau U;E)\xrightarrow{\mathrm{sp}}\mathscr{C}^\flat(iS^*M-U^\circ;E).$$

2) $\Gamma(V, {}^E\mathscr{A}^{\flat,\beta})\longrightarrow\Gamma(U, {}^E\mathscr{A}^{\flat,\beta})$ *is an isomorphism.*

Definition 9.16. We say $u\in\mathscr{B}^\flat(\Omega;E)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty)\notin$S.S.u. This is equivalent to being represented as

$$u=\sum_j\lambda(\varphi_j),\ \varphi_j\in\mathscr{A}^{\flat,\beta}(U_j;E),\ (x, i\eta\infty)\notin U_j^\circ.$$

10. Partial and modified Fourier microfunctions

10.1. Partial and modified Fourier hyperfunctions. In this subsection we recall the notion of partial and modified Fourier hyperfunctions following Ito [2].

Let $n=(n_1, n_2)$ be a pair of nonnegative integers with $|n|=n_1+n_2\neq 0$. We denote the product spaces $\boldsymbol{C}^{n_1}\times\widetilde{\widetilde{\boldsymbol{C}}}^{n_2}$ and $\boldsymbol{R}^{n_1}\times\widetilde{\widetilde{\boldsymbol{R}}}^{n_2}$ by $\boldsymbol{C}^{\natural,n}$ and $\boldsymbol{R}^{\natural,n}$ respectively. Also put $\boldsymbol{C}^{|n|}=\boldsymbol{C}^{n_1}\times\boldsymbol{C}^{n_2}$, $X=\boldsymbol{C}^{\natural,n}$ and $M=\boldsymbol{R}^{\natural,n}$. Then M is the closure of $\boldsymbol{R}^{|n|}=\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$ in X. We denote $z=(z', z'')\in\boldsymbol{C}^{|n|}$ so that $z'=(z_1, \cdots, z_{n_1})$, $z''=(z_{n_1+1}, \cdots, z_{|n|})$.

Let $\mathscr{O}^\natural$ be the sheaf of partially slowly increasing and holomorphic functions on X following Ito [2], Definition 9.1.1. Put $\mathscr{A}^\natural=\mathscr{O}^\natural|_M$. Then $\mathscr{A}^\natural$ is the sheaf of partially slowly increasing and real-analytic functions on M. Then we have $\mathscr{A}^\natural=\iota^{-1}\mathscr{O}^\natural$, where $\iota: M\hookrightarrow X$ is the canonical injection.

As in Ito [2], we define the sheaf of partial and modified Fourier hyperfunctions on M:

Definition 10.1. The sheaf $\mathscr{B}^\natural$ is, by definition,

$$\mathscr{B}^\natural=\mathscr{H}_M^{|n|}(\mathscr{O}^\natural)=\mathrm{Dist}^{|n|}(M, \mathscr{O}^\natural),$$

where the notation in the right hand side of the above equality is due to Sato [16], p. 405. A section of $\mathscr{B}^\natural$ is called a partial and modified Fourier hyperfunction.

As stated in Ito [2], $\mathscr{H}_M^k(\mathscr{O}^\natural)=0$ for $k\neq|n|$ and $\mathscr{B}^\natural$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito [4] to this case where $\mathscr{F}$, X and Y correspond to $\mathscr{O}^\natural$, X and M respectively. Then we obtain the sheaf homomorphism

$$\mathscr{A}^\natural\longrightarrow\mathscr{B}^\natural,$$

which will be proved to be injective later. This injection allows us to consider partial and modified Fourier hyperfunctions as a generalization of partially slowly increasing and real-analytic functions. The one purpose of this section is to analyse the structure of the quotient sheaf $\mathscr{B}^{\natural}/\mathscr{A}^{\natural}$ by a similar way to S. K. K. [21].

10.2. Definition of partial and modified Fourier microfunctions. Suppose that $M=\boldsymbol{R}^{\natural,n}$ and $X=\boldsymbol{C}^{\natural,n}$. We denote by $\mathscr{O}^{\natural}$ the sheaf of partially slowly increasing and holomorphic functions defined on X. The (co-)sphere bundle iSM(resp. iS^*M) are defined similarly to subsection 2. 2 of Ito[4]. We also use a similar notation to subsection 2. 2 of Ito[4]. Then we have the following diagram :

$$\begin{array}{ccccccccc}
 & & & {}^{M}\widetilde{X}^{+} & \hookleftarrow & DM & & & \\
 & \pi & \swarrow & \tau & \times & \pi & \searrow & \tau & \\
{}^{M}\widetilde{X} & \hookleftarrow & iSM & & & & {}^{M}\widetilde{X}^{*} & \hookleftarrow & iS^{*}M \\
 & \tau & \searrow & \tau & \times & \pi & \swarrow & \pi & \\
 & & & X & \hookleftarrow & M & & & .
\end{array}$$

Theorem 10.2. *We have* $\mathscr{H}^{k}_{iSM}(\tau^{-1}\mathscr{O}^{\natural})=0$ *for* $k\neq1$, *where* $\tau:{}^{M}\widetilde{X}\longrightarrow X$ *is the canonical projection.*

The following theorem is the most essential one in the theory of partial and modified Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 10.3. *We have* $\mathscr{H}^{k}_{iS^*M}(\pi^{-1}\mathscr{O}^{\natural})=0$ *for* $k\neq|n|$, *where* $\pi:{}^{M}\widetilde{X}^{*}\longrightarrow X$ *is the canonical projection.*

In the proof of the above theorem, the following theorem is essential.

Theorem 10.4 (the "Edge of the Wedge" Theorem). *Put* $G=\{z=x+iy\in\boldsymbol{C}^{|n|};y_j\geqq0(1\leqq j\leqq|n|)\}^{\mathrm{cl}}$. *Then we have, for each* $x\in M$,

$$\mathscr{H}^{k}_{G}(\mathscr{O}^{\natural})_x=0 \quad \textit{for } k\neq|n|.$$

Definition 10.5. We define the sheaf $\mathscr{C}^{\natural}$ on iS^*M by the relation

$$\mathscr{C}^{\natural}=\mathscr{H}^{|n|}_{iS^*M}(\pi^{-1}\mathscr{O}^{\natural})^{a},$$

where we denote by a the antipodal map $iS^*M\longrightarrow iS^*M$, and by $\mathscr{F}^{a}$ the inverse image under a of a sheaf $\mathscr{F}$ on iS^*M. A section of $\mathscr{C}^{\natural}$ is called a partial and modified Fourier microfunction.

Now we define the sheaves $\mathscr{Q}^{\natural}$, $\mathscr{O}^{\natural,\beta}$ and $\mathscr{A}^{\natural,\beta}$ by the relations

$$\mathscr{D}^{\natural}=\mathscr{H}^{1}_{iSM}(\tau^{-1}\mathcal{O}^{\natural}),$$
$$\mathcal{O}^{\natural,\beta}=j_{*}(\mathcal{O}^{\natural}|_{X-M}),$$
$$\mathscr{A}^{\natural,\beta}=\mathcal{O}^{\natural,\beta}|_{iSM},$$

where $j:X-M\hookrightarrow {}^{M}\widetilde{X}$, $\pi:{}^{M}\widetilde{X}^{*}\longrightarrow X$ and $\tau:{}^{M}\widetilde{X}\longrightarrow X$ are canonical maps.

By Proposition 1.3 of Ito[4] and Theorems 10.2 and 10.3, we have the following.

Proposition 10.6. *We have*

$$R^{k}\tau_{*}\pi^{-1}\mathscr{D}^{\natural}=\begin{cases}(\mathscr{C}^{\natural})^{a}, & (k=|n|-1),\\ 0, & (k\neq|n|-1).\end{cases}$$

Theorem 10.7. *We have*

$$R^{k}\pi_{*}\mathscr{C}^{\natural}=R^{k+|n|-1}\tau_{*}\mathscr{D}^{\natural}=0 \quad \textit{for } k\neq 0,$$

and we have the exact sequence

$$0\longrightarrow\mathscr{A}^{\natural}\longrightarrow\mathscr{B}^{\natural}\longrightarrow\pi_{*}\mathscr{C}^{\natural}\longrightarrow 0.$$

This is the required decomposition of singularity of partial and modified Fourier hyperfunctions.

