Signed Graphs Associated with the Lattice $A_n$

By

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Abstract

For any base of the root lattice $A_n$, we can construct a signed graph naturally. A connected graph is called a Fushimi tree if its all blocks are complete subgraphs. In the present note, a Fushimi tree is said to be simple when by deleting any cut vertex, we have always two its connected components. We prove that a signed graph corresponding to a base of $A_n$ is a simple Fushimi tree. Cameron, Seidel and Cameron [3] defined local switchings of signed graphs. We also show that a simple Fushimi tree is transformed to a line by a sequence of local switchings.

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Introduction

A signed graph is a graph whose edges are signed by +1 or -1. In [3], for a given signed graph, Cameron, Seidel and Cameron construct the corresponding root lattice. In the present note, we treat with signed graphs corresponding to the root lattice $A_n$. A connected graph is called a Fushimi tree if its all blocks are complete subgraphs. A Fushimi tree is said to be simple when by deleting any cut vertex, we have always two its connected components. Switching and local switching of signed graphs are also introduced by Cameron, Seidel and Cameron in [3]. A signed Fushimi tree is said to be a Fushimi tree with standard sign if it can be transformed to a signed Fushimi tree whose all edges are signed by +1, by a switching. We prove that any signed graph corresponding to $A_n$ is a simple Fushimi tree with standard sign. Switching defines an equivalent relation in the set of all signed graphs. A equivalent class is called a switching class. Local switching partitions the all signed graphs on n vertices into clusters of
switching classes. Our main result is that a simple Fushimi tree with standard sign is contained in the cluster given by the line.

1. Signed graphs

Following 3], we state basic facts about signed graphs. A graph $G = (V, E)$ consists of an n-set $V$ (the vertices) and a set $E$ of unordered pairs from $V$ (the edges). A signed graph $(G, f)$ is a graph $G$ with a signing $f : E \to \{1, -1\}$ of the edges. For any subset $U \subseteq V$ of vertices, let $f_U$ denote the signing obtained from $f$ by reversing the sign of each edge which has one vertex in $U$. This defines on the set of signings an equivalence relation, called switching. The equivalence classes $\{f_U : U \subseteq V\}$ are the signed switching classes of the graph $G = (V, E)$. The adjacency matrix $A = (A_{ij})$ is defined by $A_{ij} = f(\{i, j\})$ for $\{i, j\} \in E$; else $A_{ij} = 0$ otherwise. The matrix $2I + A$ is called the intersection matrix, and interpreted as the Gram matrix of the inner product of $n$ vectors $a_1, \ldots, a_n$ in a (possibly indefinite) inner product space $R^{p/2}$. These vectors are roots (which have length $\sqrt{2}$) at angles $\pi/2, \pi/3$, or $2\pi/3$. Their integral linear combinations form a root lattice (an even integral lattice spanned by vectors of norm 2), which we denote by $L(G, f)$. Let $i \in V$ be a vertex of $G$, and $V(i)$ be the neighbors of $i$. The local graph of $(G, f)$ at $i$ has $V(i)$ as its vertex set, and as edges all edges $\{j, k\}$ of $G$ for which $f(i, j)f(j, k)f(k, i) = -1$. A rim of $(G, f)$ at $i$ is any union of connected components of local graph at $i$. Let $J$ be any rim at $i$, and let $K = V(i) \setminus J$. Local switching of $(G, f)$ with respect to $(i, J)$ is the following operation: (i) delete all edges of $G$ between $J$ and $K$; (ii) for any $j \in J$, $k \in K$ not previously joined, introduce an edge $\{j, k\}$ with sign chosen so that $f(i, j)f(j, k)f(k, i) = -1$; (iii) change the signs of all edges from $i$ to $J$; (iv) leave all other edges and signs unaltered. Let $\Omega_n$ be the set of switching classes of signed graphs of order $n$. Local switching, applied to any vertex and any rim at the vertex, gives a relation on $\Omega$ which is symmetric but not transitive. The equivalence classes of its transitive closure are called the clusters of order $n$.

