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Kernel Theorem for Fourier Hyperfunctions

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Abstract

An appropriate general version of the kernel theorem of L. Schwartz is formulated for Fourier hyperfunctions and a direct functional analytic proof is presented.

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Introduction

L. Schwartz' kernel theorem has found a wide range of applications in Mathematics, Applied Mathematics and Mathematical Physics. Though it is a relatively abstract result in the theory of generalized functions it seems to have been conceived from the beginning with concrete applications in mind (see Schwartz' talk at the ICM 1950 [18]). Naturally this result has been extended in various directions (for instance [5, 1, 4, 3]). Soon, the abstract core of this result has been formulated as the so-called *abstract kernel theorem* in the context of nuclear spaces (see [17]). The kernel theorem has played an important role in (standard) relativistic quantum field theory (see [21, 19, 9, 8]).

Let G_i be an open nonempty subset of $\mathbb{R}^{n_i}$ for $i = 1,2$ and denote by $\mathcal{D}(G_i)$ the standard Schwartz space of C^∞-functions with compact support in G_i. Then the kernel theorem of L. Schwartz states that every separately continuous bilinear form b on $\mathcal{D}(G_1) \times \mathcal{D}(G_2)$ comes from a unique distribution T on $G_1 \times G_2$, i.e., there is a unique $T \in \mathcal{D}(G_1 \times G_2)'$ such that

$$b(f_1, f_2) = T(f_1 \otimes f_2) \ \ \forall f_i \in \mathcal{D}(G_i),\ i = 1,2.$$

Applied to the present case the abstract kernel theorem guarantees that there is a unique $T \in (\mathcal{D}(G_1)\otimes_\pi \mathcal{D}(G_2))'$ satisfying the above identity. The proof of Schwartz' kernel theorem then requires in addition to identify the completion of the projective tensor product of $\mathcal{D}(G_1)$ and $\mathcal{D}(G_2)$ as $\mathcal{D}(G_1 \times G_2)$.

Another proof has been given by L. Hörmander [7]. This proof is based on a clever 'ansatz' and various somewhat tricky estimates. Later S. Y. Chung et al ([2]) have extended this version of the proof to cover the case of Fourier hyperfunctions on $\mathbb{R}^{n_i}$ relying on Fourier transformation for Fourier hyperfunctions and another clever 'ansatz'.

The space of Sato's hyperfunctions is much larger than the space of Schwartz distributions (for instance, hyperfunctions and Fourier hyperfunctions can be of "infinite order" locally, for this and other basic facts see the book of Kaneko, [10]). Thus it was not clear whether the kernel theorem can be expected to hold for hyperfunctions too.

Apparently the first version of the kernel theorem for Fourier hyperfunctions is due to S. Nagamachi and N. Mugibayashi, [15]. Roughly, their indirect proof goes as follows: At first it is shown that the test function space $\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^n)$ of Fourier hyperfunctions is isomorphic to the space $S_1^1(\mathbb{R}^n)$ of Gelfand of type S (see [6]). Then some results of B. S. Mityagin [12] are used: a) $S_1^1(\mathbb{R}^n)$ is a nuclear space; b) $\otimes^n S_1^1(\mathbb{R})$ is dense in $S_1^1(\mathbb{R}^n)$; c) $\tilde{\otimes}^n S_1^1(\mathbb{R}) = S_1^1(\mathbb{R}^n)$. Finally they proceed as indicated above.

If we denote by $\mathcal{B}(\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k), \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^m))$ the space of separately continuous bilinear forms on $\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k) \times \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^m)$, the result of Nagamachi and Mugibayashi can be formulated as

$$\mathcal{B}(\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k), \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^m)) \cong \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^{k+m})'.$$

However, in our recent investigations in hyperfunction quantum field theory we need a result of the form

$$\mathcal{B}(\underset{\sim}{\mathcal{O}}(K), \underset{\sim}{\mathcal{O}}(L)) \cong \underset{\sim}{\mathcal{O}}(K \times L)'$$

for general closed subsets $K \subset \boldsymbol{D}^k$, respectively $L \subset \boldsymbol{D}^m$. And it is this version of the kernel theorem which we intend to prove in this note.

There are other classes of generalized functions for which the kernel theorem is known to hold, see for instance reference [11].

Our functional analytic strategy of proof is straight forward and direct. First we collect the necessary properties of the relevant test function spaces of Fourier hyperfunctions for the abstract kernel theorem. Then we identify the completion of their projective tensor product in the correct way.

1. Topological preliminaries

The detailed definitions of the various spaces used here can be found in [10] and [15, 16]. Sufficient details about the topologies of these spaces are given below in proof of Lemma 1.2.