Corollary 10.8. *We have the exact sequence*

$$0\longrightarrow\mathscr{A}^{\natural}(M)\xrightarrow{\lambda}\mathscr{B}^{\natural}(M)\xrightarrow{\mathrm{sp}}\mathscr{C}^{\natural}(iS^{*}M)\longrightarrow 0.$$

Definition 10.9. Let $u\in\mathscr{B}^{\natural}(M)$. We call $\mathrm{sp}(u)\in\mathscr{C}^{\natural}(iS^{*}M)$ a spectrum of u. We denote by S.S.u the support supp sp(u) of sp(u) and call it a singular-ity spectrum of u. π (S.S.u) is evidently the subset where u is not partially slowly increasing nor real-analytic and is called the singular support of u.

Corollary 10.10. *Let $u\in\mathscr{B}^{\natural}(M)$. Then u is a partially slowly increasing and real-analytic function on M if and only if* S.S.$u=\varnothing$.

Since $\mathscr{A}=\mathscr{A}^{\natural}|_{\mathbf{R}^{|n|}}$, $\mathscr{B}=\mathscr{B}^{\natural}|_{\mathbf{R}^{|n|}}$ and $\mathscr{C}=\mathscr{C}^{\natural}|_{iS^{*}\mathbf{R}^{|n|}}$ hold in the notation of S. K. K. [21], we have the following Corollary by restricting the exact sequence in Theorem 10. 7.

Corollary 10.11. *Let $\pi:iS^{*}\mathbf{R}^{|n|}\longrightarrow\mathbf{R}^{|n|}$. Then we have the exact sequence*

$$0\longrightarrow\mathscr{A}\longrightarrow\mathscr{B}\longrightarrow\pi_{*}\mathscr{C}\longrightarrow 0.$$

10.3. Fundamental diagram on $\mathscr{C}^{\natural}$. We apply the arguments in the subsection 1. 2 of Ito[4] to a special case. At first we apply Proposition 1. 10 of Ito[4] to the situation $\mathscr{F}=(\mathscr{D}^{\natural})^{a}$, $X=M$, $S=iSM$. Then we have $\mathscr{G}=\mathscr{C}^{\natural}$ and $\mathscr{E}=\pi_{*}\mathscr{C}^{\natural}$. We obtain the following.

Proposition 10.12. *We have*

$$R^k\pi_*\tau^{-1}\mathscr{C}^\natural=0 \quad \text{for } k\neq 0$$

and we have the exact sequence

$$0\longrightarrow\mathscr{Q}^\natural\longrightarrow\tau^{-1}\pi_*\mathscr{C}^\natural\longrightarrow\pi_*\tau^{-1}\mathscr{C}^\natural\longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F}=\mathscr{A}^{\natural,\beta}$. Thus we obtain a homomorphism

$$\text{(10.1)} \qquad \begin{aligned}&\mathscr{A}^{\natural,\beta}\longrightarrow\tau^{-1}R^{|n|-1}\tau_*\mathscr{A}^{\natural,\beta},\\ &\mathscr{A}^{\natural,\beta}=\boldsymbol{R}j_*(\mathscr{O}^\natural|_{X-M})|_{iSM},\end{aligned}$$

where $j:X-M\hookrightarrow {}^M\widetilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1}\tau_*\mathscr{A}^{\natural,\beta}=R^{|n|-1}(\tau\circ j)_*(\mathscr{O}^\natural|_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1}\tau_*\mathscr{A}^{\natural,\beta}\longrightarrow\mathscr{B}^\natural.$$

It yields, together with (10.1), a homomorphism $\mathscr{A}^{\natural,\beta}\longrightarrow\tau^{-1}\mathscr{B}^\natural$. Summing up, we have obtained the following.

Theorem 10.13. *We have the following diagram of exact sequences of sheaves on iSM :*

$$\text{(10.2)}\qquad\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau^{-1}\mathscr{A}^\natural \arrow[r] \arrow[d, equal] & \mathscr{A}^{\natural,\beta} \arrow[r] \arrow[d, "\lambda"'] & \mathscr{Q}^\natural \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau^{-1}\mathscr{A}^\natural \arrow[r] & \tau^{-1}\mathscr{B}^\natural \arrow[r] \arrow[d] & \tau^{-1}\pi_*\mathscr{C}^\natural \arrow[r] \arrow[d] & 0 \\
 & & \pi_*\tau^{-1}\mathscr{C}^\natural \arrow[r, equal] \arrow[d] & \pi_*\tau^{-1}\mathscr{C}^\natural \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Let us transform the diagram (10.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM\longrightarrow iS^*M$ and $IM\longrightarrow iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F}\cong\boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F}\cong\pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1. 7 of Ito[4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}\mathscr{C}^\natural\cong\boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}\mathscr{C}^\natural\cong\mathscr{C}^\natural[1-|n|].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (10. 2), we obtain

$$\boldsymbol{R}^k\tau'_!\pi'^{-1}\mathscr{Q}^\natural=0 \quad \text{for } k\neq|n|-1,$$
$$\boldsymbol{R}^k\tau'_!\pi'^{-1}\mathscr{A}^{\natural,\beta}=0 \quad \text{for } k\neq|n|-1.$$

We define the sheaves $\mathscr{A}^{\natural,\vee}$ and $\mathscr{Q}^{\natural,\vee}$ on iS^*M by the relations

$$\mathscr{A}^{\natural,\vee}=R^{|n|-1}\tau'_!\pi'^{-1}\mathscr{A}^{\natural,\beta},$$
$$\mathscr{Q}^{\natural,\vee}=R^{|n|-1}\tau'_!\pi'^{-1}\mathscr{Q}^\natural.$$

Then, in this way, we obtain the following theorem.

Theorem 10.14. *We have the diagram of exact sequences of sheaves on* iS^*M :

$$\text{(10. 3)}\qquad \begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}\mathscr{A}^\natural \arrow[r] \arrow[d, equal] & \mathscr{A}^{\natural,\vee} \arrow[r] \arrow[d] & \mathscr{Q}^{\natural,\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}\mathscr{A}^\natural \arrow[r] & \pi^{-1}\mathscr{B}^\natural \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_*\mathscr{C}^\natural \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{C}^\natural \arrow[r, equal] \arrow[d] & \mathscr{C}^\natural \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

and the diagram (10. 2) *and the diagram* (10. 3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_!\pi'^{-1}$ $[|n|-1]$ *and* $\boldsymbol{R}\pi_*\tau^{-1}$.

We give a direct application of Theorem 10. 13, which gives a relation between singularity spectrum and the domain of the defining function of a partial and modified Fourier hyperfunction.

By using similar notions as in subsection 2. 3 of Ito[4], we can state the following proposition.

Proposition 10.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

1) *If* $\varphi\in\Gamma(U,\mathscr{A}^{\natural,\beta})$, *then* S.S.$(\lambda(\varphi))\subset U^\circ$. *Conversely, if* $f(x)\in\Gamma(\tau U,\mathscr{B}^\natural)$ *satisfies* S.S.$(f)\subset U^\circ$, *then there exists a unique* $\varphi\in\Gamma(U,\mathscr{A}^{\natural,\beta})$ *such that*

$f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$0 \longrightarrow \mathscr{A}^{\natural,\beta}(U) \xrightarrow{\lambda} \mathscr{B}^{\natural}(\tau U) \xrightarrow{\mathrm{sp}} \mathscr{C}^{\natural}(iS^*M-U^{\circ}).$$

2) $\Gamma(V, \mathscr{A}^{\natural,\beta}) \longrightarrow \Gamma(U, \mathscr{A}^{\natural,\beta})$ *is an isomorphism.*

Definition 10.16. We say $u\in\mathscr{B}^{\natural}(\Omega)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty)\notin$ S.S.u. This is equivalent to being represented as

$$u=\sum_j \lambda(\varphi_j), \varphi_j \in\mathscr{A}^{\natural,\beta}(U_j), (x, i\eta\infty)\notin U_j^{\circ}.$$

11. Vector-valued, partial and modified Fourier microfunctions

11.1. Vector-valued, partial and modified Fourier hyperfunctions. In this subsection we recall the notion of vector-valued, partial and modified Fourier hyperfunctions following Ito[2].

