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On Some Numerical Relations of tetragonal Linear Systems

By

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Abstract

Let $\mathcal{L}$ be a pencil of degree 4 on a curve C and let e_1, e_2, e_3 be scrolar invariants. We prove that $e_1 \leq e_2 + e_3 + 2$ if and only if e_1, e_2, e_3 are scrollar invariants of some tetragonal curve.

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Introduction

Let C be a complete non-singular curve defined over an algebraically closed field k with $\mathrm{char}(k) \neq 2$. Assume that C is a tetragonal curve. We assume that C is non-hyperelliptic of genus g.

Let $F_i = \Gamma(C, \omega \otimes \mathcal{O}(-ig_4^1))$. The modules F_i $(i = 1, 2, \cdots)$ give a filtration,

$$F_0 \supset F_1 \supset \cdots \supset F_n \supset \cdots$$

and by the definition of $\{F_i\}_{i=0}^{\infty}$ we have injective maps

$$F_0/F_1 \hookrightarrow F_1/F_2 \hookrightarrow \cdots \hookrightarrow F_n/F_{n+1} \hookrightarrow \cdots.$$

By Riemann-Roch's Theorem, $\dim F_0/F_1 = 3$. We define the scrollar invariants $e_i = e_i(g_4^1)$ $(i = 1, 2, 3)$ by

$$e_i = e_i(g_d^1) = \#\{j \in \mathbb{N}; dim\ (F_{j-1}/F_j) \geq i\} - 1 \ (i = 1, 2, 3).$$

It is clear that $e_1 \geq e_2 \geq e_3 \geq 0$ and $e_1 + e_2 + e_3 = g - 3$. We now consider another description of scrollar invariants. For any $i \in \mathbb{N} \cup \{0\}$, we define $\alpha_i =$

$\dim\Gamma(C, \mathcal{O}((i+1)g_4^1)) - \dim\Gamma(C, \mathcal{O}(ig_4^1))$. Then we have

$$e_i = min\{j; \alpha_j \geq 4 - i + 1\} - 1$$

where $1 \leq i \leq 3$. This is proved by Riemann-Roch Theorem. Therefore we have that

$$\dim\Gamma(C, \mathcal{O}(g_4^1) = 2,$$
$$\dim\Gamma(C, \mathcal{O}(2g_4^1) = 3,$$
$$\cdots,$$
$$\dim\Gamma(C, \mathcal{O}((e_3+1)g_4^1) = e_3 + 2,$$
$$\dim\Gamma(C, \mathcal{O}((e_3+2)g_4^1) = (e_3+3) + 1,$$
$$\cdots,$$
$$\dim\Gamma(C, \mathcal{O}((e_2+1)g_4^1) = (e_2+2) + (e_2 - e_3),$$
$$\dim\Gamma(C, \mathcal{O}((e_2+2)g_4^1) = (e_2+3) + (e_2 - e_3 + 1) + 1,$$
$$\cdots,$$
$$\dim\Gamma(C, \mathcal{O}((e_1+1)g_4^1) = (e_1+2) + (e_1 - e_3) + (e_1 - e_2),$$
$$\dim\Gamma(C, \mathcal{O}((e_1+2)g_4^1) = (e_1+3) + (e_1 - e_3 + 1) + (e_1 - e_2 + 1) + 1,$$
$$\cdots.$$

We know the following result (see [4] p.4588 Theorem 1).

Theorem 1 *Let $e_1 \geq e_2 \geq e_3 \geq 0$ be integers such that $e_1 + e_2 + e_3 = g - 3$. Then thete is a 4-gonal curve $C = (C, g_4^1)$ with the scrollar invariants (e_1, e_2, e_3) such that $|(e_3+2)g_4^1|$ is birationally very ample , if and only if $e_2 \leq 2e_3 + 2$ and $e_1 \leq e_2 + e_3 + 2$.*

In this paper we shall consider a generalization of this result. The Main result is the following.

Theorem 2 (Main Theorem) *Let $e_1 \geq e_2 \geq e_3 \geq 0$ be integers such that $e_1 + e_2 + e_3 = g - 3$. Then thete is a tetragonal curve $C = (C, g_4^1)$ with the scrollar invariants (e_1, e_2, e_3), if and only if $e_1 \leq e_2 + e_3 + 2$.*

1 The Proof of Main Theorem

We use the following result.