Lemma 1.1 *Let K be a closed subset of $\boldsymbol{D}^k$ and L a closed subset of $\boldsymbol{D}^m$. Then $E = \underset{\sim}{\mathcal{O}}(K) \otimes \underset{\sim}{\mathcal{O}}(L)$ is dense in $\underset{\sim}{\mathcal{O}}(K \times L)$.*

Proof. According to Theorem 2.7 of [16] the space $\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k)$ is (sequentially) dense in the space $\underset{\sim}{\mathcal{O}}(K)$ and according to Proposition 3.5 of [15] the tensor product $\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k) \otimes \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^m)$ is dense in the space $\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^{k+m})$. Observe that $K \times L$ is not a subset of $\boldsymbol{D}^{k+m}$, but of $\boldsymbol{D}^k \times \boldsymbol{D}^m \neq \boldsymbol{D}^{k+m}$. Nevertheless one can prove (see the proof of Theorem IV.1 of [14])

$$\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^{k+m}) = \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k \times \boldsymbol{D}^m).$$

It follows that $\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^{k+m})$ is dense in $\underset{\sim}{\mathcal{O}}(K \times L)$, in the same way as in Theorem 2.7 of [16]. Thus we have the following dense inclusions

$$\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k) \otimes \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^m) \subset \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^{k+m}) \subset \underset{\sim}{\mathcal{O}}(K \times L)$$

$$\underset{\sim}{\mathcal{O}}(\boldsymbol{D}^k) \otimes \underset{\sim}{\mathcal{O}}(\boldsymbol{D}^m) \subset \underset{\sim}{\mathcal{O}}(K) \otimes \underset{\sim}{\mathcal{O}}(L).$$

And the statement of the lemma follows. □

Lemma 1.2 *Let K be a closed subset of $\boldsymbol{D}^k$ and L a closed subset of $\boldsymbol{D}^m$. Then $\underset{\sim}{\mathcal{O}}(K \times L)$ induces on $E = \underset{\sim}{\mathcal{O}}(K) \otimes \underset{\sim}{\mathcal{O}}(L)$ a topology τ which is finer than the ε-topology.*

Proof. Recall

$$\underset{\sim}{\mathcal{O}}(K) = \operatorname{ind} \lim_{j \to \infty} \mathcal{O}_j^c(\tilde{U}_j)$$

with a fundamental sequence of neighborhoods $\tilde{U}_j$ of $K \subset \boldsymbol{D}^k$ and similarly

$$\underset{\sim}{\mathcal{O}}(L) = \operatorname{ind} \lim_{j \to \infty} \mathcal{O}_j^c(\tilde{V}_j)$$

with a fundamental sequence of neighborhoods $\tilde{V}_j$ of $L \subset \boldsymbol{D}^m$. Hence a neighborhood of zero in $\underset{\sim}{\mathcal{O}}(K)$ is of the form $U = \operatorname{ach}(\bigcup_{j=1}^{\infty} U_j)$ with neighborhoods of zero U_j in the Banach space $\mathcal{O}_j^c(\tilde{U}_j)$ which can be assumed to be given as follows:

$$U_j = \{g \in \mathcal{O}_j^c(\tilde{U}_j);\ p_j(g) \leq r_j\}$$

for some $r_j > 0$ where the norms p_j are defined by

$$p_j(g) = \sup_{\tilde{U}_j \cap \mathbb{C}^k} e^{\frac{1}{j}|z|}|g(z)|.$$

The absolutely convex hull of a set M is denote by $\mathrm{ach}(M)$.

Similarly a neighborhood of zero V in $\underset{\sim}{\mathcal{O}}(L)$ is of the form $V = \mathrm{ach}(\bigcup_{j=1}^{\infty} V_j)$ with neighborhoods of zero V_j in the Banach space $\mathcal{O}_j^c(\tilde{V}_j)$ given by

$$V_j = \{h \in \mathcal{O}_j^c(\tilde{V}_j);\ q_j(h) \leq \rho_j\}$$

for some $\rho_j > 0$ where the norms q_j are defined by

$$q_j(h) = \sup_{\tilde{V}_j \cap \mathbb{C}^m} e^{\frac{1}{j}|\zeta|}|h(\zeta)|.$$

Then a neighborhood of zero in E for the topology ε is of the form

$$H = \{f \in E;\ q_{U,V}(f) < \delta\} \tag{1}$$

for some $\delta > 0$, some neighborhoods of zero U in $\underset{\sim}{\mathcal{O}}(K)$, respectively V in $\underset{\sim}{\mathcal{O}}(L)$ where

$$q_{U,V}(f) = \sup_{T \in U^o} \sup_{S \in V^o} |\langle T \otimes S, f \rangle|. \tag{2}$$

U^o, respectively V^o denote the absolute polar of U in $\underset{\sim}{\mathcal{O}}(K)'$, respectively of V in $\underset{\sim}{\mathcal{O}}(L)'$.

