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On the Blowup Problem for Nonlinear Kirchhoff Equations with Nonlinear Dissipative Terms

By

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Abstract

We study the blowup problem for the integro-differential equation of hyperbolic type with a nonlinear dissipative term. When the initial energy is smaller than the depth of an associated potential well and the initial displacement is in the exterior of this well, the solution is not global in time.

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1. Introduction and results

In this paper we investigate the global non-existence of solutions to the initial boundary value problem for the following nonlinear integro-differential equation of hyperbolic type with a nonlinear dissipative term :

$$(1.1)\qquad \begin{cases} u'' + M(\|A^{1/2}u(t)\|^2)Au + g(u') = f(u) & \text{in} \quad \Omega \times \mathbf{R}^+ \\ u(x,0) = u_0(x)\,, \quad u'(x,0) = u_1(x)\,, \quad \text{and} & u(x,t)|_{\partial\Omega} = 0\,, \end{cases}$$

where Ω is a bounded domain in N-dimensional Euclidean space $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial_t$, $A = -\Delta \equiv \sum_{j=1}^{N} \partial_{x_j}^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\|\cdot\|$ is the norm of $H = L^2(\Omega)$

$(\|\phi\|^2 = \int_\Omega |\phi(x)|^2\,dx)$, $M(r)$ is a non-negative C^1-function on $\mathbf{R}^+$ satisfying

$$(1.2) \qquad M(\|A^{1/2}u\|^2) \equiv a + b\|A^{1/2}u\|^{2\gamma}$$

with $a, b \geq 0$, $a + b > 0$, and $\gamma \geq 1$, and

$$(1.3) \qquad g(u') = |u'|^\beta u' \quad \text{and} \quad f(u) = |u|^\alpha u$$

with $\alpha, \beta > 0$.

When $N = 1$, Eq.(1.1) describes a small amplitude vibration of an elastic string. Kirchhoff [6] first studied such integro-differential equations without any dissipation, and then, we call Eq.(1.1) the Kirchhoff equation (see also [2], [3], [11]). The non-negative function $M(r)$ means the axial tension of a string. In the case $a = 0$ in (1.2), Eq.(1.1) is called degenerate type.

By applying the Banach contraction mapping theorem, we obtain the following local existence theorem (see [16] for the proof, and also [4], [9]) :

Proposition 1.1. *Let the initial data $\{u_0, u_1\}$ belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$. Suppose that $\alpha \leq 2/(N-4)$ if $N \geq 5$ and*

$$M(\|A^{1/2}u_0\|^2) > 0 \qquad (\text{i.e. } a > 0 \text{ or } u_0 \neq 0 \text{ if } a = 0).$$

Then, there exists a unique local solution u of Eq.(1.1) such that

$$u \in C_w([0,T); \mathcal{D}(A)) \cap C([0,T); \mathcal{D}(A^{1/2})),$$
$$u' \in C_w([0,T); \mathcal{D}(A^{1/2})) \cap C([0,T); H) \cap L^{\beta+2}((0,T) \times \Omega)$$

for some $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$.

We note that Eq.(1.1) is local ill-posed in the first energy space $D(A^{1/2}) \times H$ (see Arosio & Garavaldi [1]).

Now, we define the energy associated with Eq.(1.1) as

$$(1.4) \quad E(u, u') \equiv \|u'\|^2 + \Big(a + \frac{b}{\gamma+1}\|A^{1/2}u\|^{2\gamma}\Big)\|A^{1/2}u\|^2 + \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2},$$

where $\|\cdot\|_p$ is the usual $L^p(\Omega)$-norm (we often use $\|\cdot\| = \|\cdot\|_2$). In what follows, we denote $E(t) \equiv E(u(t), u'(t))$ and $E(0) \equiv E(u_0, u_1)$ in order to simplicity the notation. Immediately, we find that the energy has the so-called energy identity :

$$(1.5) \qquad E'(t) + 2\|u'(t)\|_{\beta+2}^{\beta+2} = 0$$

or

$$(1.6) \qquad E(t) + 2\int_0^t \|u'(s)\|_{\beta+2}^{\beta+2}\,ds = E(0).$$

We introduce the K-positive set (i.e. the modified potential well) and the K-negative set related to the global existence and the global non-existence of solutions as