We use a similar notation to section 10.1. Let E be a Fréchet space over the complex number field.

Let ${}^E\mathscr{O}^{\natural}$ be the sheaf of E-valued, partially slowly increasing, and holomorphic functions on X following Ito[2], Definition 10.1.1, and put ${}^E\mathscr{A}^{\natural}={}^E\mathscr{O}^{\natural}|_M$. Then ${}^E\mathscr{A}^{\natural}$ is the sheaf of E-valued, partially slowly increasing, and real-analytic functions on M. Then we have ${}^E\mathscr{A}^{\natural}=\iota^{-1}\,{}^E\mathscr{O}^{\natural}$, where $\iota: M\hookrightarrow X$ is the canonical injection.

As in Ito[2], we define the sheaf of E-valued, partial and modified Fourier hyperfunctions on M:

Definition 11.1. The sheaf ${}^E\mathscr{B}^{\natural}$ is, by definition,

$${}^E\mathscr{B}^{\natural}=\mathscr{H}^{|n|}_M({}^E\mathscr{O}^{\natural})=\mathrm{Dist}^{|n|}(M, {}^E\mathscr{O}^{\natural}).$$

A section of ${}^E\mathscr{O}^{\natural}$ is called an E-valued, partial and modified Fourier hyperfunction.

As stated in Ito[2], $\mathscr{H}^k_M({}^E\mathscr{O}^{\natural})=0$ for $k\neq|n|$ and ${}^E\mathscr{B}^{\natural}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito[4] to this case where $\mathscr{F}$, X and Y correspond to ${}^E\mathscr{B}^{\natural}$, X and M respectively. Then we obtain the sheaf homomorphism

$${}^E\mathscr{A}^{\natural}\longrightarrow {}^E\mathscr{B}^{\natural},$$

which will be proved to be injective later. This injection allows us to consider E-valued, partial and modified Fourier hyperfunctions as a generalization of E-valued, partially slowly increasing, and real-analytic functions. The one purpose of this section is to analyse the structure of the quotient sheaf ${}^E\mathscr{B}^{\natural}/{}^E\mathscr{A}^{\natural}$ by a similar way to S. K. K. [21].

11.2. Definition of vector-valued, partial and modified Fourier microfunctions. We use a similar notation to subsection 10.2. Let E be a Fréchet space over the complex number field. We denote by ${}^{E}\mathscr{O}^{\natural}$ the sheaf of E-valued, partially slowly increasing, and holomorphic functions defined on X. We have the following.

Theorem 11.2. *We have* $\mathscr{H}^{k}_{iSM}(\tau^{-1}{}^{E}\mathscr{O}^{\natural})=0$ *for* $k\neq 1$, *where* $\tau: {}^{M}\widetilde{X}\longrightarrow X$ *is the canonical projection.*

The following theorem is the most essential one in the theory of E-valued, partial and modified Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 11.3. *We have* $\mathscr{H}^{k}_{iS^{*}M}(\pi^{-1}{}^{E}\mathscr{O}^{\natural})=0$ *for* $k\neq|n|$, *where* $\pi: {}^{M}\widetilde{X}^{*}\longrightarrow X$ *is the canonical projection,*

In the proof of the above theorem, the following theorem is essential.

Theorem 11.4 (the "Edge of the Wedge" Theorem). *Put* $G=\{z=x+iy\in \boldsymbol{C}^{|n|};y_{j}\geqq 0\ (1\leqq j\leqq|n|)\}^{c1}$. *Then we have, for each* $x\in M$,

$$\mathscr{H}^{k}_{G}({}^{E}\mathscr{O}^{\natural})_{x}=0 \quad \text{for } k\neq|n|.$$

Definition 11.5. We define the sheaf ${}^{E}\mathscr{C}^{\natural}$ in $iS^{*}M$ by the relation

$${}^{E}\mathscr{C}^{\natural}=\mathscr{H}^{|n|}_{iS^{*}M}(\pi^{-1}{}^{E}\mathscr{O}^{\natural})^{a}.$$

A section of ${}^{E}\mathscr{C}^{\natural}$ is called an E-valued, partial and modified Fourier microfunctions.

Now we define the sheaves ${}^{E}\mathscr{Q}^{\natural}$, ${}^{E}\mathscr{O}^{\natural,\beta}$, ${}^{E}\mathscr{A}^{\natural,\beta}$ by the relations

$$\begin{aligned}
&{}^{E}\mathscr{Q}^{\natural}=\mathscr{H}^{1}_{iSM}(\tau^{-1}{}^{E}\mathscr{O}^{\natural}),\\
&{}^{E}\mathscr{O}^{\natural,\beta}=j_{*}({}^{E}\mathscr{O}^{\natural,\beta}|_{X-M}),\\
&{}^{E}\mathscr{A}^{\natural,\beta}={}^{E}\mathscr{O}^{\natural,\beta}|_{iSM},
\end{aligned}$$

where $j: X-M \hookrightarrow {}^{M}\widetilde{X}$, $\pi: {}^{M}\widetilde{X}^{*}\longrightarrow X$, $\tau: {}^{M}\widetilde{X}\longrightarrow X$ are canonical maps.

By Proposition 1.3 of Ito[4] and Theorems 11.2 and 11.3, we have the following.

Proposition 11.6. *We have*

$$R^{k}\tau_{*}\pi^{-1}{}^{E}\mathscr{Q}^{\natural}=\begin{cases}({}^{E}\mathscr{C}^{\natural})^{a}, & (k=|n|-1),\\ 0, & (k\neq|n|-1).\end{cases}$$

Theorem 11.7. *We have*

$$R^{k}\pi_{*}{}^{E}\mathscr{C}^{\natural}=R^{k+|n|-1}\tau_{*}{}^{E}\mathscr{Q}^{\natural}=0 \text{ for } k\neq 0,$$

and we have the exact sequence

$$0 \longrightarrow {}^{E}\mathscr{A}^{\natural} \longrightarrow {}^{E}\mathscr{B}^{\natural} \longrightarrow \pi_* {}^{E}\mathscr{C}^{\natural} \longrightarrow 0 .$$

This is the required decomposition of singularity of E-valued, partial and modified Fourier hyperfunctions.

Corollary 11.8. *We have the exact sequence*

$$0 \longrightarrow \mathscr{A}^{\natural}(M;\, E) \xrightarrow{\lambda} \mathscr{B}^{\natural}(M;\, E) \xrightarrow{\mathrm{sp}} \mathscr{C}^{\natural}(iS^*M;\, E) \longrightarrow 0 .$$

Definition 11.9. Let $u \in \mathscr{B}^{\natural}(M;\, E)$. We call $\mathrm{sp}(u) \in \mathscr{C}^{\natural}(iS^*M;\, E)$ a spectrum of u. We denote by S.S.u the support supp sp(u) of sp(u) and call it a singularity spectrum of u. π (S.S.u) is evidently the subset where u is not slowly increasing nor real-analytic, and it is called the singular support of u.

Corollary 11.10. *Let $u \in \mathscr{B}^{\natural}(M;\, E)$. Then u is an E-valued, partially slowly increasing, and real-analytic function on M if and only if* S.S.$u=\emptyset$.

Put ${}^{E}\mathscr{A} = {}^{E}\mathscr{A}^{\natural}|_{R^{|n|}}$, ${}^{E}\mathscr{B} = {}^{E}\mathscr{B}^{\natural}|_{R^{|n|}}$ and ${}^{E}\mathscr{C} = {}^{E}\mathscr{C}^{\natural}|_{iS^*R^{|n|}}$. Then we have the following Corollary by restricting the exact sequence in Theorem 11. 7.