Neighborhoods of zero W in

$$\underset{\sim}{\mathcal{O}}(K \times L) = \mathrm{ind} \lim_{j \to \infty} \mathcal{O}_j^c(\tilde{U}_j \times \tilde{V}_j)$$

are specified similarly

$$W = \mathrm{ach}(\bigcup_{j=1}^{\infty} W_j) \tag{3}$$

$$W_j = \{f \in \mathcal{O}_j^c(\tilde{U}_j \times \tilde{V}_j);\ \|f\|_j < \sigma_j\} \tag{4}$$

for some $\sigma_j > 0$ and norms $\|\cdot\|_j$ given by

$$\|f\|_j = \sup_{z \in \tilde{U}_j \cap \mathbb{C}^k} \sup_{\zeta \in \tilde{V}_j \cap \mathbb{C}^m} e^{\frac{1}{j}(|z|+|\zeta|)}|f(z,\zeta)|. \tag{5}$$

The topology τ on E is finer than the topology ε if for every neighborhood of zero H for ε there is a neighborhood of zero W as above such that $W \cap E \subset H$ or, since H is absolutely convex,

$$W_j \cap E \subset H, \quad j = 1, 2, \cdots. \tag{6}$$

For $f \in E \cap W_j$, $T \in U^o$, and $S \in V^o$ an elementary calculation shows

$$\langle T \otimes S, f \rangle = \langle T_x, \langle S_y, f(x,y) \rangle \rangle. \tag{7}$$

In order to estimate the right hand side we note first that $\langle S_y, f(\cdot, y) \rangle$ belongs to the space $\underset{\sim}{\mathcal{O}}(K)$ and therefore

$$|\langle T_x, \langle S_y, f(x,y) \rangle \rangle| \leq p_j'(T) p_j(\langle S_y, f(\cdot, y) \rangle).$$

According to the definitions of the norms involved we can continue this estimate as follows:

$$\begin{aligned} p_j(\langle S_y, f(\cdot,y)\rangle) &= \sup_{\tilde{U}_j\cap\mathbb{C}^k} e^{\frac{1}{j}|z|}|(\langle S_y, f(z,y)\rangle| \\ &\le \sup_{\tilde{U}_j\cap\mathbb{C}^k} e^{\frac{1}{j}|z|} q_j'(S) \sup_{\tilde{V}_j\cap\mathbb{C}^m} e^{\frac{1}{j}|\zeta|}|f(z,\zeta)| \\ &= q_j'(S)\|f\|_j \end{aligned}$$

and thus, for $j = 1, 2, \ldots,$

$$|\langle T\otimes S, f\rangle| \le p_j'(T) q_j'(S)\,\|f\|_j \tag{8}$$

where the prime indicates the dual norm. Using standard properties of polar sets and the explicit definitions of the neighborhoods U and U_j respectively V and V_j one deduces that $T \in U^o$, respectively $S \in V^o$, implies $T \in U_j^o$ respectively $S \in V_j^o$ for all $j = 1, 2, \ldots$. According to the definitions of the dual norm and the absolute polar set we get

$$p_j'(T) = \frac{1}{r_j}\sup\{|\langle T, g\rangle| \; g \in V_j\} \le \frac{1}{r_j}$$

and similarly for $S \in V_j^o$: $q_j'(S) \le \frac{1}{\rho_j}$. It follows

$$q_{U,V}(f) \le \frac{1}{r_j\rho_j}\,\|f\|_j\,. \tag{9}$$

Now, given the $\delta > 0$ in the definition of the neighborhood H, choose $\sigma_j > 0$ such that $\sigma_j \le r_j\rho_j\delta$. Then, for all $f \in W_j \cap E$ we know $q_{U,V}(f) < \delta$, hence $W_j \cap E \subset H$ and we conclude. □

Proposition 1.3 *Let K be a closed subset of $\boldsymbol{D}^k$ and L a closed subset of $\boldsymbol{D}^m$. Then*

$$\underset{\sim}{\mathcal{O}}(K)\tilde{\otimes}_{\pi,\varepsilon}\underset{\sim}{\mathcal{O}}(L) = \underset{\sim}{\mathcal{O}}(K\times L)$$

where $\cdots\tilde{\otimes}_{\pi,\varepsilon}\cdots$ denotes the completion of the tensor product $E = \underset{\sim}{\mathcal{O}}(K)\otimes\underset{\sim}{\mathcal{O}}(L)$ with respect to the projective tensor product topology π, respectively with respect to the tensor product topology ε.

