J. Math. Tokushima Univ.
Vol. 4. (1970), 1-17

On Tensor-Product Structures and Grassmannian Structures

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(Received September 30, 1970)

As it was shown by several authors, the tangent bundle of a Grassmann manifold is a tensor product of two certain vector bundles. On the other hand, Th. Hangan studied a manifold with a structure on which the tangent bundle was isomorphic to the tensor product of two vector bundles. He called this structure a tensor-product structure. Th. Hangan's study was aimed mainly at flat tensor-product structures and the natural tensor-product structure on the Grassmann manifold.

In this paper, some of his results for flat tensor-product structure are extended to general tensor-product structures.

In §3, the notion of grassmannian structures, which is a extension of that of projective structures due to S. Kobayashi and T. Nagano [4] is defined. The natural correspondence between grassmannian structures and tensor-product structures are established. This correspondence leads us to the unique existence of a certain grassmannian structure for a give tensor-product structure. In this situation, we say that this grassmannian structure is determined by the given tensor-product structure.

The notion of Cartan connection in a grassmannian structure is also introduced. Particularly, there exists uniquely so-called normal connection in a grassmannian structure determined by a tensor-product structure. Lastly, the local flatness of grassmannian structures is discussed.

The consideration is made only for the real cases, but a similar discussion seems to be possible for the complex cases.

The present author wishes to express his hearty thanks to Prof. Dr. M. Matsumoto for his kind encouragement. The author is also thankful to Prof. Dr. Y. Ichijyo who attracted my interests in this direction.

§1. Tensor-product structures

Let V^p, V^q be vector spaces of dimension p and q respectively over R and $V^{pq}=V^p\otimes V^q$ be the tensor product of V^p and V^q. We define a linear transformation J of $V^{pq}\otimes V^{pq}$ by

$$J((u\otimes v)\otimes(u'\otimes v'))=(u\otimes v')\otimes(u'\otimes v), \tag{1.1}$$

where $u,\ u'\in V^p$ and $v,\ v'\in V^q$. Let $e^1, e^2, \cdots, e^p$ and $e_{p+1}, e_{p+2}, \cdots, e_n\ (n=p+q)$ be bases of V^p and V^q respectively, then $e^i\otimes e_\alpha$, $i=1, 2, \cdots, p, \alpha=p+1, p+2, \cdots, n$ is a base of V^{pq}. With respect to this base, the linear transformation J is written as

$$J((e^i\otimes e_\alpha)\otimes(e^j\otimes e_\beta))=\Sigma\delta^{\gamma_k}_{\beta_i}\delta^{\delta_l}_{\alpha_j}(e^k\otimes e_\gamma)\otimes(e^l\otimes e_\delta).$$

Unless otherwise mentioned, indices in this paper take the following values.

$$a,\ b,\ c,\ \cdots\cdots=1,\ 2,\ \cdots,\ n(=p+q);$$

$$i,\ j,\ k,\ l,\ \cdots=1,\ 2,\ \cdots,\ p;$$

$$\alpha,\ \beta,\ \gamma,\ \delta,\cdots=p+1,\ p+2,\ \cdots,\ n;$$

$$\alpha_i,\ \beta_j,\ \gamma_k,\cdots=p+1_1,\ p+1_2,\ \cdots,\ p+1p,\ p+2_1,\ p+2_2,\ \cdots,\ p+2p,\ \cdots$$
$$\cdots,\ n_1,\ n_2;\ \cdots,\ n_p.$$

Sometimes, we shall denote $A, B, C, \cdots$ instead of $\alpha_i, \beta_j, \gamma_k, \cdots$.

$GL(pq, R)$ denotes the linear transformation group of V^{pq} as usual. Let $g\in GL(pq, R)$, then g can be extended linearly to a linear transformation $g\otimes g$ of $V^{pq}\otimes V^{pq}$. Elements of $GL(pq, R)$ which satisfy

$$(g\otimes g)J=J(g\otimes g) \tag{1.2}$$

form a subgroup $G_t(p, q)$ of $GL(pq, R)$. This subgroup is isomorphic to $GL(p, R)\otimes GL(q, R)$ [2].

In the following sections, M will denote a C^∞-manifold of dimension pq. *A tensor-product structure (t-p structure) on* M is, by definition [2], a subbundle of the frame bundle of M with structure group $G_t(p, q)$. We shall denote a t-p structure on M by $P_t(M)$, and the natural projection of $P_t(M)$ by π. A frame P contained in $P_t(M)$ is a linear isomorphism of V^{pq} onto the tangent vector space $T_x(M)$ at x and can be extended to a linear isomorphism of $V^{pq}\otimes V^{pq}$ onto $T_x(M)\otimes T_x(M)$, where $x=\pi(P)$.

A tensor-product structure $P_t(M)$ defines a (2, 2)-tensor on M by

$$\psi(X\otimes Y)=(P\otimes P)J(P^{-1}X\otimes P^{-1}Y), \tag{1.3}$$

where $X, Y\in T_x(M)$ and $P\in\pi^{-1}(x)$. It follows from (1.2) that this definition of ψ does not depend on the choice of $P\in\pi^{-1}(x)$. The tensor defined above is called *the tensor determined by the tensor-product structure* $P_t(M)$. Sometimes, the tensor ψ itself is called *the tensor-product structure.* A manifold with a fixed tensor-product structure is called *a tensor-product manifold.*

We shall describe some properties of the (2, 2)-tensor ψ which follow immediately from the definition. First of all, let $P=(X_{\alpha_i})$ be a frame contained in $P_t(M)$, then we have

$$\psi(X_{\alpha_i}\otimes X_{\beta_j})=X_{\beta_i}\otimes X_{\alpha_j}. \tag{1.4}$$

Furthermore, if ψ^{AB}_{CD} are components of ψ with respect to local coordinate systems, ψ satisfies the following identities.

$$\Sigma\psi^{AB}_{CD}\psi^{CD}_{EF}=\delta^A_E\delta^B_F, \tag{1.5}$$

$$\psi^{AB}_{CD}=\psi^{BA}_{DC}, \tag{1.6}$$

$$\Sigma\psi^{AC}_{AD}=p\delta^C_D,\ \Sigma\psi^{AB}_{CA}=q\delta^B_C, \tag{1.7}$$

$$\Sigma\psi^{A_1A_2}_{B_1C}\psi^{C\ A_3}_{B_2B_3}=\Sigma\psi^{A_2A_3}_{B_2C}\psi^{C\ A_1}_{B_3B_1}=\Sigma\psi^{A_3A_1}_{B_3C}\psi^{C\ A_2}_{B_1B_2}. \tag{1.8}$$

A tangent vector X is called *a tensor-product vector* (*t-p vector*) if it satisfies

$$\psi(X\otimes X)=X\otimes X. \tag{1.9}$$

Suppose (X_{α_i}) is a frame contained in $P_t(M)$, then (1.4) implies that each X_{α_i} is a t-p vector and any vector expressed by $\Sigma\lambda^\alpha\mu_i X_{\alpha_i}$ is also a t-p vector. Conversely, a t-p vector always arises in the fashion. In fact, let $X=\Sigma\lambda^{\alpha_i}X_{\alpha_i}$ be a t-p vector, then from (1.9), it follows $\lambda^{\alpha_i}\lambda^{\beta_j}=\lambda^{\beta_i}\lambda^{\alpha_j}$ and this shows the result. If non-zero tangent vectors X and Y satisfy $\psi(X\otimes Y)=X\otimes Y$ or $\psi(X\otimes Y)=Y\otimes X$, both X and Y are necessarily t-p vectors.

