A Note on Higher Deflections of a Local Ring

By

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Introduction.

Let $R$ be a commutative Noetherian local ring with maximal ideal $m$ and residue field $K$. To $R$ we associate homological invariants of $R$ so called "deflections" $\varepsilon_i (i=1, 2, \ldots)$. It is well known that $\varepsilon_1 = 0$ (resp. $\varepsilon_2 = 0$) if and only if $R$ is regular (resp. complete intersection). $\varepsilon_1$ and $\varepsilon_2$ are computed in terms of the homology algebra $H(E)$ of the Koszul complex $E$ of $R$: $\varepsilon_1 = \dim_K H_1(E)$ and $\varepsilon_2 = \dim H_2(E)/H_1(E)^2$.

In this note, after proving some lemmas (§1), we will calculate $\varepsilon_3$ and, in some restricted case, $\varepsilon_4$ by means of $H(E)$. As an application we give in §3 an expression of the form of Betti series of $R$ assuming its embedding dimension is 3 and that $H(E)$ has trivial multiplication. This gives an alternating proof of a theorem due to Golod [2].

Unless otherwise specified, we shall use the same notations and the same terminology which appeared in [5].

§1. Preliminary lemmas.

Let $(R, m)$ be a local ring of embedding dimension $n$ and residue field $K$ and let $\{t_1, \ldots, t_n\}$ be a minimal system of generators of $m$. By the method of killing cycles, we have a following sequence of $R$-algebras $X^{(i)} (i=0, 1, 2, \ldots)$ [5, §1];

$X^{(0)} = R, \; X^{(1)} = E = R < T_1, \ldots, T_n >; \; dT_i = t_i,$

$X^{(2)} = X^{(1)} < S_1, \ldots, S_{\varepsilon_1} >; \; dS_i = s_i, \; X^{(3)} = X^{(2)} < U_1, \ldots, U_{\varepsilon_2} >; \; dU_i = u_i,$

$X^{(4)} = X^{(3)} < V_1, \ldots, V_{\varepsilon_3} >; \; dV_i = v_i; \; \cdots$

where $T_i, S_i, U_i, V_i, \ldots$ are variables of degree $1, 2, 3, 4, \ldots$ which kill cycles $t_i, s_i, u_i, v_i, \ldots$ respectively. The $i$-th deflection $\varepsilon_i (i=1, 2, \ldots)$ is defined by $\dim_K H_i(X^{(i)})$ but this is also equal to $\dim_K H_i(X^{(i-1)})$, if $i \geq 3$, as we see in the following lemma.
Lemma 1. Let $X$ be an $R$-algebra and $\rho \geq 3$ be an integer such that $H_0(X) = K$ and $H_i(X) = 0$ for $0 < i \leq \rho - 2$. If $s \in Z_{\rho - 1}(X)$ is not a boundary, then $Y = X < S$; $dS = s$, deg $S = \rho$, satisfies $H_{\rho + \rho}(X) \approx H_{\rho + \rho}(Y)$ for $\rho - 3 \geq \mu \geq 0$. Consequently, $\varepsilon_i = \dim_K H_i(X^{(i-1)})$, if $i \geq 3$.

Proof. $\rho$ odd: From the exact sequence $0 \to X_\rho \to Y_\rho \to X_{\rho - \rho} \to 0$, we obtain the exact sequence

$$\cdots \to H_{\mu + 1}(X) \to H_{\rho + \mu}(X) \to H_{\rho + \mu}(Y) \to H_{\rho}(X) \to \cdots$$

$$\cdots \to H_1(X) \to H_0(X) \to H_0(Y) \to H_0(X)^{d_{0,0}} \to \cdots.$$  

Since $H_0(X) = K$ and $d_{0,0}$ is the multiplication by $\sigma$, the homology class of $s$, $d_{0,0}$ is injective so that $H_0(X) \approx H_0(Y)$. If $\rho - 3 \geq \mu > 0$, we have $H_{\rho + 1}(X) = H_0(X) = 0$ and hence $H_{\rho + \mu}(X) \approx H_{\rho + \mu}(Y)$.

$\rho$ even: In this case the sequence $0 \to X_\rho \to Y_\rho \to Y_{\rho - \rho} \to 0$ is exact, hence

$$\cdots \to H_1(Y) \to H_0(Y) \to H_0(Y) \to H_{\rho - 1}(X) \to H_{\rho - 1}(Y) \to 0 \text{ (exact).}$$

On one hand, since $d_{0,0} \cdot \sigma$: $H_0(X) \to H_0(Y) \to H_{\rho - 1}(X)$ is obtained by the multiplication by $\sigma$, $d_{0,0}$ is injective so that $H_0(X) \approx H_0(Y)$ since clearly we have $H_1(Y) = 0$. If $\rho - 3 \geq \mu > 0$, we have $H_\mu(Y) \approx H_\mu(X) = 0$ and $H_{\mu + 1}(X) \approx H_{\mu + 1}(X) = 0$. Whence, the exact sequence

$$\cdots \to H_{\mu + 1}(Y) \to H_{\rho + \mu}(X) \to H_{\rho + \mu}(Y) \to H_\mu(Y) \to \cdots$$

implies $H_{\rho + \mu}(X) \approx H_{\rho + \mu}(Y)$.

We need also the following lemma for calculating higher deflections.

Lemma 2. Let $w_1, \ldots, w_{\varepsilon}$ be a set of $\varepsilon_i$ cycles in $X^{(i)}$ whose homology classes constitute a base of the $K$-vector space $H_i(X^{(i)})$ ($i = 2, 3, \ldots$). Then, $w_j$ ($j = 1, 2, \ldots, \varepsilon$) can be selected in $X^{(i-1)}$.