Corollary 11.11. *Let $\pi : iS^*\boldsymbol{R}^{|n|} \longrightarrow \boldsymbol{R}^{|n|}$. Then we have the exact sequence*

$$0 \longrightarrow {}^{E}\mathscr{A} \longrightarrow {}^{E}\mathscr{B} \longrightarrow \pi_* {}^{E}\mathscr{C} \longrightarrow 0 .$$

11.3. Fundamental diagram on ${}^{E}\mathscr{C}^{\natural}$. We apply the arguments in the subsection 1. 2 of Ito[4] to a special case. At first we apply Proposition 1. 10 of Ito[4] to the situation $\mathscr{F} = ({}^{E}\mathscr{Q}^{\natural})^{a}$, $X=M$, $S=iSM$. Then we have $\mathscr{G} = {}^{E}\mathscr{C}^{\natural}$ and $\mathscr{E} = \pi_* {}^{E}\mathscr{C}^{\natural}$. We obtain the following.

Proposition 11.12. *We have*

$$R^{k}\pi_* \tau^{-1} {}^{E}\mathscr{C}^{\natural} = 0 \quad for\ k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow {}^{E}\mathscr{Q}^{\natural} \longrightarrow \tau^{-1}\pi_* {}^{E}\mathscr{C}^{\natural} \longrightarrow \pi_* \tau^{-1} {}^{E}\mathscr{C}^{\natural} \longrightarrow 0 .$$

Now we apply the same proposition to the case where $\mathscr{F} = {}^{E}\mathscr{A}^{\natural,\beta}$. Thus we obtain a homomorphism

$$(11.1) \qquad \begin{gathered} {}^{E}\mathscr{A}^{\natural,\beta} \longrightarrow \tau^{-1} R^{|n|-1} \tau_* {}^{E}\mathscr{A}^{\natural,\beta}, \\ {}^{E}\mathscr{A}^{\natural,\beta} = \boldsymbol{R}j_*({}^{E}\mathscr{O}^{\natural}|_{X-M})|_{iSM}, \end{gathered}$$

where $j : X-M \hookrightarrow {}^{M}\widetilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1}\tau_* {}^{E}\mathscr{A}^{\natural,\beta} = R^{|n|-1}(\tau \circ j)_*({}^{E}\mathscr{O}^{\natural}|_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1}\tau_* {}^{E}\mathscr{A}^{\natural,\beta} \longrightarrow {}^{E}\mathscr{B}^{\natural} .$$

It yields, together with (11. 1), a homomorphism ${}^{E}\mathscr{A}^{\natural,\beta} \longrightarrow \tau^{-1\,E}\mathscr{B}^{\natural}$. Summing up, we have obtained the following.

Theorem 11.13. *We have the following diagram of exact sequences of sheaves on iSM :*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau^{-1\,E}\mathscr{A}^{\natural} \arrow[r] \arrow[d, equal] & {}^{E}\mathscr{A}^{\natural,\beta} \arrow[r] \arrow[d, "\lambda"'] & {}^{E}\mathscr{Q}^{\natural} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau^{-1\,E}\mathscr{A}^{\natural} \arrow[r] & \tau^{-1\,E}\mathscr{B}^{\natural} \arrow[r] \arrow[d] & \tau^{-1}\pi_*{}^{E}\mathscr{C}^{\natural} \arrow[r] \arrow[d] & 0 \\
 & & \pi_*\tau^{-1\,E}\mathscr{C}^{\natural} \arrow[r, equal] \arrow[d] & \pi_*\tau^{-1\,E}\mathscr{C}^{\natural} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{11.2}
$$

Let us transform the diagram (11. 2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM \longrightarrow iS^*M$ and $IM \longrightarrow iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1. 7 of Ito[4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1\,E}\mathscr{C}^{\natural} \cong \boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1\,E}\mathscr{C}^{\natural} \cong {}^{E}\mathscr{C}^{\natural}[1-|n|].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (11. 2), we obtain

$$\boldsymbol{R}^k\tau'_!\pi'^{-1\,E}\mathscr{Q}^{\natural} = 0 \text{ for } k \neq |n|-1,$$
$$\boldsymbol{R}^k\tau'_!\pi'^{-1\,E}\mathscr{A}^{\natural,\beta} = 0 \text{ for } k \neq |n|-1.$$

We define the sheaves ${}^{E}\mathscr{A}^{\natural,\vee}$ and ${}^{E}\mathscr{C}^{\natural,\vee}$ on iS^*M by

$${}^{E}\mathscr{A}^{\natural,\vee} = R^{|n|-1}\tau'_!\pi'^{-1\,E}\mathscr{A}^{\natural,\beta},$$
$${}^{E}\mathscr{Q}^{\natural,\vee} = R^{|n|-1}\tau'_!\pi'^{-1\,E}\mathscr{Q}^{\natural,\beta}.$$

Then, in this way, we obtain the following theorem.

Theorem 11.14. *We have the diagram of exact sequences of sheaves on* iS^*M :

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}\,{}^{E}\mathscr{A}^{\natural} \arrow[r] \arrow[d, equal] & {}^{E}\mathscr{A}^{\natural,\vee} \arrow[r] \arrow[d] & {}^{E}\mathscr{Q}^{\natural,\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}\,{}^{E}\mathscr{A}^{\natural} \arrow[r] & \pi^{-1}\,{}^{E}\mathscr{B}^{\natural} \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_*\,{}^{E}\mathscr{C}^{\natural} \arrow[r] \arrow[d] & 0 \\
 & & {}^{E}\mathscr{C}^{\natural} \arrow[r, equal] \arrow[d] & {}^{E}\mathscr{C}^{\natural} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{11.3}
$$

and the diagram (11. 2) *and the diagram* (11. 3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_!\pi'^{-1}[|n|-1]$ *and* $\boldsymbol{R}\pi_*\tau^{-1}$.

We give a direct application of Theorem 11. 13, which gives a relation between singularity spectrum and the domain of the defining function of an E-valued, partial and modified Fourier hyperfunction.

Proposition 11.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

1) *If* $\varphi\in\Gamma(U,\ {}^{E}\mathscr{A}^{\natural,\beta})$, *then* S.S.$(\lambda(\varphi))\subset U^\circ$. *Conversely, if* $f(x)\in\Gamma(\tau U,\ {}^{E}\mathscr{B}^{\natural})$ *satisfies* S.S. $(f)\subset U^\circ$, *then there exists a unique* $\varphi\in\Gamma(U,\ {}^{E}\mathscr{A}^{\natural,\beta})$ *such that* $f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$0\longrightarrow\mathscr{A}^{\natural,\beta}(U;\,E)\xrightarrow{\ \lambda\ }\mathscr{B}^{\natural}(\tau U;\,E)\xrightarrow{\ \mathrm{sp}\ }\mathscr{C}^{\natural}(iS^*M-U^\circ;\,E).$$

2) $\Gamma(V,\ {}^{E}\mathscr{A}^{\natural,\beta})\longrightarrow\Gamma(U,\ {}^{E}\mathscr{A}^{\natural,\beta})$ *is an isomorphism.*

Definition 11.16. We say $u\in\mathscr{B}^{\natural}(\Omega\,;E)$ to be micro-analytic at $(x,\ i\eta\infty)$ in iS^*M if $(x,\ i\eta\infty)\notin$ S.S.u. This is equivalent to being represented as

$$u=\sum_j\lambda(\varphi_j),\ \ \varphi_j\in\mathscr{A}^{\natural,\beta}(U_j;E),\ \ (x,\ i\eta\infty)\notin U_j^\circ.$$

12. Partial and mixed Fourier microfunctions

12.1. Partial and mixed Fourier hyperfunctions. In this section we recall the notion of partial and mixed Fourier hyperfunctions following Ito[2].