PROPOSITION 1. *For a given t-p vector* $X\in T_x(M)$, *the set of vectors* Y *which satisfy* $\psi(X\otimes Y)=X\otimes Y$ *forms p-dimensional subspace* $\Pi^p(X)$ *of* $T_x(M)$ *and the set of vectors* Z *which satisfy* $\psi(X\otimes Z)=Z\otimes X$ *forms q-dimensional subspace* $\Pi^q(X)$ *of* $T_x(M)$.

PROOF. We take a frame $\{X_{\alpha_i}\}\in P_t(M)$ such that $X_{\alpha_i}\in T_x(M)$. Let $X=\Sigma\lambda^\alpha\mu_i X_{\alpha_i}$, then each Y is written as $\Sigma\lambda^\alpha y_i X_{\alpha_i}$ and each Z as $\Sigma z^\alpha\mu_i X_{\alpha_i}$. Then, the statement follows easily.

$\Pi^p(X)$ and $\Pi^q(X)$ are called respectively *p-plane of* ψ *determined by* X and *q-plane of* ψ *determined by* X. We shall describe elementary properties of $\Pi^p(X)$ and $\Pi^q(X)$. First, $\Pi^p(X)\cap\Pi^q(X)=\{X\}$. For $Y(\neq 0)\in\Pi^p(X)$, we have $\Pi^p(X)=\Pi^p(Y)$ and for $Z(\neq 0)\in \Pi^q(X)$, $\Pi^q(X)=\Pi^q(Z)$. If Y_1, $Y_2\in \Pi^p(X)$, it follows $\psi(Y_1\otimes Y_2)=Y_1\otimes Y_2$ and if Z_1, $Z_2\in \Pi^q(X)$, $\psi(Z_1\otimes Z_2)=Z_2\otimes Z_1$.

As it is described above, a tensor ψ determined by a *t-p* structure satisfies the identities (1.6), (1.7) and (1.8). Moreover, the following is evident from (1.4).

(*A*) *For every point* $x\in M$, *there is at least one vector field* X *on some local neighborhood* U *of* x *which satisfies* $\psi(X\otimes X)=X\otimes X$ *and* $X\neq 0$ *at every point of* U.

Conversely, we have

THEOREM 1. *Let* ψ *be a* (2, 2)*-tensor field on a pq-dimensional manifold* M *which satisfies* (1.6), (1.7), (1.8) *and the condition* (*A*), *then* ψ *defines a t-p structure* $P_t(M)$ *on* M. *Also,* ψ *coincides with the tensor determined by* $P_t(M)$.

PROOF. We shall show that, for every point x of M, there exists a field of frames $\{X_{\alpha_i}\}$ on some local neighborhood U of x which satisfies $\psi(X_{\alpha_i}\otimes X_{\beta_j})=X_{\beta_i}\otimes X_{\alpha_j}$ at every point of U.

Let $x\in M$ and X be a local vector field on some neighborhood U of x given by the condition (*A*). A required field of frames will be constructed on U. We may take local coordinates x^{α_i}, $\alpha=p+1, p+2, \cdots, n(=p+q)$, $i=1, 2, \cdots, p$ on U. Let X^A and ψ^{AB}_{CD} components of X and ψ with respect to the coordinates. Let us fix a index A_0 with $X^{A_0}\neq 0$, then

$$\psi^B_D(A_0)=\Sigma\frac{\psi^{A_0B}_{C\ D}}{X^{A_0}}X^C$$

is a linear transformation of the tangent space at each point of U. From (1.6) and (1.8), it follows that the eigenvalues of $(\psi^B_D(A_0))$ are 1 or 0 at every point. The relation (1.7) implies that its trace is p at every point of U. So, the dimension of the eigenspace corresponding to the eigenvalue 1 is not over p. Really, it is exactly p at every point of U. Let A_1 be another index with $X^{A_1}\neq 0$ and define $\psi^B_D(A_1)$ in the same way, then $(\psi^B_D(A_0))$ and $(\psi^B_D(A_1))$ take eigenvectors with eigenvalue 1 in common at every point. Consequently, there are local vector fields $X_{p+1_1}(=X)$, X_{p+1_2}, $\cdots$, X_{p+1_p} on U which are linearly independent at every point of U and satisfy $\psi(X_{p+1_1}\otimes X_{p+1_i})=X_{p+1_1}\otimes X_{p+1_i}$.

Similarly, it turns out that there are local vector fields $X_{p+1_1}(=X)$, X_{p+2_1}, $\cdots$, X_{n_1} on U which are linearly independent at every point of U and satisfy $\psi(X_{p+1_1}\otimes X_{\alpha_1})=X_{\alpha_1}\otimes X_{p+1_1}$.

Now, put

$$X^D_{\alpha_i}(A_0)=\Sigma\frac{\varphi^{A_0D}_{A\ B}}{X^{A_0}}X^A_{\alpha_1}X^B_{p+1_i},$$

then, they depend formally on the index A_0 with $X^{A_0}\neq 0$, but at least for $\alpha=p+1$ or $i=1$, they do not depend on it. Let $X_{\alpha_i}(A_0)=\Sigma X^D_{\alpha_i}(A_0)\frac{\partial}{\partial x^D}$. It follows $\varphi(X_{\alpha_1}(A_0)\otimes X_{\beta_1}(A_0))=X_{\beta_1}(A_0)\otimes X_{\alpha_1}(A_0)$. From this and (1.8), it is obtained

$$\Sigma\varphi^{CD}_{AB}X^A_{\alpha_i}(A_0)X^B_{\beta_j}(A_0)=X^C_{\beta_i}(A_0)X^D_{\alpha_j}(A_0). \tag{1.10}$$

If $i=1$ and $\alpha=p+1$, the relation (1.10) implies $\varphi^{AB}_{CD}X^C_{\alpha_1}X^D_{p+1_1}=X^A_{p+1_1}X^D_{\alpha_j}(A_0)$, so $X^D_{\alpha_j}(A_0)$ do not depend on the index A_0. Hence, we can write X_{α_i} instead of $X_{\alpha_i}(A_0)$.

Now, local vector fields X_{α_i} satisfy the identity $\psi(X_{\alpha_i}\otimes X_{\beta_j})=X_{\beta_i}\otimes X_{\alpha_j}$. The linear independence of X_{α_i} at every point of U follows by induction from the independence of $X_{p+1_1},\cdots,X_{p+1_p}$, and that of $X_{p+1_1},\cdots,X_{n_1}$. We have shown the assertion at the first part of the proof.