Theorem 3 *Let C be a tetragonal curve with the scrollar invariants (e_1, e_2, e_3). Let g_4^1 be a base point free complete linear system on C. Let $\phi : C \to \mathbb{P}(\Gamma(C, \mathcal{O}((e_3+2)g_4^1)))$ be the morphism defined by $|(e_3+2)g_4^1|$. If ϕ is not birational onto its image, then $e_2 \neq e_3$, $\deg(\phi) = 2$ and $\phi(C)$ is a complete non-singular curve of genus $e_3 + 1$ admitting a g_2^1 with $\phi^* g_2^1 = g_4^1$.*

From Theorem 1 and Theorem 3, we may assume C is a two-sheeted cover of a hyperelliptic curve D of genus $e_3 + 1$ such that $e_3 \neq e_2$. And put $\pi : C \to D$ be a double-covering and we put $g_4^1 = \pi^* g_2^1$. Under these assumptions, we prove that Main Theorem.

We know that $\pi_* \mathcal{O}(ng_4^1) \cong \mathcal{O}(ng_2^1) \oplus \mathcal{O}(ng_2^1 - E)$ for some divisor D such that $2E$ is linearly equivalent to some effective divisor (see [6] p.326-p.328). We now consider (C, g_4^1) such that $g_4^1 = \pi^* g_2^1$ and let e_1, e_2, e_3 be scrollar invariants. Then

$$\dim \Gamma(C, \mathcal{O}((e_3+2)g_4^1) = (e_3+3)+1$$

implies $\dim \Gamma(D, \mathcal{O}((e_3+2)g_2^1 - E) = 0$ because $\pi_* \mathcal{O}(ng_4^1) \cong \mathcal{O}(ng_2^1) \oplus \mathcal{O}(ng_2^1 - E)$. And

$$\dim \Gamma(C, \mathcal{O}((e_2+2)g_4^1) = (e_2+3) + (e_2 - e_3 + 1) + 1$$

implies $\dim \Gamma(D, \mathcal{O}((e_2+2)g_2^1 - E) = 1$. Therefore we have an effective divisor $T = P_1 + \cdots + P_t$ such that $\iota(P_i) \notin \{P_1, \ldots, P_t\}$, for every $i = 1, \ldots, t$ and $(e_2+2)g_2^1 - T \sim E$ where ι is a hyperelliptic involution on D. As $2E$ is linearly equivalent to a ramification divisor of ϕ (see [6] p.326-p.328), so we have

$$2g - 2 = 2(2(e_3+1) - 2) + 2(2(e_2+2) - t).$$

Hence we have

$$t = -e_1 + e_2 + e_3 + 2$$

because $e_1 + e_2 + e_3 = g - 3$. As $t \geq 0$, we have $e_1 \leq e_2 + e_3 + 2$. Now let $e_1 \geq e_2 \geq e_3 \geq 0$ be integers such that $e_1 + e_2 + e_3 = g - 3$ and assume that $e_1 \leq e_2 + e_3 + 2$. Let D be a hyperelliptic curve of genus $e_3 + 1$. Let $t = -e_1 + e_2 + e_3 + 2$. Take points $P_1 \ldots, P_t \in D$ such that $\iota(P_i) \notin \{P_1, \ldots, P_t\}$, for every $i = 1, \ldots, t$. Let $T = P_1 + \cdots + P_t$ and $E = (e_2+2)g_2^1 - T$. Let take an effective divisor R such that $R \sim 2E$. Because $2E \sim 2(e_2+2)g_2^1 - 2T$, so $e_2 + 2 \geq 2t = 2(-e_1 + e_2 + e_3 + 2)$ implies $2E$ is linearly equivalent to some effective dividor R. Therefore an isomorphism $\mathcal{O}(-2E) \cong \mathcal{O}(-R) \hookrightarrow \mathcal{O}$ induces an algebra structure on $\mathcal{O} \oplus \mathcal{O}(-E)$. Let $C = \mathrm{Spec}(\mathcal{O} \oplus \mathcal{O}(-E))$. Then R is a ramification divisor of $\pi : C \to D$ (see [6] p.326-p.328), therefore C is of genus g. Let $g_4 = \pi^*(g_2^1)$. Then it is clear that scrolar invariants of g_4^1 are e_1, e_2, e_3.

Q.E.D.

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