$$\mathcal{W}_* \equiv \{u \in \mathcal{D}(A^{1/2})\,;\, K(u) > 0\} \cup \{0\}$$

and

$$\mathcal{V}_* \equiv \{u \in \mathcal{D}(A^{1/2})\,;\, K(u) < 0\}\,,$$

respectively, where we set

$$K(u) \equiv \begin{cases} a\|A^{1/2}u\|^2 - \|u\|_{\alpha+2}^{\alpha+2} & \text{if } a > 0 \\ b\|A^{1/2}u\|^{2(\gamma+1)} - \|u\|_{\alpha+2}^{\alpha+2} & \text{if } a = 0 \end{cases}$$

(see Nakao & Ono [10], Ono [15], and also [5], [17], [18]). We note that $E(u, u') > 0$ if $u \in \mathcal{W}_*$ and $u \neq 0$, and $u \in \mathcal{V}_*$ if $E(u, u') < 0$.

For the linear case $g(u') = u'$ (i.e. $\beta = 0$ in (1.3)), in previous paper [15], we have shown the global existence of solutions under the conditions that the initial data $\{u_0, u_1\} \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ are suitably small and $u_0 \in \mathcal{W}_*$, and derived the exponential and polynomial energy decay estimates for non-degenerate and degenerate equations, respectively. Moreover, under the conditions that $u_0 \in \mathcal{V}_*$ and the initial energy $E(0) < D_a$ (D_a is a positive constant given by (1.7)), we have proved the global non-existence theorem using the concavity method and the potential well method (see [5], [7], [9], [12], [13], [17], [18]).

In previous paper [16], combining the modified potential well method and the energy method, we have shown the global existence of a unique solution for Eq.(1.1) with $a > 0$ under the assumptions that $u_0 \in \mathcal{W}_*$ and the initial energy $E(0)$ is suitably small, and derived the polynomial decay estimates, e.g.,

$$\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2 \leq C(1+t)^{-2/\beta}.$$

On the other hand, when the initial energy $E(0)$ is sufficiently negative ($E(0) \ll -1$), Georgiev & Todorova [4] have shown the global non-existence of solutions for the semilinear wave equation (i.e. $b = 0$ in (1.2)) with the nonlinear dissipative term in the first energy space. By improving their method, when $E(0)$ is negative, we have proved the following global non-existence theorem in [16] (cf. [8], [9], [13]).

Theorem 1.2. *In addition to the assumptions of Proposition 1.1, suppose that*

$$E(0) < 0 \qquad \textit{and} \qquad \alpha > \max\{\beta\,, 2\gamma\}\,.$$

Then, the local solution u cannot be continued to some finite time T.

However, when $E(0)$ is positive, this theorem cannot give any results for the blowup problem. When $g(u')$ is nonlinear, similar blowup problem as the linear case arises. That is our problem of this paper.

Recently, Ohta [12] has studied the blowup problem for the related semilinear wave equation with the nonlinear dissipative term with the initial data belonging to the energy space and the so-called potential well (see [9]). In the proof, he proved that the energy becomes negative at some finite time and applied the result of Georgiev & Todorova [4]. In this paper, combining his idea [12] and the results of the previous papers [15, 16], we solve the blowup problem under the positive initial energy $E(0) < D_a$.

To state our result, we define a positive constant D_a as

$$
(1.7)\quad D_a = \begin{cases} \dfrac{\alpha}{\alpha+2}\left\{\dfrac{a}{C(\Omega,\alpha+2)}\right\}^{(\alpha+2)/\alpha} & \text{if } a > 0 \\[2ex] \dfrac{\alpha-2\gamma}{(\alpha+2)(\gamma+1)}\left\{\dfrac{b}{C(\Omega,\alpha+2)^{2(\gamma+1)}}\right\}^{(\alpha+2)/(\alpha-2\gamma)} & \text{if } a = 0\,, \end{cases}
$$

where we set

$$
C(\Omega,\alpha+2) = \sup\{\|u\|_{\alpha+2}/\|A^{1/2}u\|\,;\, u \in \mathcal{D}(A^{1/2})\,,\, u \neq 0\}
$$

for $0 \leq \alpha \leq 4/(N-2)$.

Our main result is as follows.