If V be a local neighborhood with $U\cap V\neq\emptyset$ and if $\{Y_{\alpha_i}\}$ such a field of frames on V as $\{X_{\alpha_i}\}$, then it holds $Y_{\alpha_i}=\Sigma g^\beta_\alpha g^i_j X_{\beta_j}$ on $U\cap V$. Thus we have a $t-p$ structure on M.

It is evident that the tensor determined by the t-p structure coincides with ψ.

§ 2. Connections on a tensor-product manifold

Let $P_t(M)$ be a t-p structure and ψ the tensor determined by $P_t(M)$. We shall give a condition for connections ∇ and ∇' on M to satisfy the relation $\nabla\psi=\nabla'\psi$. For a flat t-p structure and symmetric connections, Th. Hangan obtained the condition. In this section, we shall consider in a more general setting. As to connections, it is not necessary that they are symmetric.

Let T^A_{BC} be a (1, 2)-tensor expressed in terms of local coordinate systems, T^A_{BC} is called *semi-symmetric*, when

$$\Sigma\psi^{AE}_{BF}(T^F_{EC}-T^F_{CE})=0,\ \Sigma\psi^{AE}_{FB}(T^F_{EC}-T^F_{CE})=0 \tag{2.1}$$

hold, where ψ^{AE}_{BF} are components of ψ. As $\nabla-\nabla'$ is a (1, 2)-tensor for connections ∇ and ∇', it has a meaning to suppose that $\nabla-\nabla'$ is semi-symmetric.

THEOREM 2. *Let ∇ and ∇' be connections on a differentiable manifold M with a tensor-product structure ψ. If $\nabla\psi=\nabla'\psi$ and $\nabla-\nabla'$ is semi-symmetric, then there exists a covariant vector field Φ_A which satisfies*

$$\Gamma'^{A}_{BC}-\Gamma^{A}_{BC}=\Sigma\psi^{AD}_{BC}\Phi_D+\Sigma\psi^{AD}_{CB}\Phi_D, \tag{2.2}$$

where Γ'^{A}_{BC}, Γ^{A}_{BC} *and* ψ^{AD}_{BC} *are components of* ∇', ∇ *and* ψ *with respect to local coordinate systems. Conversely, if* (2.2) *holds for some covariant vector* Φ_A, *then* $\nabla\psi=\nabla'\psi$ *and* $\nabla-\nabla'$ *is symmetric.*

Proof. Let U be a neighborhood on which there is a local section $s_U=\left(\Sigma X^A_B\frac{\partial}{\partial x^A}\right)$. Put

$$\Psi^{A}_{BC}=\Sigma Y^{A}_{A_1}(\Gamma'^{\,A_1}_{B_1C_1}-\Gamma^{\,A_1}_{B_1C_1})X^{B_1}_{B}X^{C_1}_{C},$$

where $(Y^{A}_{A_1})$ is the inverse matrix of $(X^{A_1}_{A})$. Then, it follows from $\nabla\psi=\nabla'\psi$

$$\Psi^{\alpha_1 i_1}_{\gamma_k\beta_2 j_1}\delta^{\alpha_2}_{\beta_1}\delta^{i_2}_{j_2}+\Psi^{\alpha_2 i_2}_{\gamma_k\beta_1 j_2}\delta^{\alpha_1}_{\beta_2}\delta^{i_1}_{j_1}-\Psi^{\alpha_2 i_1}_{\gamma_k\beta_1 j_1}\delta^{\alpha_1}_{\beta_2}\delta^{i_2}_{j_2}-\Psi^{\alpha_1 i_2}_{\gamma_k\beta_2 j_2}\delta^{\alpha_2}_{\beta_1}\delta^{i_1}_{j_1}=0. \tag{2.3}$$

The semi-symmetry of $\nabla'-\nabla$ means

$$\Sigma\Psi^{\alpha_l}_{\gamma_k\beta_l}=\Sigma\Psi^{\alpha_l}_{\beta_l\gamma_k},\quad \Sigma\Psi^{\delta_i}_{\gamma_k\delta_j}=\Sigma\Psi^{\delta_i}_{\delta_j\gamma_k}. \tag{2.4}$$

By means of (2.4), we obtain from (2.3)

$$\Psi^{\alpha_i}_{\beta_j\gamma_k}=\delta^{\alpha_i}_{\gamma_j}\Psi_{\beta_k}+\delta^{\alpha_i}_{\beta_k}\Psi_{\gamma_j},$$

where $\Psi_{\beta_k}=\dfrac{1}{p+q}\Sigma\Psi^{\delta_l}_{\beta_k\delta_l}$.
Consequently we have

$$\Gamma'^{A}_{BC}-\Gamma^{A}_{BC}=\Sigma\psi^{AD}_{BC}\Phi_D+\Sigma\psi^{AD}_{CB}\Phi_D,$$

where $\Phi_D=\dfrac{1}{p+q}\,\Sigma(\Gamma'^{A}_{DA}-\Gamma^{A}_{DA})$.

If two linear connections gratify the relation (3.3) for some covariant vector Φ_D, we say that they are *ψ-related.* Next, we introduce linear connections which may be fundamental for *t-p* structures.

Theorem 3. *Let* ψ *be a t-p structure on a differentiable manifold M. Then there exists a connection* ∇ *satisfying the following conditions*

(*a*) $\nabla\psi=0$;

(*b*) *The torsion tensor of* ∇ *is semi-symmetric.*

If ∇' *be another connection satisfying above conditions,* ∇ *and* ∇' *are* ψ*-related.*

Proof. We shall give a required connection in an explicit form with respect to local coordinate systems. Take an arbitrary connection $\bar{\Gamma}^A_{BC}$ on M. We define a connection by

$$\Gamma^A_{BC}=\frac{1}{p+q}(\Sigma\psi^{AD}_{BC}\bar{\Gamma}^E_{DE}+\Sigma\psi^{AD}_{BC}\bar{\Gamma}^E_{DE})+N^A_{BC}, \tag{2.5}$$

where

$$\begin{aligned} N^A_{BC}=&-\Sigma\frac{\partial\psi^{D_1A}_{D_3D_4}}{\partial x^{D_1}}\psi_{B}{}^{D_3}{}_{C}{}^{D_4}-\frac{pq-1}{(p^2-1)(q^2-1)}\delta^A_C\Sigma\frac{\partial\psi^{D_1D_2}_{D_3D_4}}{\partial x^{D_1}}\psi_{B}{}^{D_3}{}_{D_2}{}^{D_4} \\ &+\frac{1}{p^2-1}\Sigma\frac{\partial\psi^{D_1D_2}_{B\ D_4}}{\partial x^{D_1}}\psi^{D_4A}_{D_2C}+\frac{1}{q^2-1}\Sigma\frac{\partial\psi^{D_1D_2}_{D_3B}}{\partial x^{D_1}}\psi^{A\ D_3}_{D_2C} \\ &+\Sigma\frac{\partial\psi^{D_1D_2}_{D_3D_4}}{\partial x^{D_1}}\left(\frac{p}{p^2-1}\psi^{D_4A}_{D_2E}\psi^{ED_3}_{CB}+\frac{q}{q^2-1}\psi^{A\ D_3}_{D_2E}\psi^{ED_4}_{BC}\right. \\ &\left.-\frac{1}{(p^2-1)(p+q)}\psi^{D_4D_3}_{D_2E}\psi^{EA}_{BC}-\frac{1}{(q^2-1)(p+q)}\psi^{D_4D_3}_{D_2E}\psi^{EA}_{CB}\right). \end{aligned} \tag{2.6}$$

A direct calculation shows the connection defined by (2.5) satisfies the conditions (a) and (b). The last statement of the theorem is immediate from the theorem 2.