Let $n=(n_1,\ n_2,\ n_3)=(n_1,\ n')$ be a triplet of nonnegative integers with $|n|=n_1+n_2+n_3\neq 0$. We denote the product spaces $\boldsymbol{C}^{n_1}\times\widetilde{\boldsymbol{C}}^{n_2}\times\widetilde{\widetilde{\boldsymbol{C}}}^{n_3}$ and $\boldsymbol{R}^{n_1}\times\widetilde{\boldsymbol{R}}^{n_2}\times\widetilde{\widetilde{\boldsymbol{R}}}^{n_3}$ by $\boldsymbol{C}^{*\cdot n}$ and $\boldsymbol{R}^{*\cdot n}$ respectively. Also put $\boldsymbol{C}^{|n|}=\boldsymbol{C}^{n_1}\times\boldsymbol{C}^{n_2}\times\boldsymbol{C}^{n_3}$, $X=\boldsymbol{C}^{*\cdot n}$

and $M=\boldsymbol{R}^{*\cdot n}$. Then M is the closure of $\boldsymbol{R}^{|n|}=\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}\times\boldsymbol{R}^{n_3}$ in X. We denote $z=(z', z'', z''')\in\boldsymbol{C}^{|n|}$ so that $z'=(z_1, \cdots, z_{n_1})$, $z''=(z_{n_1+1}, \cdots, z_{n_1+n_2})$ and $z'''=(z_{n_1+n_2+1}, \cdots, z_{|n|})$.

Let $\mathcal{O}^*$ be the sheaf of partially slowly increasing and holomorphic functions on X following Ito [2], Definition 11.1.1. Put $\mathscr{A}^*=\mathcal{O}^*|_M$. Then $\mathscr{A}^*$ is the sheaf of partially slowly increasing and real-analytic functions on M. Then we have $\mathscr{A}^* =\iota^{-1}\mathcal{O}^*$, where $\iota : M\hookrightarrow X$ is the canonical injection.

As in Ito [2], we define the sheaf of partial and mixed Fourier hyperfunctions on M :

Definition 12.1. The sheaf $\mathscr{B}^*$ is, by definition,

$$\mathscr{B}^*=\mathscr{H}_M^{|n|}(\mathcal{O}^*)=\mathrm{Dist}^{|n|}(M, \mathcal{O}^*),$$

where the notation in the right hand side of the above equality is due to Sato [16], p. 405. A section of $\mathscr{B}^*$ is called a partial and mixed Fourier hyperfunction.

As stated in Ito [2], $\mathscr{H}_M^k(\mathcal{O}^*)=0$ for $k\neq|n|$ and $\mathscr{B}^*$ constitutes a flabby sheaf on M.

Now we apply Lemma 1.1 of Ito [4] to this case where $\mathscr{F}$, X and Y correspond to $\mathcal{O}^*$, X and M respectively. Then we obtain the sheaf homomorphism

$$\mathscr{A}^*\longrightarrow\mathscr{B}^*,$$

which will be proved to be injective later. This injection allows us to consider partial and mixed Fourier hyperfunctions as a generalization of partially slowly increasing and real-analytic functions. The one purpose of this section is to analyse the structure of the quotient sheaf $\mathscr{B}^*/\mathscr{A}^*$ by a similar way to S. K. K. [21].

12.2. Definition of partial and mixed Fourier microfunctions. Suppose that $M=\boldsymbol{R}^{*\cdot n}$ and $X=\boldsymbol{C}^{*\cdot n}$. We denote by $\mathcal{O}^*$ the sheaf of partially slowly increasing and holomorphic functions defined on X. The (co-)sphere bundle iSM (resp. iS^*M) are defined similarly to subsection 2.2 of Ito [4]. We also use a similar notation to subsection 2.2 of Ito [4]. Then we have the following diagram :

$$
\begin{array}{ccccccc}
 & & {}^{M}\widetilde{X}^{+} & \hookleftarrow & DM & & \\
\pi & \swarrow & \tau & \times & \pi & \searrow & \tau \\
{}^{M}\widetilde{X} \hookleftarrow iSM & & & & & & {}^{M}\widetilde{X}^{*} \hookleftarrow iS^{*}M \\
\tau & \searrow & \tau & \times & \pi & \swarrow & \pi \\
 & & X & \hookleftarrow & M & & .
\end{array}
$$

Theorem 12.2. We have $\mathscr{H}^{k}_{iSM}(\tau^{-1}\mathscr{O}^{*})=0$ *for* $k\neq 1$, *where* $\tau:{}^{M}\widetilde{X}\longrightarrow X$ *is the canonical projection.*

The following theorem is the most essential one in the theory of partial and mixed Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 12.3. *We have* $\mathscr{H}^{k}_{iS^{*}M}(\pi^{-1}\mathscr{O}^{*})=0$ *for* $k\neq|n|$, *where* $\pi:{}^{M}\widetilde{X}^{*}\longrightarrow X$ *is the canonical projection.*

In the proof of the above theorem, the following theorem is essential.

Theorem 12.4 (the "Edge of the Wedge" Theorem). *Put* $G=\{z=x+iy\in \boldsymbol{C}^{|n|};y_{j}\geqq 0(1\leqq j\leqq|n|)\}^{\mathrm{cl}}$. *Then we have, for each* $x\in M$,

$$\mathscr{H}^{k}_{G}(\mathscr{O}^{*})_{x}=0 \text{ for } k\neq|n|.$$

Definition 12.5. We define the sheaf $\mathscr{C}^{*}$ on $iS^{*}M$ by the relation

$$\mathscr{C}^{*}=\mathscr{H}^{|n|}_{iS^{*}M}(\pi^{-1}\mathscr{O}^{*})^{a},$$

where we denote by a the antipodal map $iS^{*}M\longrightarrow iS^{*}M$, and by $\mathscr{F}^{a}$ the inverse image under a of a sheaf $\mathscr{F}$ on $iS^{*}M$. A section of $\mathscr{O}^{*}$ is called a partial and mixed Fourier microfunction.

Now we define the sheaves $\mathscr{Q}^{*}$, $\mathscr{O}^{*,\beta}$ and $\mathscr{A}^{*,\beta}$ by the relations

$$
\begin{aligned}
\mathscr{Q}^{*}&=\mathscr{H}^{1}_{iSM}(\tau^{-1}\mathscr{O}^{*}),\\
\mathscr{O}^{*,\beta}&=j_{*}(\mathscr{O}^{*}|_{X-M}),\\
\mathscr{A}^{*,\beta}&=\mathscr{O}^{*,\beta}|_{iSM},
\end{aligned}
$$

where $j:X-M\hookrightarrow {}^{M}\widetilde{X}$, $\pi:{}^{M}\widetilde{X}^{*}\longrightarrow X$ and $\tau:{}^{M}\widetilde{X}\longrightarrow X$ are canonical maps.

By Proposition 1.3 of Ito [4] and Theorems 12.2 and 12.3, we have the following.

Proposition 12.6. *We have*

$$R^{k}\tau_{*}\pi^{-1}\mathscr{Q}^{*}=\begin{cases}(\mathscr{C}^{*})^{a}, & (k=|n|-1),\\ 0, & (k\neq|n|-1).\end{cases}$$

Theorem 12.7. *We have*

$$R^k\pi_*\mathscr{C}^* = R^{k+|n|-1}\tau_*\mathscr{C}^* = 0 \quad \textit{for } k \neq 0,$$

and we have the exact sequence

$$0 \longrightarrow \mathscr{A}^* \longrightarrow \mathscr{B}^* \longrightarrow \pi_*\mathscr{C}^* \longrightarrow 0.$$

This is the required decomposition of singularity of partial and mixed Fourier hyperfunctions.

Corollary 12.8. *We have the exact sequence*

$$0 \longrightarrow \mathscr{A}^*(M) \xrightarrow{\lambda} \mathscr{B}^*(M) \xrightarrow{\mathrm{sp}} \mathscr{C}^*(iS^*M) \longrightarrow 0.$$

Definition 12.9. Let $u \in \mathscr{B}^*(M)$. We call $\mathrm{sp}(u) \in \mathscr{C}^*(iS^*M)$ a spectrum of u. We denote by S.S.u the support supp sp(u) of sp(u) and call it a singularity spectrum of u. π (S.S.u) is evidently the subset where u is not partially slowly increasing nor real-analytic and is called the singular support of u.