2. The lattice $A_n$ and signed Fushimi trees

A connected graph $G = (V, E)$ is called Fushimi tree if each block of $G$ is a perfect graph. In the present paper, a Fushimi tree $G$ is said to be a simple Fushimi tree if $G$ is divided exactly two connected components when each cut vertex in $G$ is deleted.

A signed simple Fushimi tree is called a special Fushimi tree with standard sign if we can switch all signs of edges into +1.

The lattice $A_n$ is spanned by vectors $e_i - e_j, 1 \leq i \neq j \leq n + 1$, where $\{e_1, \ldots, e_{n+1}\}$ is the orthonormal base of the euclidean $(n + 1)$-space $R^{n+1}$. 
There is the one-to-one correspondence between ordered root bases of $A_n$ and connected signed graphs associated with $A_n$.

**Theorem 1** Any connected signed graph is a signed graph associated with $A_n$ if and only if it is a simple Fushimi tree with standard sign.

Proof. Let $G$ be a signed graph corresponding to a ordered base $\{a_1, a_2, \ldots, a_n\}$. If we replace $a_i$ by $-a_i$, then the sign of $G$ is switched with respect to $\{a_i\}$. Hence there is no problem whether we take $a_i$ or $-a_i$. There is no cycle in $G$ whose length is more than 3. In fact, if $a_{i_1}, a_{i_2}, \ldots, a_{i_m}, m > 3$ make a cycle, then we can assume that $a_{i_1} = e_{j_1} - e_{j_2}, a_{i_2} = e_{j_2} - e_{j_3}, a_{i_3} = e_{j_3} - e_{j_4}, \ldots, a_{i_m} = e_{j_{m-1}} - e_{j_1}$. But this implies that $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ are not linearly independent. If $a_{i_1}, a_{i_2}, a_{i_3}$ make a cycle, then we can assume that $a_{i_1} = e_{j} - e_{j_1}, a_{i_2} = e_{j} - e_{j_2}, a_{i_3} = e_{j} - e_{j_3}$. We have cycles of this type only in $G$. Now take a block $B$ of $G$ consisting of vertices $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$. Two vertices $a_{i_1}$ and $a_{i_2}$ must be on a cycle in $B$. We may assume that $a_{i_1}, a_{i_2}, a_{i_3}$ make a cycle. Then we can put $a_{i_1} = e_{j} - e_{j_1}, a_{i_2} = e_{j} - e_{j_2}, a_{i_3} = e_{j} - e_{j_3}$. Two vertices $a_{i_2}$ and $a_{i_4}$ are also on a cycle in $B$, which may be $a_{i_1}, a_{i_4}, a_{i_6}$. Then we can put $a_{i_4} = e_{j} - e_{j_4}, a_{i_6} = e_{j} - e_{j_5}$ or $a_{i_4} = e_{j_1} - e_{j_4}, a_{i_6} = e_{j_5} - e_{j_6}$, where $j_4 \neq j$. Assume that $a_{i_4} = e_{j_4} = e_{j} - e_{j_4}, a_{i_6} = e_{j_5} - e_{j_6}$. Two vertices $a_{i_2}$ and $a_{i_4}$ are also on a cycle in $B$. Then we have $j_4 = j_2$, a contradiction. Hence, we get $a_{i_4} = e_{j_4} = e_{j} - e_{j_6}$. By this way, we get $a_{i_k} = e_{j} - e_{j_k}, 1 \leq k \leq m$. Hence any block of $G$ is a perfect graph whose edges have sign $+1$. Suppose that $a_{i_1} = e_{j} - e_{j_1}$ of a block $B$ is a cut vertex. If two vertex $a_{j}, a_{k}$ which are not in $B$ are adjacent with $a_{i_1}$, then we can put $a_{j} = e_{j_1} - e_{j_1}, a_{k} = e_{j_1} - e_{k_1}$. Hence $a_{i_1}, a_{j}, a_{k}$ are contained in another block of $G$. Hence we show that $G - a_{i_1}$ has two connected components. Thus $G$ is a simple Fushimi tree with standard sign.