It should be noticed that $\Sigma\bar{\Gamma}^A_{DA}$ coincides with $\Sigma\Gamma^A_{DA}$.

For a given t-p structure, linear connections satisfying the condition (a) and (b) in the theorem have the same torsion tensor. So, this tensor is called *the torsion of the t-p structure.*

THEOREM 4. *Let M be a differentiable manifold with a t-p structure ψ and ∇ be a linear connection on M. Then, the covariant differential $\nabla\psi$ of ψ vanishes if and only if the parallel displacement of any p-plane of ψ and any q-plane of ψ along any curve gives a p-plane of ψ and a q-plane of ψ respectively.*

PROOF. If $\nabla\psi=0$, then for any family $X(t)$ and any family $Y(t)$ parallel along any curve $C(t)$, $\psi(X(t)\otimes Y(t))$ is parallel along $C(t)$. The converse is also true.

Let $X(t)$ and $Y(t)$ be parallel along a curve $C(t)(0\leq t\leq 1)$, then $X(t)\otimes Y(t)$ is parallel along $C(t)$. Now, if $\nabla\psi=0$, both $\psi(X(t)\otimes Y(t))$ and $X(t)\otimes Y(t)$ are parallel along $C(t)$. When $X(0)$ and $Y(0)$ belong to some p-plane Π^p of ψ, $\psi(X(0)\otimes Y(0))$ coincides with $X(0)\otimes Y(0)$, so $\psi(X(t)\otimes Y(t))=X(t)\otimes Y(t)$. This implies that Π^p is parallelly displaced along $C(t)$. Similarly, it is seen that any q-plane of ψ is parallelly displaced.

Conversely, let $C(t)(0\leqq t\leqq 1)$ be any curve in M and $\{X^{\alpha_i}_{\beta_j}(0)\}$ be some frame at $C(0)$. Then the family of frames $\{X^{\alpha_i}_{\beta_j}(t)\}$ $(0\leqq t\leqq 1)$ parallel along $C(t)$ satisfies

$$\psi(X_{\alpha_i}(t)\otimes X_{\beta_j}(t))=X_{\beta_i}(t)\otimes X_{\alpha_j}(t), \qquad 0\leqq t\leqq 1.$$

This implies $\nabla\psi=0$ in terms of the first statement of the proof.

§3. Grassmannian structures

Let p and q be positive integers and put $n=p+q$. $G_{n,p}(R)$ denotes the Grassmann manifold over R, consisting of all p-dimensional linear subspaces of R^n. Let $GL(n, R)$ be the general linear group, then $G_{n,p}(R)=GL(n, R)/H$, where

$$H=\left\{\begin{matrix} p\{ \\ q\{ \end{matrix}\begin{pmatrix} g^i_j & g^i_\alpha \\ \hline \underbrace{0}_{p} & \underbrace{g^\alpha_\beta}_{q} \end{pmatrix} \in GL(n, R)\right\}.$$

Let $G^2(pq, R)$ be a set of 2-frames $j^2_0(f)$ at O in R^{pq}, where f is a diffeomorphism from a neighborhood of O in R^{pq} onto a neighborhood of O in R^{pq}. Then $G^2(pq, R)$ is a Lie group and it holds

$$G^2(pq, R)=\{(S^{\alpha_i}_{\beta_j}, S^{\alpha_i}_{\beta_j\gamma_k}); (S^{\alpha_i}_{\beta_j})\in GL(pq, R), S^{\alpha_i}_{\beta_j\gamma_k}=S^{\alpha_i}_{\gamma_k\beta_j}\}.$$

A mapping Φ of H to $G^2(pq, R)$ is defined by

$$\Phi\begin{pmatrix} g^i_j & g^i_\alpha \\ \hline 0 & g^\alpha_\beta \end{pmatrix}=(g^{*j}_{\ i}g^\alpha_\beta, -\Sigma(g^{*k}_{\ i}g^{*j}_{\ h}g^h_\gamma g^\alpha_\beta+g^{*j}_{\ i}g^{*k}_{\ h}g^h_\beta g^\alpha_\gamma)),$$

where (g^{*j}_i) is the inverse matrix of (g^i_j). If $\tilde{H}$ be the image $\Phi(H)$, then we have by a calculation

PROPOSITION 2. *Φ is a homomorphism of H into G^2 (pq, R) and the kernel K of Φ is the center of H, i.e,*

$$K=\left\{\alpha\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}; \alpha(\neq 0)\in R\right\}.$$

From the above proposition, it follows $H/K\simeq\tilde{H}$, so we shall sometimes identify $\tilde{H}$ with H/K.

DEFINITION. *Let M be a pq-dimensional manifold and $P^2(M)$ be the 2-*

frame bundle over M *[4]. A grassmannian structure (gr-structure) is a subbundle of* $P^2(M)$ *with structure group* $\tilde{H}$.

Let $P_g(M)$ be a grassmannian structure on M, then $\pi_1^2(p_g(M))$ gives a *t-p* structure on M, where π_1^2 is the natural projection of $P^2(M)$ onto $P(M)$. This *t-p* structure is called *the underlying t-p structure of* $P_g(M)$.

Conversely, for a given *t-p* structure we can construct a grassmannian structure in a natural manner. Let $P_t(M)$ be the given *t-p* structure and ∇ be an arbitrary but fixed symmetric linear connection on M. We may take local coordinate systems $\{(x_U^{\alpha_i}); U\}$ on M satisfying that there is a local section of $P_t(M)$

$$s_U = (x_U^{\alpha_i}, \Sigma X_{\beta_j}^{\alpha_i})$$

on each local neighborhood U. Now, we difine a local section of $P^2(M)$ on each U by

$$\tilde{s}_U = (x_U^{\alpha_i}, X_{\beta_j}^{\alpha_i}, -\Sigma \Gamma_{\delta_l \lambda_h}^{\alpha\, i} X_{\beta_j}^{\delta_l} X_{\gamma_k}^{\lambda_h}). \tag{3.1}$$

For another local neighborhood V with $U \cap V \neq \emptyset$, as s_U and s_V are local sections of $P_t(M)$, it holds on $U \cap V$

$$s_V = s_U g, \qquad g = (g_\beta^\alpha\, g^{*j}_{\ i}) \in G_t(p, q).$$

From this, it follows with ease

$$\tilde{s}_V = \tilde{s}_U \tilde{g}, \qquad \tilde{g} = (g_\beta^\alpha\, g^{*j}_{\ i}, 0) \in \tilde{H}$$

on $U \cap V$. Thus we get a *gr*-structure on M. This *gr*-structure does not depend on the choise of local coordinate systems and that of local sections. The underlying *t-p* structure of the *gr*-structure is also $P_t(M)$. We call this *gr*-structure *the gr-structure determined by* $P_t(M)$ *and* ∇, while we say that ∇ *belongs to the gr-structure.*

It has been shown that a linear symmetric connection determines a *gr*-structure together with a *t-p* structure, but it is possible that many linear symmetric connections determine the same *gr*-structure together with the *t-p* structure and these symmetric connections are characterized as follows.

