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On the Betti Numbers of Certain Local Rings of Embedding Dimension 3

By

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Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k and let K be the Koszul complex associated with R. The relationship between the homology algebra $H(K)$ and the homological invariants of R such as the Betti numbers $B_p=\dim_k \mathrm{Tor}_p^R(k, k)$ and the Betti series $\mathscr{B}(R)=\sum B_p Z^p$ of R has been investigated. Especially, for the local ring of embedding dimension $n\leqq 2$, the Betti series of R is completely determined by the multiplicative property of the homology algebra $H(K)$ [5, 7]. In the case when $n=3$ Wiebe [9] proved the rationality of $\mathscr{B}(R)$ under the assumption that R is a Gorenstein ring and is not complete intersection by calculating the syzygy modules of k.

In this paper, mostly we assume that $n=3$ and we calculate the Betti numbers by using the spectral sequence associated with the Koszul complex introduced by T. H. Gulliksen and G. Levin [3]. Then, as an application of this, we give the recurrence relation between the Betti numbers and give the explicit form of Betti series under the additional assumption that $H_1(K)^2=0$ and $H_1(K)H_2(K)=H_3(K)$. This gives an alternating proof of a theorem due to Wiebe above in some extended form. As a second application, we will calculate the fourth deflection ε_4 which is also an invariant of R by means of $H(K)$ in a similar restricted case.

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Unless otherwise specified, we shall use the similar notations and the similar terminology which appeared in [6].

1. Let $(R, \mathfrak{m})$ be a local ring of embedding dimension n with residue field k. First we recall some properties of the exact couples defined by means of the Koszul complex K of R [3].

Let $D=\sum_{p,q} D_{p,q}$ and $E=\sum_{p,q} E_{p,q}$ be two graded R-modules where $E_{p,q}=\mathrm{Tor}_p^R(k, H_q(K))$ for $p\geqq 0$, $q\geqq 0$ and is zero for $p<0$ or $q<0$ and where

$$D_{p,q}=\begin{cases}\mathrm{Tor}_p^R(k,\ B_q(K)) & \text{if } p\geqq 0,\ q\geqq 0,\\ 0 & \text{if } q<0,\\ E_{0,0} & \text{if } p<0 \text{ and } p+q=-1,\\ D_{0,p+q} & \text{if } p<0 \text{ and } p+q\neq -1.\end{cases}$$

Then the exact couple $\mathfrak{C}=\{D,\ E;\ f,\ g,\ h\}$ is defined by a pair of bigraded modules D, E together with three homomorphisms f, g, h such that

$$\mathfrak{C}:\quad \begin{array}{ccc} D & \xrightarrow{f} & D \\ & {\scriptstyle h}\nwarrow \quad \swarrow{\scriptstyle g} & \\ & E & \end{array},$$

where f, g and h are defined as follows. Let

$$\begin{array}{ccc} \mathrm{Tor}^R(k,\ B(K)) & \xrightarrow{i} & \mathrm{Tor}^R(k,\ Z(K)) \\ & {\scriptstyle h}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & \mathrm{Tor}^R(k,\ H(K)) & \end{array}$$

be an exact triangle induced by the exact sequence $0\to B(K)\to Z(K)\to H(K)\to 0$ and let ∂ be the isomorphism $D_{p,q}\to \mathrm{Tor}_{p-1}(k,\ Z_{q+1}(K))$ which is induced by the exact sequence $0\to Z_{q+1}(K)\to K_{q+1}\to B_q(K)\to 0$ for $p\geqq 1$, $q\geqq 0$. Then $f\colon D_{p,q}\to D_{p+1,q-1}$ is defined by $f=\partial^{-1}\cdot i$ if $p\geqq 0$, $q\geqq 1$ and identity if $p<0$. And $g\colon D_{p,q}\to E_{p-1,q+1}$ is defined by $g=j\cdot\partial$ if $p\geqq 1$, $q\geqq 0$ and zero if $p<1$. Finally, $h\colon E_{p,q}\to D_{p-1,q}$ is defined as identity for $p=0$, $q=0$ and zero for $p=0$, $q\neq 0$.