Theorem 1.3. *In addition to the assumptions of Proposition 1.1, suppose that* $\alpha \leq 4/(N-2)$ *if* $N \geq 3$, $u_0 \in \mathcal{V}_*$,

$$
E(0) < D_a \qquad \textit{and} \qquad \alpha > \max\{\beta\,,\, 2\gamma\}\,.
$$

Then, the local solution u *cannot be continued to some finite time* T.

2. Proof

In this section we devote to the proof of Theorem 1.3. To this, we need the following result related to the K-negative set $\mathcal{V}_*$ (see Proposition 5.1 in [15] for the proof).

Proposition 2.1. *Let* u *be a solution of Eq.*(1.1). *Suppose that* $\alpha \leq 4/(N-2)$ *if* $N \geq 3$, $u_0 \in \mathcal{V}_*$,

$$
E(0) < D_a \qquad \textit{and} \qquad \alpha > 2\gamma\,.
$$

Then,

$$
(2.1)\qquad K(u(t)) < 0 \qquad \textit{and} \qquad u(t) \neq 0
$$

and

$$E(t) < D_a \le \frac{a\alpha}{\alpha+2}\|A^{1/2}u(t)\|^2 + \frac{b(\alpha-2\gamma)}{(\alpha+2)(\gamma+1)}\|A^{1/2}u(t)\|^{2(\gamma+1)}. \tag{2.2}$$

Proof of Theorem 1.3. From a veiwpoint of Theorem 1.2, it is sufficient to show that there exists a real number $T_* \ge 0$ such that $E(T_*) < 0$. We put

$$P(t) = \|u(t)\|^2,$$

and then, we see $P'(t) = 2(u(t), u'(t))$, where $(\cdot,\cdot)$ is the inner product of $H = L^2(\Omega)$, and

$$\begin{aligned} P''(t) &= 2\{(u(t), u''(t)) + \|u'(t)\|^2\} \\ &= 2\{\|u'(t)\|^2 - K(u(t)) - (u(t), g(u'(t)))\}. \end{aligned} \tag{2.3}$$

First, we shall estimate the term $K(u(t))$.

(i) *Case of* $a > 0$. For any $1 < q < (\alpha+2)/2$, we see

$$\begin{aligned} K(u(t)) = qE(t) - a(q-1)\|A^{1/2}u(t)\|^2 - \frac{bq}{\gamma+1}\|A^{1/2}u(t)\|^{2(\gamma+1)} \\ -q\|u'(t)\|^2 - (1-2q/(\alpha+2))\|u(t)\|_{\alpha+2}^{\alpha+2}. \end{aligned}$$

From (2.1), it follows

$$\begin{aligned} &qE(t) - a(q-1)\|A^{1/2}u(t)\|^2 - \frac{bq}{\gamma+1}\|A^{1/2}u(t)\|^{2(\gamma+1)} \\ &\le qE(0) - (q-1)(\alpha+2)\alpha^{-1}D_a - \frac{b(\alpha+2(q-1)\gamma)}{\alpha(\gamma+1)}\|A^{1/2}u(t)\|^{2(\gamma+1)} \\ &\le (\alpha+2)\alpha^{-1}D_a - \{(\alpha+2)\alpha^{-1}D_a - E(0)\}q \le 0 \end{aligned}$$

for $q \ge (\alpha+2)D_a/((\alpha+2)D_a - \alpha E(0))$. Thus, we obtain

$$-K(u(t)) \ge q\|u'(t)\|^2 + (1-2q/(\alpha+2))\|u(t)\|_{\alpha+2}^{\alpha+2} \tag{2.4}$$

for $(\alpha+2)D_a/((\alpha+2)D_a - \alpha E(0)) \le q < (\alpha+2)/2$.