THEOREM 5. *Two linear symmetric connections* ∇ *and* ∇' *belong to the same grassmannian structure* $P_g(M)$ *if and only if they are* ϕ*-related, where* ϕ *is the underlying t-p structure of* $P_g(M)$.

PROOF. Suppose that ∇ and ∇' belong to the same *gr*-structure. Let

$\Gamma^{\alpha_i}_{\beta_j\gamma_k}$ and $\Gamma'^{\alpha_i}_{\beta_j\gamma_k}$ be components of ∇ and ∇' with respect to local coordinate systems. Let s_U be a local section of the *t-p* structure over a local coordinate neighborhood U. Corresponding to ∇ and ∇', $\tilde{s}_U$ and $\tilde{s}'_U$ are local sections of the *gr*-structure given by (3.1) respectively. Then we have

$$\tilde{s}'_U(x)=\tilde{s}_U(x)\,S(x),\ S(x)\in\tilde{H},\ x\in U. \tag{3.2}$$

It turns out from the definition of $\tilde{s}_U$ and $\tilde{s}'_U$ that S has a form

$$(\delta^{\alpha}_{\beta}\delta^{j}_{i},\ -(\delta^{\alpha_i}_{\beta_k}S^{l}_{r}+\delta^{\alpha_i}_{\gamma_l}S^{k}_{\beta})).$$

Hence, (3.2) implies

$$\Gamma'^{\alpha_i}_{\beta_j\gamma_k}-\Gamma^{\alpha_i}_{\beta_j\gamma_k}=\Sigma X^{\alpha_i}_{\alpha_{1_{i_1}}}Y^{\alpha_1 j_1}_{\beta_j}Y^{\gamma_1 i_1}_{\gamma_k}S^{j_1}_{\gamma_1}+\Sigma X^{\alpha_i}_{\gamma_{1_{j_1}}}Y^{\gamma_1 k_1}_{\gamma_k}\ Y^{\beta_1 j_1}_{\beta_j}S^{k_1}_{\beta_1},$$

where $s_U=(x^{\alpha_i}_U,\ X^{\alpha_i}_{\beta_j})$ and $(Y^{\alpha_i}_{\beta_j})$ is the inverse matrix of $(X^{\alpha_i}_{\beta_j})$. On putting $S^{j}_{\beta}=\Sigma\Phi_{\alpha_i}X^{\alpha_i}_{\beta_j}$. We find

$$\Gamma'^{\alpha_i}_{\beta_j\gamma_k}-\Gamma^{\alpha_i}_{\beta_j\gamma_k}=\Sigma\psi^{\alpha_i\tau_t}_{\beta_j\gamma_k}\Phi_{\tau_t}+\Sigma\psi^{\alpha_i\tau_t}_{\gamma_k\beta_j}\ \Phi_{\tau_t}. \tag{3.3}$$

Conversely, from the relation (3.3), (3.2) follows easily.

The next theorem asserts that the set of *gr*-structures consists of *gr*-structures determined by *t-p* structures and linear symmetric connections.

Theorem 6. *Let M be a paracompact manifold carrying a grassmannian structure $P_g(M)$. Then $P_g(M)$ is a gr-structure determined by the underlying t-p structure of $P_g(M)$ and a certain linear symmetric connection.*

Proof. Let $\{(x^{\alpha_i}_U),\ U\}$ be local coordinate systems of M. We may assume there exists a local section s_U over each coordinate neighborhood U and there exists the partition of unity $\{f_U\}$ subordinate to $\{U\}$. Let $s_U=(x^{\alpha_i}_U,\ X^{\alpha_i}_{\beta_j},\ X^{\alpha_i}_{\beta_j\gamma_k})$, we define a linear symmetric connection ∇_U on U by

$$\Gamma_U{}^{\alpha_i}_{\beta_j\gamma_k}=-\Sigma X^{\alpha_i}_{\beta_{1_{j_1}}\gamma_{1_{k_1}}}Y^{\beta_1 j_1}_{\beta_j}Y^{\gamma_1 k_1}_{\gamma_k},$$

where $(Y^{\beta_j}_{\gamma_k})$ is the inverse matrix of $(X^{\alpha_i}_{\beta_j})$. Then $\nabla=\Sigma f_U\nabla_U$ is a linear symmetric connection on M. Moreover, it belongs to $P_g(M)$ from the theorem 5.

We shall introduce a *gr*-structure which seems to correspond naturally to a given *t-p* structure ψ. Let ∇ be one of connections given by the theorem 3 for the *t-p* structure ψ. We do not know whether ∇ is symmetric or not.

So we take the symmetric connection $\bar{\nabla}$ given by $\bar{\nabla}=\nabla-\frac{1}{2}T$, where T is the torsion tensor of ψ. In this situation, we say simply that ∇ *belongs to the gr-structure* determined by $\bar{\nabla}$ and ψ. If ∇' is the same connection as ∇, $\bar{\nabla}$ and $\bar{\nabla}'$ are symmetric connections and ψ-related. Hence, they determine the same *gr*-structure together with the *t-p* structure ψ. Summing up, we have

THEOREM 7. *Let ψ be a t-p structure on M. Then there exists uniquely the gr-structure with underlying t-p structure ψ, to which any connection given by the theorem* 3 *for ψ belongs.*

The above *gr*-structure is called *the gr-structure determined by ψ.*

§4. Cartan connections on a grassmannian structure

As a *gr*-structure has the structure group H/K, it is better to consider the Grassmann manifold $GL(n, R)/H$ projectively. So, $PL(n-1, R)$ is taken instead of $GL(n, R)$. Moreover, $PL(n-1, R)$ may be identified with $SL^*(n, R)/\{\pm I\}$, where $SL^*(n, R)=\{A \in GL(n, R);\ \det A=\pm 1\}$. This identification leads us to

PROPOSITION 3. *There are left invariant* 1*-forms ω^a_b, $a, b=1, 2, \cdots, n$ on $PL(n-1, R)$ which span the space of left invariant* 1*-forms on $PL(n-1, R)$ and satisfy*

$$\Sigma\omega^a_a=0 \tag{4.1}$$

$$d\omega^a_b=-\Sigma\omega^a_c\wedge\omega^c_b \tag{4.2}$$

Although left invariant 1-forms ω^a_b, $a, b=1, 2, \cdots, n$ are not linearly independent at each point of $PL(n-1, R)$, we treat them as if they were a base of the space of left invariant 1-forms and the equation (4.2) were that of Maurer-Cartan, under consideration of the relation (4.1).