Proof. Let $X^{(i)} = X^{(i-1)} < P_1, \ldots, P_{\varepsilon_{i-1}}$; $dP_j = \pi_j$, deg $P_j = i$. Put $w_j = w$ and $\varepsilon_{i-1} = \varepsilon$. Write $w = w' + \sum_{k=1}^{\varepsilon} r_k P_k$, where $w' \in X^{(i-1)}$ and $r_k \in R$. Then, $0 = dw = dw' + \sum_{k=1}^{\varepsilon} r_k \pi_k$. Since $\pi_1, \ldots, \pi_\varepsilon$ are linearly independent cycles (modulo $B(X^{(i-1)})$, each $r_k \in m$. Take $P_k \in X^{(i-1)}$ such that $r_k = dP_k$. Then, $\sum r_k P_k = \sum (dP_k) P_k = d(\sum P_k P_k) + \sum P_k \pi_k$. Hence, $w = w' + \sum P_k \pi_k + d(\sum P_k P_k)$ and consequently $w' + \sum P_k \pi_k$ is a cycle in $X^{(i-1)}$ and is homologous to $w$.

Corollary. $u_i$ ($i = 1, 2, \ldots, \varepsilon_2$) can be selected in $Z_2(E)$. $v_i$ ($i = 1, 2, \ldots, \varepsilon_3$) can be selected in $Z_3(X^{(2)})$.

First we construct special cycles in $Z_3(X^{(2)})$. For this we fix a set $I$ of pairs of integers $(p, q)$ ($1 \leq p < q \leq \varepsilon_1$) such that homology classes of $s_{p,q}$,
$(p, q) \in I$, constitute a base of the vector space $H_1(E)^2$. And, we put $J=\{(i, j) \mid 1 \leq i < j \leq \varepsilon_1, (i, j) \in I\}$.

Now, for any set $\{x_1^{(j)} \in Z_1(E) \mid j = 1, \ldots, \varepsilon_1\}$, we can find $r_{pq} \in R, (p, q) \in I$, and $x_3 \in E_3$ such that

$$v_x = x_3 + \sum_{i=1}^{\varepsilon_1} x_1^{(i)} S_i - \sum_{(p, q) \in I} r_{pq} s_p s_q$$

belongs to $Z_3(X^{(2)})$ and moreover $x_3 - \sum_{(p, q) \in I} r_{pq} s_p s_q$ is defined uniquely up to modulo $B_3(X^{(2)}) + Z_3(E)$. In fact, we have $\sum x_1^{(i)} S_i = \sum r_{pq} s_p s_q + d x_3$, for some $r_{pq} \in R$ and $x_3 \in E_3$ and hence $v_x \in Z_3(X^{(2)})$. To see the second part, it is enough to show that the relation, $\sum r_{pq} s_p s_q + d x_3 = 0$, implies $x_3 - \sum r_{pq} s_p s_q \in B_3(X^{(2)}) + Z_3(E)$. Now, by the definition of $I$, each $r_{pq} \in m$ and hence $r_{pq} = d P_{pq}$ for some $P_{pq} \in E_1$ and whence $\sum r_{pq} s_p s_q + d x_3 = d (\sum P_{pq} s_p s_q) + x_3$. Therefore $\sum P_{pq} s_p s_q + x_3 \in Z_3(E)$ and, consequently, $x_3 - \sum r_{pq} s_p s_q = x_3 - d (\sum P_{pq} s_p s_q) + \sum P_{pq} s_p s_q \in Z_3(E) + B_3(X^{(2)})$.

We remark that, if $x_1^{(i)} \in B_1(E) \ (i = 1, \ldots, \varepsilon_1)$, then $v_x \in Z_3(E) + B_3(X^{(2)})$.

For, we put $x_1^{(i)} = d y_2^{(i)}$ with $y_2^{(i)} \in E_2$. Then, the relation, $0 = d v_x = d (x_3 + \sum (d y_2^{(i)} S_i - \sum r_{pq} s_p s_q) = d (x_3 - \sum y_2^{(i)} s_i) + \sum r_{pq} s_p s_q)$, implies that each $r_{pq} \in m$. Take $P_{pq} \in E_1$ such that $d P_{pq} = r_{pq}$. Then, we see easily that $x_3 - \sum y_2^{(i)} s_i + \sum P_{pq} s_p s_q \in Z_3(E)$. Hence, we have $v_x = x_3 + \sum (d y_2^{(i)} S_i - \sum (d P_{pq}) s_p s_q) = x_3 + \{d (\sum y_2^{(i)} S_i - \sum y_2^{(i)} s_i) - \{d (\sum P_{pq} s_p s_q) - \sum P_{pq} s_p s_q\} = \{x_3 - \sum y_2^{(i)} s_i + \sum P_{pq} s_p s_q\} + \sum (d y_2^{(i)} S_i - \sum P_{pq} s_p s_q) \in Z_3(E) + B_3(X^{(2)})$.