Corollary 12.10. *Let* $u \in \mathscr{B}^*(M)$. *Then u is a partially slowly increasing and real-analytic function on M if and only if* S.S.$u = \phi$.

Since $\mathscr{A} = \mathscr{A}^*|_{R^{|n|}}$, $\mathscr{B} = \mathscr{B}^*|_{R^{|n|}}$ and $\mathscr{C} = \mathscr{C}^*|_{iS^*R^{|n|}}$ hold in the notation of S. K. K. [21], we have the following Corollary by restricting the exact sequence in Theorem 12. 7.

Corollary 12.11. Let $\pi: iS^*\boldsymbol{R}^{|n|} \longrightarrow \boldsymbol{R}^{|n|}$. *Then we have the exact sequence*

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_*\mathscr{C} \longrightarrow 0.$$

12.3. Fundamental diagram on $\mathscr{C}^*$. We apply the arguments in the subsection 1. 2 of Ito [4] to a special case. At first we apply Proposition 1. 10 of Ito [4] to the situation $\mathscr{F} = (\mathscr{Q}^*)^a$, $X=M$, $S=iSM$. Then we have $\mathscr{G} = \mathscr{C}^*$ and $\mathscr{E} = \pi_*\mathscr{C}^*$. We obtain the following.

Proposition 12.12. *We have*

$$R^k\pi_*\tau^{-1}\mathscr{C}^* = 0 \ \textit{for } k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow \mathscr{Q}^* \longrightarrow \tau^{-1}\pi_*\mathscr{C}^* \longrightarrow \pi_*\tau^{-1}\mathscr{C}^* \longrightarrow 0.$$

Now we apply the same proposition to the case where $\mathscr{F} = \mathscr{A}^{*,\beta}$. Thus we obtain a homomorphism

(12. 1)
$$\mathscr{A}^{*,\beta} \longrightarrow \tau^{-1}R^{|n|-1}\tau_*\mathscr{A}^{*,\beta},$$
$$\mathscr{A}^{*,\beta} = \boldsymbol{R}j_*(\mathscr{O}^*|_{X-M})|_{iSM},$$

where $j: X-M \hookrightarrow {}^M\widetilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1}\tau_*\mathscr{A}^{*,\beta}=R^{|n|-1}(\tau\circ j)_*(\mathscr{O}^*|_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1}\tau_*\mathscr{A}^{*,\beta}\longrightarrow\mathscr{B}^*.$$

It yields, together with (12.1), a homomorphism $\mathscr{A}^{*,\beta}\longrightarrow\tau^{-1}\mathscr{B}^*$. Summing up, we have obtained the following.

Theorem 12.13. *We have the following diagram of exact sequences of sheaves on iSM :*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau^{-1}\mathscr{A}^* \arrow[r] \arrow[d, equal] & \mathscr{A}^{*,\beta} \arrow[r] \arrow[d] & \mathscr{Q}^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau^{-1}\mathscr{A}^* \arrow[r] & \tau^{-1}\mathscr{B}^* \arrow[r] \arrow[d] & \tau^{-1}\pi_*\mathscr{C}^* \arrow[r] \arrow[d] & 0 \\
 & & \pi_*\tau^{-1}\mathscr{C}^* \arrow[r, equal] \arrow[d] & \pi_*\tau^{-1}\mathscr{C}^* \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{12.2}$$

Let us transform the diagram (12.2) of the sheaves on iSM to a diagram of the sheaves on iS^*M by the functor $\boldsymbol{R}\tau'_!\pi'^{-1}$, where τ', π' are projections $IM\longrightarrow iS^*M$, $IM\longrightarrow iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_!\pi'^{-1}\tau^{-1}\mathscr{F}\cong\boldsymbol{R}\tau'_!\tau'^{-1}\pi^{-1}\mathscr{F}\cong\pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1.7 of Ito[4],

$$\boldsymbol{R}\tau'_!\pi'^{-1}\pi_*\tau^{-1}\mathscr{C}^*\cong\boldsymbol{R}\tau'_!\pi'^{-1}\boldsymbol{R}\pi_*\tau^{-1}\mathscr{C}^*\cong\mathscr{C}^*[1-|n|].$$

By operating $\boldsymbol{R}\tau'_!\pi'^{-1}$ on exact columns in (12.2), we obtain

$$R^k\tau'_!\pi'^{-1}\mathscr{Q}^*=0 \text{ for } k\neq|n|-1,$$
$$R^k\tau'_!\pi'^{-1}\mathscr{A}^{*,\beta}=0 \text{ for } k\neq|n|-1.$$

We define the sheaves $\mathscr{A}^{*,\vee}$ and $\mathscr{Q}^{*,\vee}$ on iS^*M by the relations

$$\mathscr{A}^{*,\vee}=R^{|n|-1}\tau'_!\pi'^{-1}\mathscr{A}^{*,\beta},$$
$$\mathscr{Q}^{*,\vee}=R^{|n|-1}\tau'_!\pi'^{-1}\mathscr{Q}^*.$$

Then, in this way, we obtain the following theorem.

Theorem 12.14. *We have the diagram of exact sequences of sheaves on iS^*M :*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}\mathscr{A}^* \arrow[r] \arrow[d, equal] & \mathscr{A}^{*,\vee} \arrow[r] \arrow[d] & \mathscr{Q}^{*,\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}\mathscr{A}^* \arrow[r] & \pi^{-1}\mathscr{B}^* \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_*\mathscr{C}^* \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{C}^* \arrow[r, equal] \arrow[d] & \mathscr{C}^* \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{12.3}
$$

and the diagram (12.2) *and the diagram* (12.3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_!\pi'^{-1}[|n|-1]$ *and* $\boldsymbol{R}\pi_*\tau^{-1}$.

We give a direct application of Theorem 12.13, which gives a relation between singularity spectrum and the domain of the defining function of a partial and mixed Fourier hyperfunction.

By using a similar notion to subsection 2.3 of Ito[4], we can state the following proposition.

Proposition 12.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

1) *If* $\varphi\in\Gamma(U, \mathscr{A}^{*,\beta})$, *then* S.S.$(\lambda(\varphi))\subset U^\circ$. *Conversery, if* $f(x)\in\Gamma(\tau U, \mathscr{B}^*)$ *satisfies* S.S.$(f)\subset U^\circ$, *then there exists a unique* $\varphi\in\Gamma(U, \mathscr{A}^{*,\beta})$ such that $f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$0\longrightarrow\mathscr{A}^{*,\beta}(U)\longrightarrow\mathscr{B}^*(\tau U)\xrightarrow{\mathrm{sp}}\mathscr{C}^*(iS^*M-U^\circ).$$

2) $\Gamma(V, \mathscr{A}^{*,\beta})\longrightarrow\Gamma(U, \mathscr{A}^{*,\beta}_{\lambda})$ *is an isomorphism.*

Definition 12.16. We say $u\in\mathscr{B}^*(\Omega)$ to be micro-analytic at $(x, i\eta\infty)$ in iS^*M if $(x, i\eta\infty)\notin$S.S.u. This is equivalent to being represented as

$$u=\sum_j\lambda(\varphi_j),\ \varphi_j\in\mathscr{A}^{*,\beta}(U_j),\ (x, i\eta\infty)\notin U_j^\circ.$$

13. Vector-valued, partial and mixed Fourier microfunctions

13.1. Vector-valued, partial and mixed Fourier hyperfunctions. In this subsection we recall the notion of vector-valued, partial and mixed Fourier hyperfunctions following Ito[2]. This is equivalent to the notion of Fourier hyperfunctions of general type called in my Thesis[3].

We use a similar notation to subsection 12. 1. Let E be a Fréchet space over the complex number field.