The Lie algebra $\mathfrak{pl}(n-1, R)$ of $PL(n-1, R)$ is regarded as $\mathfrak{gl}(n, R)/\{\alpha I;\ \alpha \in R\}$. Let $\boldsymbol{E}^b_a$ be the matrix whose (a, b)-element equals to 1 and all other elements vanish. Then $\boldsymbol{E}^b_a$, $a, b=1, 2, \cdots, n$ form a base of $\mathfrak{gl}(n, R)$. If $\bar{\boldsymbol{E}}^a_b$ is the class containing $\boldsymbol{E}^a_b$ modulo $\{\alpha I;\ \alpha \in R\}$, $\bar{\boldsymbol{E}}^a_b$, $a, b=1, 2, \cdots, n$ span $\mathfrak{pl}(n, R)$ and have a relation $\Sigma\bar{\boldsymbol{E}}^a_a=0$. They will behave as if they formed a base in this case too.

Suppose that a *gr*-structure $P_g(M)$ is given, we introduce the notion of Cartan connection in $P_g(M)$. It is a 1-form ω on $P_g(M)$ with values in $\mathfrak{pl}(n-1, R)$ satisfying the following conditions

$$
(4.3)\qquad \begin{cases} (1)\quad \omega(A^*)=A & \text{for every } A\in\tilde{\mathfrak{h}}\,(\text{Lie algebra of } \tilde{H}) \\ (2)\quad R_S^*\omega=ad(S^{-1})\,\omega & \text{for every } S\in\tilde{H}, \\ (3)\quad \omega(X)\neq 0 & \text{for every nonzero vector } X, \end{cases}
$$

where A^* is the fundamental vector field corresponding to A. This 1-form ω is called *a grassmannian connection* (*gr-connection*) in $P_g(M)$. With reference to $\overline{\boldsymbol{E}}_b^a$, ω is represented as

$$\omega=\Sigma\omega_b^a\overline{\boldsymbol{E}}_a^b,$$

where ω_b^a are real-valued 1-forms on $P_g(M)$. From the linear dependence of $\overline{\boldsymbol{E}}_a^b$, ω_b^a are not uniquely determined. Indeed, it holds

$$\Sigma\omega_b^a\overline{\boldsymbol{E}}_a^b=\Sigma\tilde{\omega}_b^a\overline{\boldsymbol{E}}_a^b,$$

if and only if $\omega_b^a=\tilde{\omega}_b^a$ $(a\neq b)$ and $\omega_a^a-\tilde{\omega}_a^a=\omega_b^b-\tilde{\omega}_b^b$. However, it turns out that any gr-connection ω determines uniquely real-valued 1-forms ω_b^a such that

$$\omega=\Sigma\omega_b^a\overline{\boldsymbol{E}}_a^b,\quad \Sigma\omega_a^a=0.$$

Now, we have from (4.3)

Proposition 4. *Let ω be a gr-connection in $P_g(M)$ and put*

$$\omega=\Sigma\omega_b^a\overline{\boldsymbol{E}}_a^b,\quad \Sigma\omega_a^a=0.$$

Then it follows

$$d\omega_b^a=-\Sigma\omega_c^a\wedge\omega_b^c+\Omega_b^a,$$

where Ω_b^a are 2-forms generated by ω_i^α.

§ 5. Normal connections on a gr-structure determined by a $t-p$ structure

Let $P_t(M)$ be a $t-p$ structure. In the sequel, we confine our attention to the gr-structure $P_g(M)$ determined by $P_t(M)$. We take a local coordinate system (x^{α_i}) on a local coordinate neighborhood U. Then we can introduce a coordinate system $(x^{\alpha_i}, x^{\alpha_i}_{\beta_j}, x^{\alpha_i}_{\beta_j\gamma_k})$ in $(\pi^2)^{-1}(U)$ naturally, where π^2 is the natural projection of $P^2(M)$ onto M. Let P be any point of $\pi^{-1}(U)$ and i is the natural injection of $P_t(M)$ to $P^2(M)$. Here, π is the natural projection of $P_t(M)$ to M. Suppose s_U is a local section of $P_t(M)$ over U and represented as

$$s_U=(x^{\alpha_i},\ X^{\alpha_i}_{\beta_j}).$$

From the theorem 7, it is shown that local coordinates $x^{\alpha_i}(i(p))$, $x^{\alpha_i}_{\beta_j}(i(P))$, $x^{\alpha_i}_{\beta_j\gamma_k}(i(P))$ of $i(P)$ can be represented as

$$x^{\alpha_i}(i(P))=u^{\alpha_i},$$

$$x^{\alpha_i}_{\beta_j}(i(P))=\Sigma X^{\alpha_i}_{\gamma_k}(i(P))\,u^j_k\,u^\gamma_\beta,\ \det|u^j_k|=\pm\det|u^\gamma_\beta|,$$

$$x^{\alpha_i}_{\beta_j\gamma_k}(i(P))=-\Sigma\left\{\frac{1}{p+q}\,(\phi^{\alpha_iA}_{BC}+\phi^{\alpha_iA}_{CB})\,u_A+\frac{1}{2}\,(N^{\alpha_i}_{BC}+N^{\alpha_i}_{CB})\right\}x^B_{\beta_j}\,x^C_{\gamma_k},$$

where $\phi^{\alpha_iA}_{BC}$ is the tensor determined by $P_t(M)$ and $N^{\alpha}_{BC}{}^{i}$ is the object given by (2.6). Now we shall represent points of $\pi^{-1}(U)$ by

$$(u^{\alpha_i},\ u^j_k,\ u^\gamma_\beta,\ u_{\beta_j}),$$

although it is not a coordinate system in $\pi^{-1}(U)$ in the strict sense.