In particular, if $\{x_1^{(i)} \ldots, x_1^{(\varepsilon_1)}\} = \{0, \ldots, 0, s_i, 0, \ldots, 0\}$, then

$$v_{ij} = w_{ij} + s_i S_j - \sum_{(p, q) \in I} r_{pq}^{(ij)} s_p s_q$$

belongs to $Z_3(X^{(2)})$ for some $w_{ij}$ in $E_3$ and $r_{pq}^{(ij)}$ in $R$. And, moreover, these $v_{ij}$ can be imposed on the following conditions:

$$v_{ij} + v_{ji} = d (S_i S_j) \text{ if } i \neq j \text{ and } v_{ii} = d (S_i^{(2)})$$

For, by the above construction of $v_{ij}$, $w_{ij}$ and $r_{pq}^{(ij)}$ can be selected so as to $s_i S_j = \sum r_{pq}^{(ij)} s_p s_q + d w_{ij}$ holds and therefore we can assume $w_{ij}$ and $r_{pq}^{(ij)}$ satisfy the relations, $w_{ij} + w_{ji} = 0$, $w_{ii} = 0$, $r_{pq}^{(i) + r_{pq}^{(j)}} = 0$ and $r_{pq}^{(i) + r_{pq}^{(j)}} = 0$, by virtue of $s_i S_j = -s_j S_i$ and $s_i S_i = 0$. Consequently, $v_{ij} + v_{ji} = s_i S_j + s_j S_i = d (S_i S_j)$ and $v_{ii} = s_i S_i = d (S_i^{(2)})$.

**Lemma 3.** $Z_3(X^{(2)}) = B_3(X^{(2)}) + Z_3'(E) + \sum_{(i, j) \in I} R_{v_{ij}}$, where $Z_3'(E)$ is an $R$-submodule of $Z_3(E)$ generated by cycles in $E_3$ whose homology classes constitute a base of $H_3(E)$ modulo $H_1(E)H_2(E)$. 
PROOF. We first remark that $Z_1(E)Z_2(E) \subset B_3(X^{(2)})$. In fact, let $x \in Z_1(E)$ and $y \in Z_2(E)$. Then, $x = \sum \lambda_i s_i + x'$ where $\lambda_i \in R$ and $x' \in B_1(E)$. Fix $x''$ in $E_2$ such that $dx'' = x'$. Then, $xy = (\sum \lambda_i s_i + x')y = (\sum \lambda_i (dS_i))y + (dx'')y = d((\sum \lambda_i S_i)y + x''y) \in B_3(X^{(2)})$. Hence, to prove the lemma, it is enough to show that $Z_3(X^{(2)}) \subset B_3(X^{(2)}) + Z_2(E) + \sum_i Rv_{ij}$. Let $v = x_3 + \sum_{i=1}^{\xi_1} x_1^{(i)} s_i \in Z_3(X^{(2)})$, then $x_1^{(i)} s_i \in Z_1(E) (i=1, \ldots, \xi_1)$ and hence each $x_1^{(i)}$ is an $R$-linear combination of $s_1, \ldots, s_{\xi_1}$ modulo $B_1(E)$. Therefore, subtracting suitable $R$-linear combinations of $v_{ij}$ from $v$, we can assume each $x_1^{(i)}$ is contained in $B_1(E)$. Hence our conclusion follows from the remarks stated before the lemma.

§ 2. The calculation of higher deflections.

By making use of the lemmas of the preceding section, we can prove the following theorem.

THEOREM 1. $\varepsilon_3 = \dim_K H_3(E)/H_1(E)H_2(E) = \frac{\xi_1}{2} \dim_K H_1(E)^2$.

In particular, if $H_1(E)^2 = H_1(E)H_2(E) = 0$, then $\varepsilon_3 = \dim_K H_3(E) + \frac{\xi_1}{2}$.

PROOF. Let $h = \dim_K H_3(E)/H_1(E)H_2(E)$ and let $\pi_1, \ldots, \pi_h$ be cycles in $Z_3(E)$ such that whose homology classes constitute a base of the vector space $H_3(E)$ modulo $H_1(E)H_2(E)$. To prove our theorem, in view of lemma 1 and 3, it is enough to show that $\pi_i$ $(i=1, \ldots, h)$ and $v_{ij}$, $(i, j) \in J$, are linearly independent over $K$ modulo $B_3(X^{(2)})$. For this, suppose that

$$x = \sum_{i=1}^{\xi_2} \alpha_i \pi_i + \sum_{(i, j) \in J} \beta_{ij} v_{ij} \in B_3(X^{(2)}) = d(E_4 + \sum_{k=1}^{\xi_1} E_k s_k + \sum_{1 \leq i < j \leq \xi_1} RS_i s_j + \sum_{i=1}^{\xi_1} R S^{(2)}_i),$$

where $\alpha_i$ and $\beta_{ij}$ are elements in $R$ and we contend that $\alpha_i$, $\beta_{ij} \in m$.

Now,

$$x = d(x_4 + \sum x_2^{(i)} s_i + \sum \mu_{ij} s_i s_j + \sum \nu_i S_i^{(2)}) (x_4 \in E_4, x_2^{(i)} \in E_2 \text{ and } \mu_{ij}, \nu_i \in R)$$

$$= (dx_2^{(i)} + \nu_1 s_1 + \mu_{12} s_2 + \cdots + \mu_{1, \xi_1} s_{\xi_1}) S_1 + (d x_2^{(2)} + \mu_{12} s_1 + \nu_2 s_2 + \mu_{23} s_3 + \cdots + \mu_{2, \xi_1} s_{\xi_1}) S_2 + \cdots + (dx_2^{(i)} + \mu_{\xi_1} s_1 + \cdots + \mu_{\xi_1-1, \xi_1} s_{\xi_1} + \nu_{\xi_1} s_{\xi_1}) S_{\xi_1} + dx_4$$