Let ${}^{E}\mathcal{O}^{*}$ be the sheaf of E-valued, partially slowly increasing, and holomorphic functions on X following Ito[2], Definition 12. 1. 1, and put ${}^{E}\mathscr{A}^{*}={}^{E}\mathcal{O}^{*}|_{M}$. Then ${}^{E}\mathscr{B}^{*}$ is the sheaf of E-valued, partially slowly increasing, and real-analytic functions on M. Then we have ${}^{E}\mathscr{A}^{*}=\iota^{-1}{}^{E}\mathcal{O}^{*}$, where $\iota:M\hookrightarrow X$ is the canonical injection.

As in Ito [2], we define the sheaf of E-valued, partial and mixed Fourier hyperfunctions on M :

Definition 13.1. The sheaf ${}^{E}\mathscr{B}^{*}$ is, by definition,

$$ {}^{E}\mathscr{B}^{*}=\mathscr{H}^{|n|}_{M}({}^{E}\mathcal{O}^{*})=\mathrm{Dist}^{|n|}(M,{}^{E}\mathcal{O}^{*}). $$

A section of ${}^{E}\mathscr{B}^{*}$ is called an E-valued, partial and mixed Fourier hyperfunction.

As stated in Ito[2], $\mathscr{H}^{k}_{M}({}^{E}\mathcal{O}^{*})=0$ for $k\neq|n|$ and ${}^{E}\mathscr{B}^{*}$ constitutes a flabby sheaf on M.

Now we apply Lemma 1. 1 of Ito[4] to this case where $\mathscr{F}$, X and Y correspond to ${}^{E}\mathcal{O}^{*}$, X and M respectively. Then we obtain the sheaf homomorphism

$$ {}^{E}\mathscr{A}^{*}\longrightarrow{}^{E}\mathscr{B}^{*}, $$

which will be proved to be injective later. This injection allows us to consider E-valued, partial and mixed Fourier hyperfunctions as a generalization of E-valued, partially slowly increasing, and real-analytic functions. Then one purpose of this section is to analyse the structure of the quotient sheaf ${}^{E}\mathscr{B}^{*}/{}^{E}\mathscr{A}^{*}$ by a similar way to S. K. K. [21].

13.2. Definition of vector-valued, partial and mixed Fourier microfunctions. We use a similar notation to subsection 12. 2. Let E be a Fréchet space over the complex number field. We denote by ${}^{E}\mathcal{O}^{*}$ the sheaf of E-valued, partially slowly increasing, and holomorphic functions defined on X. We have the following.

Theorem 13.2. *We have* $\mathscr{H}^{k}_{iSM}(\tau^{-1}{}^{E}\mathcal{O}^{*})=0$ *for* $k\neq1$, *where* $\tau:{}^{M}\widetilde{X}\longrightarrow X$ *is the canonical projection.*

The following theorem is the most essential one in the theory of E-valued, partial and mixed Fourier microfunctions. This is deeply connected with the "Edge of the Wedge" Theorem.

Theorem 13.3. *We have* $\mathscr{H}^{k}_{iS^{*}M}(\pi^{-1}{}^{E}\mathcal{O}^{*})=0$ *for* $k\neq|n|$, *where* $\pi:{}^{M}\widetilde{X}\longrightarrow X$ *is the canonical projection.*

In the proof of the above theorem, the following theorem is essential.

Theorem 13.4 (the "Edge of the Wedge" Theorem). *Put* $G=\{z=x+iy\in \boldsymbol{C}^{|n|}; y_j\geqq 0(1\leqq j\leqq |n|)\}^{cl}$. *Then we have, for each* $x\in M$,

$$\mathscr{H}_G^k({}^E\mathcal{O}^*)_x=0 \text{ for } k\neq |n|.$$

Definition 13.5. We define the sheaf ${}^E\mathscr{C}^*$ in iS^*M by the relation

$${}^E\mathscr{C}^*=\mathscr{H}^{|n|}_{iS^*M}(\pi^{-1}\,{}^E\mathcal{O}^*)^a.$$

A section of ${}^E\mathscr{C}^*$ is called an E-valued, partial and mixed Fourier microfunctions.

Now we define the sheaves ${}^E\mathscr{Q}^*$, ${}^E\mathcal{O}^{*,\beta}$, ${}^E\mathscr{A}^{*,\beta}$ by the relations

$$\begin{aligned}
&{}^E\mathscr{Q}^*=\mathscr{H}^1_{iS^*M}(\tau^{-1}\,{}^E\mathcal{O}^*),\\
&{}^E\mathcal{O}^{*,\beta}=j_*({}^E\mathcal{O}^*|_{X-M}),\\
&{}^E\mathscr{A}^{*,\beta}={}^E\mathcal{O}^{*,\beta}|_{iSM},
\end{aligned}$$

where $j:X-M\hookrightarrow {}^M\widetilde{X}^*$, $\pi:{}^M\widetilde{X}^*\longrightarrow X$ and $\tau:{}^M\widetilde{X}\longrightarrow X$ are canonical maps.

By Proposition 1. 3 of Ito[4] and Theorems 13. 2 and 13. 3, we have the following.

Proposition 13.6. *We have*

$$R^k\tau_*\pi^{-1}\,{}^E\mathscr{Q}^*=\begin{cases}({}^E\mathscr{C}^*)^a, & (k=|n|-1),\\ 0, & (k\neq|n|-1).\end{cases}$$

Theorem 13.7. *We have*

$$R^k\pi_*\,{}^E\mathscr{C}^*=R^{k+|n|-1}\tau_*\,{}^E\mathscr{Q}^*=0 \text{ for } k\neq 0$$

and we have the exact sequence

$$0\longrightarrow {}^E\mathscr{A}^*\longrightarrow {}^E\mathscr{B}^*\longrightarrow \pi_*\,{}^E\mathscr{C}^*\longrightarrow 0.$$

This is the required decomposition of singularity of E-valued, partial and mixed Fourier hyperfunctions.

Corollary 13.8. *We have the exact sequence*

$$0\longrightarrow \mathscr{A}^*(M;E)\xrightarrow{\lambda}\mathscr{B}^*(M;E)\xrightarrow{\mathrm{sp}}\mathscr{C}^*(iS^*M;E)\longrightarrow 0.$$

Definition 13.9. Let $u\in\mathscr{B}^*(M;E)$. We call $\mathrm{sp}(u)\in\mathscr{C}^*(iS^*M;E)$ a spectrum of u. We denote by S.S.u the support supp sp(u) of sp(u) and call it a singularity spectrum of u. π (S.S.u) is evidently the subset where u is not slowly increasing nor real-analytic, and it is called the singular support of u.

Corollary 13.10. *Let* $u\in\mathscr{B}^*(M;E)$. *Then u is an E-valued, partially slowly increasing, and real-analytic function on M if and only if* S.S.$u=\phi$.

Put ${}^E\mathscr{A}={}^E\mathscr{A}^*|_{R^{|n|}}$, ${}^E\mathscr{B}={}^E\mathscr{B}^*|_{R^{|n|}}$. and ${}^E\mathscr{C}={}^E\mathscr{C}^*|_{iS^*R^{|n|}}$. Then we have the following Corollary by restricting the exact sequence in Theorem 13. 7.

Corollary 13.11. *Let* $\pi : iS^*\boldsymbol{R}^{|n|} \longrightarrow \boldsymbol{R}^{|n|}$. *Then we have the exact sequence*

$$0 \longrightarrow {}^E\mathscr{A} \longrightarrow {}^E\mathscr{B} \longrightarrow \pi_* {}^E\mathscr{C} \longrightarrow 0\,.$$

13.3. Fundamental diagram on ${}^E\mathscr{C}^*$**.** We apply the arguments in the subsection 1. 2 of Ito[4] to a special case. At first we apply Proposition 1. 10 of Ito[4] to the situation $\mathscr{F} = ({}^E\mathscr{Q}^*)^a$, $X=M$, $S=iSM$. Then we have $\mathscr{G} = {}^E\mathscr{C}^*$ and $\mathscr{E} = \pi_* {}^E\mathscr{C}^*$. We obtain the following.