In the above situation, we define 1-forms θ^α_β, θ^i_j, α, $\beta=p+1, \cdots, n$, $i, j=1, 2, \cdots, p$ on $\pi^{-1}(U)$ by

$$\omega^\alpha_\beta=\Sigma u^{-1}{}^{\alpha}_{\alpha_1}du^{\alpha_1}_{\beta}+\frac{1}{p+q}\Sigma u^{-1}{}^{\alpha}_{\alpha_1}u^{\beta_1}_{\beta}X^F_{\beta_1l}\,X^{-1}{}^{\alpha_1l}_{E}u_F\,du^E$$

$$+\frac{1}{p^2-1}\Sigma\left\{\frac{p^2}{p+q}\frac{\partial X^D_{\beta_1k}}{\partial x^D}X^{-1}{}^{\alpha_1k}_{E}-\frac{1}{p+q}\frac{\partial X^L_G}{\partial x^D}X^{-1}{}^G_LX^D_{\beta_1l}X^{-1}{}^{\alpha_1l}_{E}\right.$$

$$+\varPhi^{\alpha_1l}_{\delta_l\beta_{1k}}X^{-1}{}^{\delta_k}_{E}+\frac{p}{p+q}\,\varPhi^{\delta_k}_{\delta_l\beta_{1k}}X^{-1}{}^{\alpha_1l}_{E}+p\varPhi^{\alpha_1k}_{\beta_{1k}G}\,X^{-1}{}^G_E$$

$$\left.+\frac{p}{p+q}\,\varPhi^D_{GD}X^{-1}{}^G_E\delta^{\alpha_1}_{\beta_1}+\frac{1}{p+q}\,\varPhi^{\gamma_l}_{\gamma_k\delta_l}X^{-1}{}^{\delta_k}_{E}\,\delta^{\alpha_1}_{\beta_1}\right\}u^{-1}{}^{\alpha}_{\alpha_1}\,u^{\beta_1}_{\beta}\,du^E,$$

$$\omega^j_i=-\Sigma u^{-1}{}^{i_1}_{i}du^j_{i_1}-\frac{1}{p+q}\,\Sigma u^{-1}{}^{i_1}_{i}\,u^j_{j_1}X^F_{\delta_{j_1}}\,X^{-1}{}^{\delta_{i1}}_{E}u_F\,du^E$$

$$-\frac{1}{q^2-1}\Sigma\left\{\frac{q^2}{p+q}\frac{\partial X^D_{\delta_{j_1}}}{\partial x^D}X^{-1}{}^{\delta_{i_1}}_{E}-\frac{1}{p+q}\frac{\partial X^L_G}{\partial x^D}X^{-1}{}^G_LX^D_{\delta_{j_1}}X^{-1}{}^{\delta_{i_1}}_{E}\right.$$

$$+\varPhi^{\gamma_{i1}}_{\gamma_k\delta_{j_1}}X^{-1}{}^{\delta_k}_{E}+\frac{q}{p+q}\,\varPhi^{\delta_l\delta}_{\gamma_l\delta_{j_1}}X^{-1}{}^{\gamma_{i1}}_{E}+q\varPhi^{\delta_{i1}}_{\delta_{j_1}G}\,X^{-1}{}^G_E$$

$$+\frac{q}{p+q}\Phi^{D}_{GD}X^{-1}{}^{G}_{E}\delta^{j_1}_{i_1}-\frac{1}{p+q}\Phi^{\gamma_l}_{\gamma_k\delta_l}X^{-1}{}^{\delta_k}_{E}\delta^{j_1}_{i_1}\Big\}u^{-1}{}^{i_1}_{i}u^{j}_{j_1}du^{E},$$

where $\Phi^{A}_{BC}=\Sigma X^{-1}{}^{A}_{L}\Big(\frac{\partial X^{L}_{B}}{\partial x^{D}}X^{D}_{C}-\frac{\partial X^{L}_{C}}{\partial x^{D}}X^{D}_{B}\Big)$.

For every local coordinate neighborhood, we can similarly construct 1-forms which are fit together to form 1-forms on $P_t(M)$. We denote them by ω^{α}_{β}, ω^{i}_{j}.

In order to describe properties of these 1-forms ω^{α}_{β}, ω^{i}_{j}, we introduce some notations. Let $(\theta^{A}, \theta^{A}_{B})$ be the restriction to $P_g(M)$ of the canonical form of $P^2(M)$ [4]. Then, it satisfies

$$d\theta^{A}=-\Sigma\theta^{A}_{B}\wedge\theta^{B}. \tag{5.1}$$

Next, $\tilde{T}^{A}_{BC}$ are functions on $P_g(M)$ given by

$$\tilde{T}^{A}_{BC}=\Sigma T^{D}_{EF}x^{-1}{}^{A}_{D}x^{E}_{B}x^{F}_{C},$$

where T^{D}_{EF} is the torsion tensor of $P_t(M)$. Now it holds

$$\theta^{\alpha}_{\beta}{}^{i}_{j}=\omega^{\alpha}_{\beta}\delta^{j}_{i}-\omega^{j}_{i}\delta^{\alpha}_{\beta}-\frac{1}{2}\Sigma\tilde{T}^{\alpha}_{B}{}^{i}_{\beta_j}\theta^{B}. \tag{5.2}$$

From (5.1) and (5.2), it follows

$$d\theta^{\alpha}_{i}=-\Sigma\omega^{\alpha}_{\beta}\wedge\theta^{\beta}_{i}-\Sigma\theta^{\alpha}_{j}\wedge\omega^{j}_{i}+\frac{1}{2}\Sigma\tilde{T}^{\alpha}_{\gamma}{}^{i}_{k}{}_{\beta_j}\theta^{\gamma}_{\kappa}\wedge\theta^{\beta}_{j}. \tag{5.3}$$

Next, we have

$$\Sigma\omega^{\alpha}_{\alpha}+\Sigma\omega^{i}_{i}=0. \tag{5.4}$$

We define a 1-form θ with values in $\mathfrak{pl}(n-1, R)$ by

$$\theta=\Sigma\omega^{j}_{i}\bar{\boldsymbol{E}}^{i}_{j}+\Sigma\omega^{\alpha}_{\beta}\bar{\boldsymbol{E}}^{\beta}_{\alpha}+\Sigma\theta^{\alpha}_{i}\bar{\boldsymbol{E}}^{i}_{\alpha}$$

which is called *a canonical form on* $P_g(M)$. Then, θ satisfies

$$\begin{cases} R^{*}_{A}\theta=ad(A^{-1})\theta & \text{for every } A\in\tilde{H}, \\ \theta(\bar{\boldsymbol{E}}^{\alpha}_{\beta})=\bar{\boldsymbol{E}}^{\alpha}_{\beta}, \quad \theta(\bar{\boldsymbol{E}}^{i}_{j})=\bar{\boldsymbol{E}}^{i}_{j}, \end{cases} \tag{5.5}$$

where $ad(A^{-1})$ is in fact the mapping: $\mathfrak{pl}(n-1, R)/\{\bar{\boldsymbol{E}}^{\alpha}_{i}\}\rightarrow\mathfrak{pl}(n-1, R)/\{\bar{\boldsymbol{E}}^{\alpha}_{i}\}$ induced by $ad(A^{-1})$: $\mathfrak{pl}(n-1, R)\rightarrow\mathfrak{pl}(n-1, R)$.

From (5.3) and (5.5), we get in the same manner as [4. p. 222]

Proposition 5. *There are 1-forms* ω^i_α *such that* $\omega=\theta+\Sigma\omega^i_\alpha\bar{E}^\alpha_i$ *is a gr-connection on* $P_g(M)$, *where* θ *is the canonical form on* $P_g(M)$.