$$+ \sum x_2^{(k)} s_k.$$

Since

$$\beta_{ij} v_{ij} = \beta_{ij} (w_{ij} + s_i s_j - \sum_{T} r_{pq}^{(ij)} s_p S_q),$$

we have
\[ \sum \alpha_i \pi_i = \{dx_2^{(1)} + \nu_1 s_1 + \mu_{12} s_2 + \cdots + \mu_{1 \varepsilon_i} s_{\varepsilon_i}\} S_1 + \{dx_2^{(2)} + (\mu_{12} - \beta'_{12}) s_1 \\
+ \nu_2 s_2 + \mu_{23} s_3 + \cdots + \mu_{2 \varepsilon_i} s_{\varepsilon_i}\} S_2 + \cdots + \{(dx_2^{(\ell)} + (\mu_{1 \varepsilon_i} - \beta'_{1 \varepsilon_i}) s_1 + \cdots \\
+ (\mu_{\varepsilon_i-1, \varepsilon_i} - \beta'_{\varepsilon_i-1, \varepsilon_i}) s_{\varepsilon_i-1} + \nu_{\varepsilon_i} s_1\} S_{\varepsilon_i} + dx_4 + \sum x_2^{(k)} s_k - \sum \beta_{ij} w_{ij}, \]

where

\[ \beta'_{kk} = \begin{cases} \beta_{hh} & \text{if } (k, h) \in J, \\ - \sum_{(i, j) \in J} r_{x_{ki}^{(ij)}} \beta_{ij} & \text{if } (k, h) \in I. \end{cases} \]

Considering the coefficients of \( S_i (i = 1, 2, \ldots, \varepsilon_i) \), we get

\[ \nu_1 s_1 + \mu_{12} s_2 + \cdots + \mu_{1 \varepsilon_i} s_{\varepsilon_i} = -dx_2^{(1)} \in B_1(E) \]
\[ (\mu_{12} - \beta'_{12}) s_1 + \nu_2 s_2 + \cdots + \mu_{2 \varepsilon_i} s_{\varepsilon_i} = -dx_2^{(2)} \in B_1(E) \]
\[ \cdots \]
\[ (\mu_{\varepsilon_i-1, \varepsilon_i} - \beta'_{\varepsilon_i-1, \varepsilon_i}) s_{\varepsilon_i-1} + \nu_{\varepsilon_i} s_1 = -dx_2^{(\ell_i)} \in B_1(E). \]

Hence, \( \nu_i (i = 1, \ldots, \varepsilon_i) \), \( \mu_{ij} \) and \( \beta_{ij}^{(i)} (1 \leq i < j \leq \varepsilon_i) \) each belongs to \( \mathfrak{m} \), since \( s_i (i = 1, \ldots, \varepsilon_i) \) are linearly independent over \( K \) modulo \( B_1(E) \). In particular, \( \beta_{ij} \in \mathfrak{m} \) for any \((i, j) \in J \).

Take \( P_i, Q_{ij} \) and \( R_{ij} \) in \( E_1 \) such that

\[ \nu_i = dP_i, \quad \mu_{ij} = dQ_{ij}, \quad \beta_{ij}^{(i)} = dR_{ij}. \]

It is clear that \( R_{ij} (1 \leq i < j \leq \varepsilon_i) \) can be imposed on the following relation:

\[ R_{pq} = - \sum_{(i, j) \in J} R_{ij} r_{pq}^{(ij)} \text{ for } (p, q) \in I. \]

Then, obviously,

\[ \nu_i s_i = d(P_i s_i), \quad \mu_{ij} s_k = d(Q_{ij} s_k), \quad \beta_{ij}^{(i)} s_k = d(R_{ij} s_k). \]

Hence

\[ x_2^{(1)} + P_1 s_1 + Q_{12} s_2 + \cdots + Q_{1 \varepsilon_i} s_{\varepsilon_i} \in Z_2(E) \]
\[ x_2^{(2)} + (Q_{12} - R_{12}) s_1 + P_{2} s_2 + \cdots + Q_{2 \varepsilon_i} s_{\varepsilon_i} \in Z_2(E) \]
\[ \cdots \]
\[ x_2^{(\ell_i)} + (Q_{\varepsilon_i-1, \varepsilon_i} - R_{\varepsilon_i-1, \varepsilon_i}) s_{\varepsilon_i-1} + P_{\varepsilon_i} s_1 \in Z_2(E) \]

and we denote these cycles by \( \gamma_2^{(1)}, \ldots, \gamma_2^{(\ell_i)}. \) Then,
\[ \sum_{i=1}^{h} \alpha_i \pi_i = d x_4 + \sum_{k=1}^{\varepsilon_1} \gamma_2^{(k)} x_k - \sum_j \beta_{ij} w_{ij} \]

\[ = d x_4 + \sum_j \gamma_2^{(k)} x_k - \{(P_{i_1} + Q_{1_2} s_2 + \cdots + Q_{1_{\varepsilon_1}} s_{\varepsilon_1}) s_1 + \cdots + ((Q_{1_{\varepsilon_1}} - R_{1_{\varepsilon_1}}) s_1 + \cdots + (Q_{\varepsilon_{i-1} \varepsilon_1} - R_{\varepsilon_{i-1} \varepsilon_1}) \phi_{\varepsilon_{i-1}} + P_{\varepsilon_{i}} s_{\varepsilon_{i}}) s_{\varepsilon_{i}} \} - \sum_j \beta_{ij} w_{ij} \]

\[ = d x_4 + \sum_j \gamma_2^{(k)} x_k + \sum_{1 \leq k < h \leq \varepsilon_1} R_{kk} s_k - \sum_j \beta_{ij} w_{ij}, \]

in view of \( s_{ij} + s_{ji} = 0 \) (\( i \neq j \)) and \( s_{ii} = 0 \).