Proposition 13.12. *We have*

$$R^k \pi_* \tau^{-1} {}^E\mathscr{C}^* = 0 \ \textit{for} \ k \neq 0$$

and we have the exact sequence

$$0 \longrightarrow {}^E\mathscr{Q}^* \longrightarrow \tau^{-1}\pi_* {}^E\mathscr{C}^* \longrightarrow \pi_* \tau^{-1} {}^E\mathscr{C}^* \longrightarrow 0\,.$$

Now we apply the same proposition to the case where $\mathscr{F} = {}^E\mathscr{A}^{*,\beta}$. Thus we obtain a homomorphism

$$\begin{aligned} &{}^E\mathscr{A}^{*,\beta} \longrightarrow \tau^{-1} R^{|n|-1} \tau_* {}^E\mathscr{A}^{*,\beta}, \\ &{}^E\mathscr{A}^{*,\beta} = \boldsymbol{R}j_* ({}^E\mathscr{O}^* |_{X-M}) |_{iSM}, \end{aligned} \tag{13. 1}$$

where $j : X-M \hookrightarrow {}^M\widetilde{X}$ is the canonical injection, which implies that

$$R^{|n|-1} \tau_* \, {}^E\mathscr{A}^{*,\beta} = R^{|n|-1} (\tau \circ j)_* ({}^E\mathscr{O}^* |_{X-M}).$$

Hence we can define the canonical map

$$R^{|n|-1} \tau_* {}^E\mathscr{A}^{*,\beta} \longrightarrow {}^E\mathscr{B}^*.$$

It yields, together with (13. 1), a homomorphism ${}^E\mathscr{A}^{*,\beta} \longrightarrow \tau^{-1} {}^E\mathscr{B}^*$. Summing up, we have obtained the following.

Theorem 13.13. *We have the following diagram of exact sequences of sheaves on iSM* :

$$
(13.2)\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau^{-1\,E}\mathscr{A}^{*} \arrow[r] \arrow[d, equal] & {}^{E}\mathscr{A}^{*,\beta} \arrow[r] \arrow[d] & {}^{E}\mathscr{Q}^{*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau^{-1\,E}\mathscr{A}^{*} \arrow[r] & \tau^{-1\,E}\mathscr{B}^{*} \arrow[r] \arrow[d] & \tau^{-1}\pi_{*}{}^{E}\mathscr{C}^{*} \arrow[r] \arrow[d] & 0 \\
 & & \pi_{*}\tau^{-1\,E}\mathscr{C}^{*} \arrow[r, equal] \arrow[d] & \pi_{*}\tau^{-1\,E}\mathscr{C}^{*} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\ .
$$

Let us transform the diagram (13.2) of the sheaves on iSM to a diagram of the sheaves on $iS^{*}M$ by the functor $\boldsymbol{R}\tau'_{!}\pi'^{-1}$, where τ', π' are projections $IM \longrightarrow iS^{*}M$ and $IM \longrightarrow iSM$, respectively.

For a sheaf $\mathscr{F}$ on M, we have

$$\boldsymbol{R}\tau'_{!}\pi'^{-1}\tau^{-1}\mathscr{F} \cong \boldsymbol{R}\tau'_{!}\tau'^{-1}\pi^{-1}\mathscr{F} \cong \pi^{-1}\mathscr{F}[1-|n|].$$

By Proposition 1.7 of Ito[4],

$$\boldsymbol{R}\tau'_{!}\pi'^{-1}\pi_{*}\tau^{-1\,E}\mathscr{C}^{*} \cong \boldsymbol{R}\tau'_{!}\pi'^{-1}\boldsymbol{R}\pi_{*}\tau^{-1\,E}\mathscr{C}^{*} \cong {}^{E}\mathscr{C}^{*}[1-|n|].$$

By operating $R\tau'\pi'^{-1}$ on exact columns in (13.2), we obtain

$$R^{k}\tau'_{!}\pi'^{-1\,E}\mathscr{Q}^{*} = 0 \quad \text{for } k \neq |n|-1,$$
$$R^{k}\tau'_{!}\pi'^{-1\,E}\mathscr{A}^{*,\beta} = 0 \quad \text{for } k \neq |n|-1.$$

We define the sheaves ${}^{E}\mathscr{A}^{*,\beta}$ and ${}^{E}\mathscr{Q}^{*,\beta}$ on $iS^{*}M$ by the relations

$${}^{E}\mathscr{A}^{*,\beta} = R^{|n|-1}\tau'_{!}\pi'^{-1\,E}\mathscr{A}^{*,\beta},$$
$${}^{E}\mathscr{Q}^{*,\beta} = R^{|n|-1}\tau'_{!}\pi'^{-1\,E}\mathscr{Q}^{*,\beta}.$$

Then, in this way, we obtain the following theorem.

Theorem 13.14. *We have the diagram of exact sequences of sheaves on $iS^{*}M$:*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi^{-1}\,{}^{E}\mathscr{A}^{*} \arrow[r] \arrow[d, equal] & {}^{E}\mathscr{A}^{*,\vee} \arrow[r] \arrow[d] & {}^{E}\mathscr{Q}^{*,\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi^{-1}\,{}^{E}\mathscr{A}^{*} \arrow[r] & \pi^{-1}\,{}^{E}\mathscr{B}^{*} \arrow[r] \arrow[d, "\mathrm{sp}"'] & \pi^{-1}\pi_{*}\,{}^{E}\mathscr{C}^{*} \arrow[r] \arrow[d] & 0 \\
 & & {}^{E}\mathscr{C}^{*} \arrow[r, equal] \arrow[d] & {}^{E}\mathscr{C}^{*} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{13.3}
$$

and the diagram (13. 2) *and the diagram* (13. 3) *are mutually transformed by the functors* $\boldsymbol{R}\tau'_{!}\pi'^{-1}[|n|-1]$ *and* $\boldsymbol{R}\pi_{*}\tau^{-1}$.

We give a direct application of Theorem 13. 13, which gives a relation between singularity spectrum and the domain of the defining function of an E-valued, partial and mixed Fourier hyperfunction.

Proposition 13.15. *Let U be an open subset of iSM with convex fiber, and V a convex hull of U. Then we have*

1) *If* $\varphi \in \Gamma(U,\ {}^{E}\mathscr{A}^{*,\beta})$, *then* $\mathrm{S.S.}(\lambda(\varphi)) \subset U^{\circ}$. *Conversely, if* $f(x) \in \Gamma(\tau U,\ {}^{E}\mathscr{B}^{*})$ *satisfies* $\mathrm{S.S.}(f) \subset U^{\circ}$, *then there exists a unique* $\varphi \in \Gamma(U,\ {}^{E}\mathscr{A}^{*,\beta})$ *such that* $f=\lambda(\varphi)$. *Namely, we have the exact sequence*

$$0 \longrightarrow \mathscr{A}^{*,\beta}(U;\ E) \xrightarrow{\ \lambda\ } \mathscr{B}^{*}(\tau U;\ E) \xrightarrow{\ \mathrm{sp}\ } \mathscr{C}^{*}(iS^{*}M - U^{\circ};\ E).$$

2) $\Gamma(V,\ {}^{E}\mathscr{A}^{*,\beta}) \longrightarrow \Gamma(U,\ {}^{E}\mathscr{A}^{*,\beta})$ *is an isomorphism.*

Definition 13.16. We say $u \in \mathscr{B}^{*}(\Omega;\ E)$ to be micro-analytic at $(x,\ i\eta\infty)$ in $iS^{*}M$ if $(x,\ i\eta\infty) \notin \mathrm{S.S.}u$. This is equivalent to being represented as

$$u = \sum_{j} \lambda(\varphi_j),\ \varphi_j \in \mathscr{A}^{*,\beta}(U_j;\ E),\ (x,\ i\eta\infty) \notin U_j^{\circ}.$$

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