Proof. Since the projective topology π is the finest locally convex topology on E we know $\varepsilon \le \pi$ and $\tau \le \pi$ where τ denotes the relative topology on E induced by the topology of $\underset{\sim}{\mathcal{O}}(K\times L)$. Lemma 2.2 shows that $\varepsilon \le \tau$. This then implies for the completions of the space E with respect to these three topologies the following relation

$$\tilde{E}[\pi] \subset \tilde{E}[\tau] \subset \tilde{E}[\varepsilon].$$

By Proposition 2.12 of [13] the spaces $\underset{\sim}{\mathcal{O}}(K)$ and $\underset{\sim}{\mathcal{O}}(L)$ are nuclear, hence Theorem 50.1 of [20] implies that the completions of the tensor product E with respect to the topologies π and ε are equal, i.e., in our notation $\tilde{E}[\pi] = \tilde{E}[\varepsilon]$. Now Lemma 2.1 implies $\tilde{E}[\tau] = \underset{\sim}{\mathcal{O}}(K \times L)$ and thus we conclude. □

2. The kernel theorem for Fourier hyperfunctions

Theorem 2.1 *Let K be a closed subset of $\mathbf{D}^k$ and L a closed subset of $\mathbf{D}^m$. Then, for every separately continuous bilinear form B on $G = \underset{\sim}{\mathcal{O}}(K) \times \underset{\sim}{\mathcal{O}}(L)$ there is a unique Fourier hyperfunction F_B on $K \times L$, i.e. $F_B \in \underset{\sim}{\mathcal{O}}(K \times L)'$ such that for all $(g, h) \in G$*

$$B(g,h) = F_B(g \otimes h).$$

Proof. Since $\underset{\sim}{\mathcal{O}}(K)$ and $\underset{\sim}{\mathcal{O}}(L)$ are DFS-spaces (see [15]) they are strong duals of reflexive Fréchet spaces and thus Theorem 41.1 of [20] implies that every separately continuous bilinear form B on G is continuous. Furthermore, the spaces $\underset{\sim}{\mathcal{O}}(K)$ and $\underset{\sim}{\mathcal{O}}(L)$ are nuclear according to Proposition 2.12 of [13]. Hence the abstract kernel theorem (Theorem 7.4.3 of Pietsch [17]) applies and proves that every such bilinear form B is actually nuclear, i.e., there is an equicontinuous sequence (T_j) in $\underset{\sim}{\mathcal{O}}(K)'$, an equicontinuous sequence (S_j) in $\underset{\sim}{\mathcal{O}}(L)'$, and a sequence (λ_j) of real numbers with $\sum_{j=1}^{\infty} |\lambda_j| < \infty$ such that for all $(g, h) \in G$

$$B(g,h) = \sum_{j=1}^{\infty} \lambda_j \langle T_j, g\rangle \langle S_j, h\rangle.$$

When we write the right hand side of this equation in the form

$$\sum_{j=1}^{\infty} \lambda_j (T_j \otimes S_j)(g \otimes h)$$

we see immediately that B is actually given by the continuous linear form

$$F_B = \sum_{j=1}^{\infty} \lambda_j T_j \otimes S_j$$

on $\underset{\sim}{\mathcal{O}}(K) \otimes_\pi \underset{\sim}{\mathcal{O}}(L)$ (compare Theorem 43.4 of [20]). Clearly F_B extends to a continuous linear form on the completion of this space which is equal to $\underset{\sim}{\mathcal{O}}(K \times L)$ according to Proposition 2.3. Thus we conclude. □

The general form of the kernel theorem for Fourier hyperfunctions is now a straight forward consequence of Theorem 2.1.

Theorem 2.2 *Let K_i be a closed subset of $\mathbf{D}^{k_i}$ for $i = 1, 2, \ldots, N$. Then, for every separately continuous N-linear form B on $G = \underset{\sim}{\mathcal{O}}(K_1) \times \cdots \times \underset{\sim}{\mathcal{O}}(K_N)$ there is a unique Fourier hyperfunction F_B on $K_1 \times \cdots \times K_N$, i.e., $F_B \in \underset{\sim}{\mathcal{O}}(K_1 \times \cdots \times K_N)'$ such that for all $(g_1, \ldots, g_N) \in G$*

$$B(g_1, \ldots, g_N) = F_B(g_1 \otimes \cdots \otimes g_N).$$

Proof. On the basis of Theorem 3.1 a straight forward proof of induction with respect to the number N of arguments establishes this general form. □

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