Since

\[ \sum_{1 \leq k < h \leq \varepsilon_1} R_{kk} s_k - \sum_j \beta_{ij} w_{ij} = \sum_j R_{pq} s_p s_q + \sum_j R_{ij} s_i s_j - \sum_j \beta_{ij} w_{ij} \]

\[ = \sum_j R_{pq} s_p s_q + \sum_j R_{ij} (\sum_r r_{pq}^{(i)} s_r s_q + dw_{ij}) - \sum_j \beta_{ij} w_{ij} \]

\[ = \sum_j \{ R_{pq} + (\sum_j R_{ij} r_{pq}^{(i)}) \} s_p s_q + \sum_j R_{ij} dw_{ij} - \sum_j \beta_{ij} w_{ij} \]

\[ = -d(\sum_j R_{ij} w_{ij}), \]

we finally have

\[ \sum_{i=1}^{h} \alpha_i \pi_i = d x_4 + \sum_{k=1}^{\varepsilon_1} \gamma_2^{(k)} x_k - d(\sum_j R_{ij} w_{ij}) \in Z_1(E) Z_2(E) + B_3(E) \]

and consequently \( \alpha_i \in m \) (\( i = 1, \ldots, h \)), which complete our proof.

Next, we compute \( \varepsilon_4 \) in some restricted case. Since our computation is quite similar to that of \( \varepsilon_3 \), the detail of it shall be omitted.

**Lemma 4.** If \( n \leq 3 \) and \( H_1(E)^2 = 0 \), then we have

\[ \varepsilon_4 = \varepsilon_1 \varepsilon_2 - \dim_K H_1(E) H_2(E). \]

**Proof.** We have proved that \( \varepsilon_4 = \dim H_4(X^{(3)}) \) (lemma 1) and \( \varepsilon_2 = \dim H_3(E) \) by our assumption. Let \( I \) be a set of integers \( (p, q), 1 \leq p \leq \varepsilon_1, 1 \leq q \leq \varepsilon_2 \) such that homology classes of \( s_{p q} U_q \) (\( p, q \) \( i \) \( I \)) form a base of the vector space \( H_1(E) H_2(E) \), and let \( J = \{(i, j) | 1 \leq i \leq \varepsilon_1, 1 \leq j \leq \varepsilon_2, (i, j) \in I\} \). Then, for any \( (i, j) \in J \), we can find \( r_{p q}^{(i j)} \in R, (p, q) \in I \), such that

\[ z_{ij} = s_i U_j - \sum_j r_{p q}^{(i j)} s_p U_q \]

belongs to \( Z_4(X^{(3)}) \). With these \( z_{ij}, (i, j) \in J \), we can prove \( Z_4(X^{(3)}) = B_4(X^{(3)}) + \sum_j R_{z ij} \) and, moreover, these \( z_{ij} \) are linearly independent cycles modulo \( B_4(X^{(3)}) \).
§3. An application to the Betti series of local rings of embedding dimension 3.

In this section we restrict the case when the embedding dimension $n$ is 3 and consider the Betti series of $R$ under the additional assumption that the multiplication in $H(E)$ is trivial, i.e., we assume $H_1(E)^2 = H_1(E) H_1(E) = 0$. Hence

$$Z_1(E)^2 \subset B_2(E), \quad Z_1(E)Z_2(E) \subset B_3(E) = 0, \quad H_3(E) = Z_3(E) \approx 0: m.$$ 

Our assumption also implies that, with the same notations as in §1, $I = \emptyset$ (empty set), $J = \{(i, j) | 1 \leq i < j \leq \varepsilon_1\}$ and $r_{ij} = 0$ so that

$$v_{ij} = w_{ij} + s_i S_j (1 \leq i < j \leq \varepsilon_1).$$

Let $X$ be a minimal $R$-algebra resolution of the residue field $K$ of $R$ \([3, 5, 7]\).

$$X: \cdots \rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow K \rightarrow 0,$$

where $\varepsilon$ is the augmentation homomorphism. Then, $X_i(i = 1, 2, \ldots)$ has the following form:

$$X_0 = R, \quad X_1 = E_1, \quad X_2 = E_2 + \sum_{i=1}^{\varepsilon_1} RS_i, \quad X_3 = E_3 + \sum_{j=1}^{\varepsilon_1} E_1 S_j + \sum_{i=1}^{\varepsilon_3} R U_i,$$

$$X_4 = \sum_{k=0}^{\varepsilon_1} E_2 S_k + \sum_{1 \leq i < j \leq \varepsilon_1} RS_i S_j + \sum_{i=1}^{\varepsilon_1} RS_i^{(2)} + \sum_{j=1}^{\varepsilon_3} E_1 U_j + \sum_{i=1}^{\varepsilon_3} R V_i \cdots$$

where $E = \{E_i\}_{i=0, 1, 2, 3}$ is the Koszul complex of $R$ and $S_i(i = 1, \ldots, \varepsilon_1)$, $U_i(i = 1, \ldots, \varepsilon_3)$ are variables of degree 2, 3 and 4 respectively.