Conversely, Let $G$ be a special Fushimi tree with standard sign. Assume that $G$ has $m$ blocks. If $m = 1$, it is evident that $G$ is a connected signed graph associated with $A_n$. Now suppose that the result is true for simple signed Fushimi trees with $m$ blocks. Let $G$ be a simple signed Fushimi tree with $m + 1$ blocks. Let $G'$ be a simple Fushimi tree with standard sign which is made from $G$ by deleting one block $B$ of $G$. Then $G'$ is a connected signed graph associated with $A_n$ and corresponding to an ordered base $\{a_1, a_2, \ldots, a_n\}$. Now assume that all $\ell$ vertices of $B$ are adjacent with a vertex $a_i = e_{i_1} - e_{i_2}$ and that $e_{i_2}$ is not used in any other $a_j$. Then, we can consider that the block $B$ consists of $e_{i_2} = e_{n+2}, e_{i_2} = e_{n+3}, \ldots, e_{i_2} = e_{n+\ell+1}$ and $a_i$. Hence we regard $G$ as a connected signed graph associated with $A_{n+\ell}$.

3. Local switching of simple Fushimi trees

Let $G$ be a simple Fushimi tree with standard sign. Take a block $B$ of $G$. In the present paper, $G$ is said to be $(n+k,k)$-type with respect to $B$ if the order of $B$ is $n+k$ and the number of cut vertices in $B$ is $k$. Let $a$ be a cut
vertex in a block \( B \). \( G/\{a\} \) has the two connected components. One is the component containing \( B/\{a\} \). We call the other the branch with respect to \( (B, a) \). All such components are called branches with respect to \( B \). A block of a simple Fushimi tree is said to be pendant if it has only one cut vertex. We call a simple Fushimi tree line-like if it has only one block or it has exactly two pendant blocks. A branch \( B_a \) with respect to \( (B, a) \) is called line-like if \( B_a \cup \{a\} \) is line-like. A simple Fushimi tree is said to be a line Fushimi tree if the order of its every block is 2.

**Lemma 2.** Let \( G \) be a simple Fushimi tree of \((n + k, k)\)-type with respect to a block \( B \). We can transform \( G \) into a simple Fushimi tree of \((k, k)\)-type, by a sequence of local switchings.

Proof. Let the block \( B \) consist of vertices \( a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+k} \), where \( a_{n+1}, \ldots, a_{n+k} \) are cut vertices. Set \( J = \{a_{n+1}\} \) and \( K = \{a_1, a_2, \ldots, a_{n-1}, a_{n+2}, \ldots, a_{n+k}\} \). By local switching with respect to \( (a_n, J) \), we obtain a simple Fushimi tree of \((n + k - 1, k)\)-type with respect to block \( \{a_1, a_2, \ldots, a_n, a_{n+2}, \ldots, a_{n+k}\} \), where \( a_n, a_{n+2}, \ldots, a_{n+k} \) are cut vertices. Denote by \( G_1 \) this simple Fushimi tree of \((n + k - 1, k)\)-type. Applying the same procedure to \( G_1 \), we get a simple Fushimi tree of \((n + k - 2, k)\)-type. Repeating the same method, we get a simple Fushimi tree of \((k, k)\)-type at last.

**Lemma 3.** Let \( G \) be a simple Fushimi tree of \((k, k)\)-type with respect to a block \( B \). Assume that all branches with respect to \( B \) are line-like. Then, we can transform \( G \) into a simple Fushimi tree of \((k + n, k - 1)\)-type with respect to some block \( B \) whose all branches are line-like, by a sequence of local switchings, where \( n \) is some positive integer.