We shall show there is a unique *gr*-connection that is one of connections given in the proposition 5 and satisfies a certain condition. Suppose $\omega=\theta+\Sigma\omega^i_j\bar{E}^j_i$ is a *gr*-connection, then as it was described in the proposition 4, the structure equations of ω are given by

$$(5.6)\quad \begin{cases} d\theta^\alpha_i=-\Sigma\omega^\alpha_\gamma\wedge\theta^\gamma_i-\Sigma\theta^\alpha_j\wedge\omega^j_i+\dfrac{1}{2}\Sigma\tilde{T}^{\alpha_i}_{\gamma_k\beta_j}\theta^\gamma_k\wedge\theta^\beta_j, \\ d\omega^i_j=-\Sigma\omega^i_k\wedge\omega^k_j-\Sigma\omega^i_\gamma\wedge\theta^\gamma_j+\dfrac{1}{2}\Sigma K^i_{j\gamma_k\beta_j}\theta^\gamma_k\wedge\theta^\beta_j, \\ d\omega^\alpha_\beta=-\Sigma\theta^\alpha_k\wedge\omega^k_\beta-\Sigma\omega^\alpha_\gamma\wedge\omega^\gamma_\beta+\dfrac{1}{2}\Sigma K^\alpha_{\beta\gamma_k\beta_j}\theta^\gamma_k\wedge\theta^\beta_j, \\ d\omega^i_\alpha=-\Sigma\omega^i_k\wedge\omega^k_\alpha-\Sigma\omega^i_\gamma\wedge\omega^\gamma_\alpha+\dfrac{1}{2}\Sigma K^i_{\alpha\gamma_k\beta_j}\theta^\gamma_k\wedge\theta^\beta_j. \end{cases}$$

$\tilde{T}^{\alpha_i}_{\gamma_k\beta_j}$ is called *the torsion of* ω and $K^i_{j\gamma_k\beta_j}$, $K^\alpha_{\beta\gamma_k\beta_j}$, $K^i_{\alpha\gamma_k\beta_j}$ are called *the curvatures of* ω.

Now we consider the following condition

$$(5.7)\qquad \Sigma R^{\alpha_i}_{\beta_j\alpha_i\gamma_k}=0,$$

where

$$(5.9)\qquad R^{\alpha_i}_{\beta_j\gamma_k\delta_l}=\delta^\alpha_\beta K^j_{i\gamma_k\delta_l}-\delta^j_i K^\alpha_{\beta\gamma_k\delta_l}.$$

We are now in position to prove the uniqueness of a *gr*-connection given in the proposition 5 and satisfying the condition (5.7). Let $\omega=\theta+\Sigma\omega^i_j\overline{\boldsymbol{E}}^j_i$ and $\bar{\omega}=\theta+\Sigma\bar{\omega}^i_j\overline{\boldsymbol{E}}^j_i$ be *gr*-connection of the kind. The condition (4.3) implies

$$\bar{\omega}^i_\alpha-\omega^i_\alpha=\Sigma A^{ij}_{\alpha\beta}\theta^\beta_j,$$

where $A^{ij}_{\alpha\beta}$ are functions on $P_g(M)$. A computations using the structure equations ω and $\bar{\omega}$ shows

$$-A^{jk}_{\delta\gamma}\delta^l_i\delta^\alpha_\beta+A^{il}_{\gamma\delta}\delta^k_i\delta^\alpha_\beta+A^{kl}_{\beta\delta}\delta^\alpha_\gamma\delta^j_i-A^{lk}_{\beta\gamma}\delta^\alpha_\delta\delta^j_i+\bar{R}^{\alpha_i}_{\beta_j\gamma_k\delta_l}-R^{\alpha_i}_{\beta_j\gamma_k\delta_l}=0,$$

where $\bar{R}^{\alpha_i}_{\beta_j\gamma_k\delta_l}$ and $\bar{R}^{\alpha_i}_{\beta_j\gamma_k\delta_l}$ are functions given by (5.8) for $\bar{\omega}$ and ω respectively. Together with the condition (5.7), this leads us to

$$(n^2-4)A^{jl}_{\beta\delta}=0.$$

So, $A^{jl}_{\beta\delta}=0$, if $n\geqq 3$.

In the next place, we shall show the existence of a gr-connection satisfying the conditions. Let $\omega=\theta+\Sigma\omega^i_\alpha\bar{\boldsymbol{E}}^\alpha_i$ be an arbitrary gr-connection and $R^{\alpha_i}_{\beta_j\gamma_k\delta_l}$ be functions given by (5.8) corresponding to ω. Put

$$A^{jl}_{\beta\delta}=\frac{1}{n^2-4}\left(\frac{n^2-2}{n}R_{\beta_j\delta_l}+R_{\delta_j\beta_l}+R_{\beta_l\delta_j}+\frac{2}{n}R_{\delta_l\beta_j}\right),$$

where $R_{\beta_j\delta_l}=R^{\alpha_i}_{\beta_j\alpha_i\delta_l}$. Then 1-form $\bar{\omega}$ defined by

$$\bar{\omega}=\theta+\Sigma(\omega^i_\alpha+A^{il}_{\alpha\delta}\theta^\delta_l)\bar{E}^\alpha_i$$

is a required gr-connection. Thus, we have

THEOREM 8. *Let $P_g(M)$ be a gr-structure determined by a given $t-p$ structure on a pq-dimensional manifold $M(p+q\geqq 3)$. Then, there exists a unique gr-connection $\omega=\theta+\Sigma\omega^i_\alpha\bar{\boldsymbol{E}}^\alpha_i$ satisfying the condition* (5.7), *where θ is the canonical form on $P_g(M)$.*

The connection given in the theorem 8 is called *the normal connection of* $P_g(M)$.

The Grassmann manifold $G_{n,p}(R)$ carries the natural $t-p$ structure and $PL(n-1, R)$ is regarded as the gr-structure determined by the $t-p$ structure. This gr-structure is called *the standard gr-structure.* The left invariant 1-forms given in the proposition 3 is the normal connection of the standard gr-structure. A gr-structure $P_g(M)$ with projection π, is called *flat* if and only if for every point of the base manifold M, there is a neighborhood U of the point and a local diffeomorphism of U to $G_{n,p}(R)$ whose prolongation maps $\pi^{-1}(U)$ into the standard gr-structure.

If a gr-structure $P_g(M)$ is flat, it is evident that the normal connection of $P_g(M)$ has the vanishing torsion and the vanishing curvatures. Conversely, if a gr-structure $P_g(M)$ has the normal connection with the vanishing torsion and the vanishing curvatures, the structure equations of the normal connection and the equations of Maurer-Cartan of $PL(n-1, R)$ have the same forms. Then, every point of $P_g(M)$ has a neighborhood U and a diffeomorphism of U to $PL(n-1, R)$ whose differential maps the left invariant 1-forms to the normal connection. In addition, every local diffeomorphism is a plolongation of a certain local diffeomorphism of the base manifolds. Summing up, it follows

THEOREM 9. *A gr-structure determined by a given $t-p$ structure is flat if and only if the normal connection of the gr-structure has the vanishing torsion and the vanishing curvatures.*

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