Let $M = dX_1$ and let $\{c_1, \ldots, c_3\}$ be a minimal generating system of $0: m$ ($\delta = \dim_K (0: m)$). Since we can take $c_i T_1 T_2 T_3(i = 1, \ldots, \delta)$ and $v_{ij}(1 \leq i < j \leq \varepsilon_1)$ as $v_{11}, \ldots, v_{\varepsilon_1}$ (theorem 1), $M$ can be written as $M = M_1 + M_2 + M_3$, where

$$M_1 = d(\sum E_2 S_k + \sum RS_i S_j + \sum RS_i^{(2)}) + \sum RV_{ij},$$

$$M_2 = d(\sum E_1 U_j),$$

$$M_3 = (0: m) T_1 T_2 T_3.$$

**Lemma 5.** $M_1 \cong \bigoplus N$, where $N = dX_2 = B_1(E) + \sum R s_i$.

**Proof.** Let $x \in M_1$. Then

$$x = d(\sum x_2^{(k)} S_k + \sum \mu s_i S_j + \sum \nu s_i^{(2)}) + \sum \beta v_{ij} \quad (x_2^{(k)} \in E_2, \mu, \nu, \beta \in R)$$

$$= (dx_2^{(1)} + \nu s_1 + \mu_1 s_2 + \cdots + \mu_2 s_\varepsilon) s_1 + (dx_2^{(2)} + (\mu_1 + \beta_1) s_1 + \nu s_2$$

$$+ \mu_2 s_3 + \cdots + \mu_\varepsilon s_\varepsilon) s_2 + \cdots + (dx_2^{(\varepsilon_1)} + (\mu_1 + \beta_1) s_1 + \cdots +$$

$$(\mu_\varepsilon + \beta_\varepsilon) s_\varepsilon + \nu s_\varepsilon) s_\varepsilon + \sum x_2^{(k)} s_k + \sum \beta v_{ij}.$$
If \( \phi: M_1 \rightarrow \bigoplus N \) is an \( R \)-homomorphism defined by
\[
\phi(x) = (dx^{(1)}_2 + \nu_1 s_1 + \cdots) s_1 + \cdots + (dx^{(\ell_1)}_2 + (\mu_{1,\ell_1} + \beta_{1,\ell_1}) s_1 + \cdots) s_{\ell_1},
\]
i.e., the projection of \( M_1 \) on the sum of its first \( \varepsilon_1 \)-factors, then clearly \( \phi \) is surjective.

Now, we shall show that \( \phi \) is injective. Assume \( x \in \text{Ker} \phi \). Then,
\[
x = \sum_{k=1}^{\ell_1} x^{(k)}_2 s_k + \sum_{i<j} \beta_{ij} w_{ij}
\]
and
\[
\begin{align*}
&dx^{(1)}_2 + \nu_1 s_1 + \mu_{1,2} s_2 + \cdots + \mu_{1,\ell_1} s_{\ell_1} = 0 \\
&dx^{(2)}_2 + (\mu_{1,2} + \beta_{1,2}) s_2 + \cdots + \mu_{2,\ell_1} s_{\ell_1} = 0 \\
&\cdot \cdot \cdot \\
&dx^{(\ell_1)}_2 + (\mu_{1,\ell_1} + \beta_{1,\ell_1}) s_1 + \cdots + (\mu_{\ell_1-1,\ell_1} + \beta_{\ell_1-1,\ell_1}) s_{\ell_1-1} + \nu_{\ell_1} s_{\ell_1} = 0
\end{align*}
\]
Hence, \( \nu, \mu \) and \( \beta \) \( \in M \) and
\[
\begin{align*}
\gamma^{(1)}_2 &= x^{(1)}_2 + P_1 s_1 + Q_{1,2} s_2 + \cdots + Q_{1,\ell_1} s_{\ell_1} \in Z_2(E) \\
\gamma^{(2)}_2 &= x^{(2)}_2 + (Q_{1,2} + R_{1,2}) s_1 + P_2 s_2 + \cdots + Q_{2,\ell_1} s_{\ell_1} \in Z_2(E) \\
\cdot \cdot \cdot \\
\gamma^{(\ell_1)}_2 &= x^{(\ell_1)}_2 + (Q_{1,\ell_1} + R_{1,\ell_1}) s_1 + \cdots + (Q_{\ell_1-1,\ell_1} + R_{\ell_1-1,\ell_1}) s_{\ell_1-1} + P_{\ell_1} s_{\ell_1} \in Z_2(E),
\end{align*}
\]
where \( P, Q, R \) \( \in E_1 \) such that \( dP = \nu, dQ_{ij} = \mu_{ij} \) and \( dR_{ij} = \beta_{ij} \).
Therefore, we have
\[
x = \sum \gamma^{(i)}_2 s_i = \{(P_1 s_1 + Q_{1,2} s_2 + \cdots + Q_{1,\ell_1} s_{\ell_1}) s_1 + \cdots + ((Q_{1,\ell_1} + R_{1,\ell_1}) s_1 + \cdots + (Q_{\ell_1-1,\ell_1} + R_{\ell_1-1,\ell_1}) s_{\ell_1-1} + P_{\ell_1} s_{\ell_1}) s_{\ell_1}\} + \sum \beta_{ij} w_{ij}
\]
\[
= - \sum R_{ij} s_i s_j + \sum \beta_{ij} w_{ij} \quad \text{(since \( \sum \gamma^{(i)}_2 s_i \in Z_2(E) Z_1(E) = 0 \)}
\]
\[
= d (\sum R_{ij} w_{ij})
\]
\[
= 0,
\]
since \( \sum R_{ij} w_{ij} \in E_1 = 0 \).