Proof. Let the block \( B \) consist of vertices \( a_1, a_2, \ldots, a_k \). Take a branch, for example, the branch \( B_1 \) with respect to \( (B, a_1) \). Assume the order of \( B_1 \) is \( n \). Suppose \( B_1 \cup \{a_1\} \) has \( m \) blocks. Firstly, assume \( m = 1 \). Take \( i = a_1, J = B_1, K = \{a_2, \ldots, a_k\} \). Then by local switching with respect to \( (a_1, B_1) \), \( G \) is transformed to a simple Fushimi tree of \((k + n, k - 1)\)-type with respect to the block \( \{a_2, a_3, \ldots, a_k\} \), whose branches with respect to this block are all line-like. Suppose that the result is true for \( m \). Assume that \( B_1 \cup \{a_1\} \) has \( m + 1 \) blocks. Let \( C \) be the pendant block of \( B_1 \cup \{a_1\} \) and \( c \) be its cut vertex. Put \( G_1 = \{G \backslash C\} \cup \{c\} \). Then \( G_1 \) is a simple Fushimi tree of \((k, k)\)-type with respect to a block \( B \). By the inductive hypothesis, \( G_1 \) is transformed into a simple Fushimi tree \( G_2 \) of \((k + n_1, k - 1)\)-type with respect to the block \( \{a_2, a_3, \ldots, a_k\} \), whose branches with respect to this block are all line-like, by a sequence of local switchings, where \( n_1 = n + 1 - n_2 \) and \( n_2 \) is the order of the block \( C \). By the same way, we can transform \( G \) into \( G_2 \cup C \), which is a special Fushimi tree of \((k + n_1, k)\)-type with respect to the block \( \{c, a_2, a_3, \ldots, a_k\} \), whose branches with respect to this block are all line-like and can be transform into a simple Fushimi tree of \((k + n, k - 1)\)-type with respect to the block \( \{a_2, a_3, \ldots, a_k\} \), whose branches with respect to this block are all line-like, by local switching.
Lemma 4. Let $B$ be a simple Fushimai tree with one block. Then it can be transformed into a line Fushimai tree, by a sequence of local switchings.

Proof. Let $B$ consist of vertices $a_1, a_2, \ldots, a_k$. Set $J = \{a_1\}$ and $K = \{a_3, a_4, \ldots, a_k\}$. By local switching with respect to $(a_2, J)$, we obtain a simple Fushimai tree of $(k-1,1)$-type with respect to block $\{a_2, \ldots, a_k\}$. Next, set $J = \{a_2\}$ and $K = \{a_4, \ldots, a_k\}$. By local switching with respect to $(a_3, J)$, we obtain a simple Fushimai tree of $(k-2,1)$-type with respect to block $\{a_3, \ldots, a_k\}$. By this way, we can get a line Fushimi tree, by a sequence of local switchings.

Lemma 5. Let $G$ be a simple Fushimai tree of $(n+k,k)$-type with respect to a block $B$. Assume that all branches with respect to $B$ are line-like. Then it is transformed to a line Fushimai tree, by a sequence of local switchings. Especially, a line-like special Fushimai tree is transformed to a line Fushimai tree, by a sequence of local switchings.

Proof. By the same way in the proof of lemma 2, $G$ can be transformed into a simple Fushimai tree $G_1$ of $(k,k)$-type with respect to some block $B_1$, by a sequence of local switchings, whose branches with respect to $B_1$ are all line-like. By lemma 3, we can get a simple Fushimai tree of $(n+k-1)$-type with some block $B_2$ whose branches are all line-like, by a sequence of local switchings, where $n$ is some positive integer. By a sequence of this process, we obtain a simple Fushimai tree of $(k+N,0)$, where $N$ is some positive integer, that is, a simple Fushimai tree with a one block, which is also transformed into a line Fushima tree by lemma 4.

Lemma 6. Let $G$ be a simple Fushimai tree. Then it has at least two pendant blocks.

Proof. If every block has more than one cut vertex, then, as we have no cycle, the order of the graph is infinite. Hence, $G$ has at least two pendant blocks.

Theorem 7. Let $G$ be a simple Fushimai tree. We can transform $G$ into a line Fushimai tree, by a sequence of local switchings.

Proof. Assume $G$ has $m$ blocks. If $m = 1$, we get the result by Lemma 4. Suppose the result is true for $m = k$. Let $m = k + 1$. Take a pendant block $B_1$ of $G$ with cut vertex $b$. Let $B_2$ be the other block with cut vertex $b$. Put $i = b, J = B_1 \setminus b, K = B_2 B_1 \setminus b$. By local switching with respect to $(b, J)$, we obtain a simple Fushimai tree with $k$ blocks, which can be transformed into a line Fushimai tree, by a sequence of local switchings.

References