(ii) *Case of* $a = 0$. For any $\gamma+1 < q < (\alpha+2)/2$, we see

$$\begin{aligned} K(u(t)) = qE(t) - b(q/(\gamma+1) - 1)\|A^{1/2}u(t)\|^{2(\gamma+1)} \\ -q\|u'(t)\|^2 - (1-2q/(\alpha+2))\|u(t)\|_{\alpha+2}^{\alpha+2}. \end{aligned}$$

From (2.2), it follows

$$\begin{aligned} &qE(t) - b(q/(\gamma+1) - 1)\|A^{1/2}u(t)\|^{2(\gamma+1)} \\ &\le qE(0) - (q-(\gamma+1))\frac{\alpha+2}{\alpha-2\gamma}D_a \\ &= \frac{(\gamma+1)(\alpha+2)}{\alpha-2\gamma}D_a - \left\{\frac{\alpha+2}{\alpha-2\gamma}D_a - E(0)\right\}q \le 0 \end{aligned}$$

for $q \geq (\gamma+1)(\alpha-2\gamma)D_a/((\alpha+2)D_a-(\alpha-2\gamma)E(0))$. Thus, we obtain

$$-K(u(t)) \geq q\|u'(t)\|^2 + (1-2q/(\alpha+2))\|u(t)\|_{\alpha+2}^{\alpha+2} \tag{2.5}$$

for $(\gamma+1)(\alpha+2)D_a/((\alpha+2)D_a-(\alpha-2\gamma)E(0)) \leq q < (\alpha+2)/2$.

Next, we estimate the term $(u, g(u'))$. It follows from (2.1) and (2.2) that there exists a positive constant C such that

$$\|u(t)\|_{\alpha+2} \geq C > 0\,.$$

Then we observe

$$\begin{aligned} |(u,g(u'))| &\leq \|u\|_{\beta+2}\|u'\|_{\beta+2}^{\beta+1} \leq C\|u\|_{\alpha+2}\|u'\|_{\beta+2}^{\beta+1} \\ &\leq C\|u\|_{\alpha+2}^{(\alpha+2)/(\beta+2)}\|u'\|_{\beta+2}^{\beta+1} \leq \varepsilon\|u\|_{\alpha+2}^{\alpha+2} + C_\varepsilon\|u'\|_{\beta+2}^{\beta+2} \end{aligned} \tag{2.6}$$

for small $\varepsilon > 0$, where $C_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Therefore, it follows from (2.3)–(2.6) that

$$P''(t) \geq 2\{(1+q)\|u'(t)\|^2 + (1-2q/(\alpha+2)-\varepsilon)\|u(t)\|_{\alpha+2}^{\alpha+2} - C_\varepsilon\|u'(t)\|_{\beta+2}^{\beta+2}\}\,,$$

and hence, for $0 < \varepsilon < 1 - 2q/(\alpha+2)$,

$$P''(t) \geq C_1 E'(t) + C_2$$

with $C_1 = C_\varepsilon > 0$ and $C_2 = 2(1-2q/(\alpha+2)-\varepsilon)D_a/\alpha > 0$, where we have used the identity (1.5), (2.1), and (2.2). Moreover, we see

$$P'(t) \geq C_2 t + C_3\,, \qquad C_3 = P'(0) - C_1E(0)$$

and

$$\|u(t)\|^2 = P(t) \geq (C_2/2)t^2 + C_3 t + P(0)\,. \tag{2.7}$$

On the other hand, since $\beta > 0$ and $u(t) = u_0 + \int_0^t u'(s)\,ds$, we observe

$$\begin{aligned} \|u(t)\|^2 - 2\|u_0\|^2 &\leq 2\Big(\int_0^t \|u'(s)\|\,ds\Big)^2 \leq C\Big(\int_0^t \|u'(s)\|_{\beta+2}\,ds\Big)^2 \\ &\leq Ct^{2(\beta+1)/(\beta+2)}\Big(\int_0^t \|u'(s)\|_{\beta+2}^{\beta+2}\,ds\Big)^{2/(\beta+2)}\,, \end{aligned}$$

and hence, from (2.7),

$$\begin{aligned} \textstyle\int_0^t \|u'(s)\|_{\beta+2}^{\beta+2}\,ds &\geq Ct^{-(\beta+1)}\{\|u(t)\|^2 - 2\|u_0\|^2\}^{(\beta+2)/2} \\ &\geq C_4 t \qquad \text{for large } t > 0 \end{aligned}$$

with some positive constant C_4.

Thus, we have from the energy identity (1.6) that

$$E(t) \leq E(0) - \int_0^t \|u'(s)\|_{\beta+2}^{\beta+2}\, ds \leq E(0) - C_4 t$$

for large $t > 0$, that is, there exists a real number $T_* > 0$ such that

$$E(T_*) < 0 .$$

By Theorem 1.2, we arrive at the conclusion.

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