**Lemma 6.** \( M_2 \cong \bigoplus m \) and \( M_3 \cong \bigoplus K \), where \( \delta = \dim_K (0: m) = \ell_3 - \frac{(\ell_1^2)}{2} \).

**Proof.** Since \( M_2 = d(\sum_{j=1}^{\ell_1} E_1 U_j) \), \( x \in M_2 \) can be written as
\[ x = \sum_{j=1}^{\delta_1} (dx_1^{(j)}) U_j - \sum_{j=1}^{\delta_2} x_1^{(j)} u_j, \]

where \( x_1^{(j)} \in E_1 \) and \( u_j \in Z_2(E) \) (lemma 2). Let \( \phi: M_2 \to \sum_{j=1}^{\delta_1} mU_j \) be an \( R \)-homomorphism defined by

\[ \phi(x) = \sum_{j=1}^{\delta_1} (dx_1^{(j)}) U_j. \]

Obviously, \( \phi \) is surjective. If \( x \in \text{Ker} \phi \), then \( dx_1^{(j)} = 0 \) \( (j=1, \ldots, \delta_2) \) so that \( x_1^{(j)} \in Z_1(E) \) and, consequently, \( x = - \sum x_1^{(j)} u_j \in Z_1(E)Z_2(E) = 0. \)

For the second assertion of the lemma, we consider the map \( \eta: \oplus R \to 0: m \) defined by

\[ \eta(\oplus r_i) = \sum_{i=1}^{\delta} r_i c_i, \]

where \( \{c_1, \ldots, c_3\} (\delta = \dim (0: m) = \varepsilon_3 - \left(\frac{\varepsilon_1}{2}\right)) \), by theorem 1) is a fixed minimal system of generators of \( 0: m \). It is clear that \( \eta \) induces a bijection between \( \oplus \) \( K \) and \( 0: m \).

**Lemma 7.** \( M = M_1 \oplus M_2 \oplus M_3 \) (direct).

**Proof.** Let \( \theta_i \in M_i \) \( (i=1, 2, 3) \) and suppose \( \theta_1 + \theta_2 + \theta_3 = 0. \) Then, with the same notations of the preceding lemmas, we write \( \theta_i \) as

\[ \theta_1 = d(\sum_{h=1}^{\delta_1} x_2^{(i)} S_h + \sum_{i<j} \mu_{ij} S_i S_j + \sum_{i=1}^{\delta_2} \nu_i S_i^{(2)}) + \sum_{i<j} \beta_{ij} v_{ij} \]

\[ \theta_2 = \sum_{j=1}^{\delta_2} (dx_1^{(j)}) U_j - \sum_{j=1}^{\delta_2} x_1^{(j)} u_j \]

\[ \theta_3 = \sum_{i=1}^{\delta} r_i c_i T_1 T_2 T_3, \quad \delta = \varepsilon_3 - \left(\frac{\varepsilon_1}{2}\right). \]

Considering the coefficients of \( U_j (j=1, \ldots, \varepsilon_2) \), we have \( dx_1^{(j)} = 0 \) so that \( \theta_2 \in \text{Ker} \phi \) and hence \( \theta_2 = 0 \) by lemma 6.

Now we have \( \theta_1 + \theta_3 = 0. \) Then, each coefficient of \( S_i (i=1, \ldots, \varepsilon_1) \) is equal to zero and hence \( \theta_1 \in \text{Ker} \phi \). This implies \( \theta_1 \) is actually zero by lemma 5 and \( \theta_3 = 0. \)

**Theorem 2.** Let \( (R, m) \) be a local ring of embedding dimension 3. Suppose that the multiplication in \( H(E) \) is trivial, where \( E \) is the Koszul complex of \( R \). Then, the Betti series \( B(R) \) of \( R \) has the following form:

\[ B(R) = \frac{(1+Z)^3}{1 - \varepsilon_1 Z^2 - \varepsilon_2 Z^3 - (\varepsilon_3 - \left(\frac{\varepsilon_1}{2}\right)) Z^4}, \]
where \( \varepsilon_i \) (\( i = 1, 2, 3 \)) is the \( i \)-th deflection of \( R \). Moreover,
\[
\varepsilon_4 = \varepsilon_1 \varepsilon_2 \quad \text{and} \quad \varepsilon_3 = \varepsilon_2 + \left( \frac{\varepsilon_1 - 1}{2} \right)
\]

**Proof.** By preceding lemmas, we have
\[
M = dX_4 = M_1 \oplus M_2 \oplus M_3 \approx (\oplus N) \oplus (\oplus m) \oplus (\oplus K) \quad (\delta = \varepsilon_3 - \left( \frac{\varepsilon_1}{2} \right)).
\]
Since the functor Tor is additive,
\[
\text{Tor}_p(M, K) = \left( \oplus \text{Tor}_p(N, K) \right) \oplus \left( \oplus \text{Tor}_p(m, K) \right) \oplus \left( \oplus \text{Tor}_p(K, K) \right),
\]
for \( p \geq 0 \). Hence, for \( p \geq 0 \),
\[
\text{Tor}_{p+1}(K, K) = \left( \oplus \text{Tor}_{p+2}(K, K) \right) \oplus \left( \oplus \text{Tor}_{p+1}(K, K) \right) \oplus \left( \oplus \text{Tor}_p(K, K) \right).
\]
Therefore, we have
\[
B_{p+4} = \varepsilon_1 B_{p+2} + \varepsilon_2 B_{p+1} + \delta B_p \quad (p \geq 0).
\]
Now, it is easy to see that this recurrence relation implies the representation of \( \mathcal{A}(R) \) for the first part of the theorem.

For the second part, \( \varepsilon_4 = \varepsilon_1 \varepsilon_2 \) is an immediate consequence of lemma 4. As for \( \varepsilon_3 \), the statement is true for regular local rings since in this case \( \varepsilon_i = 0 \) (\( i = 1, 2, \ldots \)). Assume \( R \) is not regular, then the Euler-Poincaré characteristic of \( E \) is 0, i.e.,
\[
\dim_K H_2(E) - \dim_K H_3(E) + \dim_K H_4(E) - \dim_K H_5(E) = 0.
\]
Combining this with \( \dim H_3(E) = \dim (0 : m) \), \( \varepsilon_2 = \dim H_2(E)/H_1(E)^2 - \dim H_5(E) \), \( \varepsilon_1 = \dim H_1(E) \), \( \dim H_0(E) = 1 \) and \( \varepsilon_3 = \dim (0 : m) + \left( \frac{\varepsilon_1}{2} \right) \) (theorem 1), we obtain
\[
\varepsilon_3 = \varepsilon_2 + \left( \frac{\varepsilon_1 - 1}{2} \right).
\]
We remark here that the rational expression of \( \mathcal{A}(R) \) obtained by G. Scheja, in the case codh \( R \geq n - 2 \) [6, Satz 9], coincides with that given in theorem 2. But, the following is a simple example of a local ring of Krull dimension 0 which satisfies the assumptions in theorem 2:
\[
R = K [[X, Y, Z]]/\alpha,
\]
where \( K \) is a field and \( \alpha \) is defined by \( (X^3 - Y^3, Y^3 - Z^3, XY^2, XZ^2, YZ^2, YX^2, ZX^2, ZY^2) \). Thus, theorem 2 is independent of Scheja's result.

Recently, H. Wiebe showed that if $R$ is a local Gorenstein ring of embedding dimension 3 and is not a complete intersection, then the Betti series of $R$ has the following form:

\[(*) \quad \mathcal{A}(R) = (1 + Z)^3 / 1 - \varepsilon_1 Z^2 - \varepsilon_1 Z^3 + Z^5, \quad \varepsilon_1 = \varepsilon_2 \quad [10].\]

It is to be mentioned that, in his argument, the multiplicative property of $H(E)$ plays an essential role. Precisely, he proved that $H(E)$ satisfies the relations, $H_1(E)^2 = 0$, $H_1(E)H_2(E) = H_3(E)$ and $\dim_k H_3(E) = 1$ and under these conditions he decided the form of the Betti series mentioned above. If we consider higher deflections in this case, we have $\varepsilon_3 = \binom{\varepsilon_1}{2}$ by theorem 1 and $\varepsilon_4 = \varepsilon_1 \varepsilon_2 - 1$ by lemma 4. Thus, (*) can be rewritten as

\[(**) \quad \mathcal{A}(R) = (1 + Z)^3 / 1 - \varepsilon_1 Z^2 - \varepsilon_2 Z^3 - (\varepsilon_3 - \binom{\varepsilon_1}{2}) Z^4 - (\varepsilon_4 - \varepsilon_1 \varepsilon_2) Z^5.\]

On one hand, in the case when $H(E)$ has trivial multiplication, which we treated in theorem 2, we have proved that $\varepsilon_4 = \varepsilon_1 \varepsilon_2$ so that the Betti series of such ring is also given by (**). If $\mathcal{A}(R)$ has the form (**), we can further calculate $\varepsilon_5$ directly and we find $\varepsilon_5 = \varepsilon_1 \varepsilon_3 - \binom{\varepsilon_1}{3} + \varepsilon_2 - \binom{\varepsilon_3}{2}$. And, it is easy to check that, if $\varepsilon_i = 0$ for $i \geq 2$, the polynomial \[(1 - \varepsilon_1 Z^2 - \varepsilon_2 Z^3 - (\varepsilon_3 - \binom{\varepsilon_1}{2}) Z^4 - (\varepsilon_4 - \varepsilon_1 \varepsilon_2) Z^5 - \varepsilon' Z^6, \quad \varepsilon' = \varepsilon_5 - \left\{ \varepsilon_1 \varepsilon_3 - \binom{\varepsilon_1}{3} + \varepsilon_2 - \binom{\varepsilon_3}{2} \right\}, \]

is equal to $1 - Z^2$, $(1 - Z^2)^2$ and $(1 - Z^2)^3$ according to $\varepsilon_1 = 1$, 2 and 3 respectively. Now, we summarize these remarks in the following

**Theorem 3.** If $R$ is of embedding dimension 3 and if $R$ is Gorenstein or $H(E)$ has trivial multiplication, then

\[\mathcal{A}(R) = (1 + Z)^3 / 1 - \varepsilon_1 Z^2 - \varepsilon_2 Z^3 - (\varepsilon_3 - \binom{\varepsilon_1}{2}) Z^4 - (\varepsilon_4 - \varepsilon_1 \varepsilon_2) Z^5 - \varepsilon' Z^6,\]

where $\varepsilon' = \varepsilon_5 - \left\{ \varepsilon_1 \varepsilon_3 - \binom{\varepsilon_1}{3} + \varepsilon_2 - \binom{\varepsilon_3}{2} \right\}$. 

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References


Addendum

Recently, we have informed from T. H. Gulliksen that the same result as Theorem 1 (§2) have been obtained by G. Levin. See T. H. Gulliksen and G. Levin: Homology of local rings, Queen's papers in pure and applied mathematics-No. 20, 1969.