We can easily see that this couple $\mathfrak{C}$ is exact and f is of bidegree $(1,\ -1)$, g bidegree $(-1,\ 1)$ and h bidegree $(-1,\ 0)$. In the following, we often write $D^1_{p,q}$(resp. $E^1_{p,q}$) in place of $D_{p,q}$(resp. $E_{p,q}$).

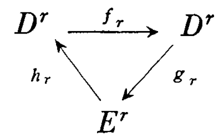
The exactness of $\mathfrak{C}$ shows that the composite map $gh\colon E^1_{p,q}\to E^1_{p-2,q+1}$ has square zero, hence is a differential on E^1. Form the homology module $H(E^1)$ for this differential and put $E^2=H(E^1)$ and $D^2=fD^1$. Then, we construct the derived couple

$$\mathfrak{C}^2:\quad \begin{array}{ccc} D^2 & \xrightarrow{f_2} & D^2 \\ & {\scriptstyle h_2}\nwarrow \quad \swarrow{\scriptstyle g_2} & \\ & E^2 & \end{array},$$

where f_2 is the map induced by f and g_2 and h_2 are defined by $g_2(fd)=gd$

$+ghE^1$ for $d \in D^1$ and $h_2(e+ghE^1)=he$ for $e \in E^1$ such that $ghe=0$. Then the diagram chase proves the derived couple $\mathfrak{C}^2$ is also exact.

Iterating this process $(r-1)$-times, we get the r-th derived exact couple $\mathfrak{C}^r$ of $\mathfrak{C}$:

$\mathfrak{C}^r$: 

where f_r is of bidegree $(1, -1)$, g_r bidegree $(-r, r)$ and h_r bidegree $(-1, 0)$. For details on this subject, see Ch. XI in [4].

The following properties concerning the exact couples are substaintially contained in [3] and we will use these properties in the later argument.

(P$_1$) *The map* $gh: E^1_{p,q} \to E^1_{p-2,q+1}$ *is* $H(K)$*-linear* [3, Theorem 4.4.2]

(P$_2$) *For a fixed integer* r, $E^{r+1}_{p,q}=E^r_{p,q}$ *if* $p \leqq r$ *and* $q<r$.

In fact, since $E^{r+1}=H(E^r)$, we have

$$E^{r+1}_{p,q}=\operatorname{Ker}(E^r_{p,q} \to E^r_{p-r-1,q+r})/\operatorname{Im}(E^r_{p+r+1,q-r} \to E^r_{p,q}).$$

On one hand, the modules $E^r_{p-r-1,q+r}$ and $E^r_{p+r+1,q-r}$ are zero if $p \leqq r$ and $q<r$, so that our assertion follows.

(P$_3$) $E^{n+1}_{p,q}=0$ *if* $q \neq 0$ *and* $E^{n+1}_{p,0} \simeq K_p \otimes_R k$ *where* n *is the embedding dimension of* R [3, Theorem 4.4.1].

(P$_4$) $E^r_{p,q}=0$ *if* i) $q>n$ *or* ii) $r>n$, $q \neq 0$ *or* iii) $n \geqq r \geqq p$ *and* $r>q>0$.

For, if $q>n$ then we have $E^1_{p,q}=0$ since the module K_q is zero. Now the first part of our assertion follows from the fact that the module $E^r_{p,q}$ is the factor module of a submodule of $E^1_{p,q}$. The second and the last part follow easily by using the properties (P$_2$) and (P$_3$).

(P$_5$) $D^r_{p,q}=0$ *if* $q<0$ *or* $r+q>n$.

In fact, since $D^r_{p,q}$ is a submodule of $D^1_{p,q}$, we get $D^r_{p,q}=0$ if $q<0$. On the other hand, we have $D^r_{p,q}=f^{r-1}D^1_{p-r+1,q+r-1}$, so that $D^r_{p,q}=0$ if $q+r>n$.

(P$_6$) *If* $n \geqq 2$, *then* $E^n_{p,n-1}=0$ *for each* p.

For, in the following exact sequence

$$D^n_{p+n,-1} \xrightarrow{g_n} E^n_{p,n-1} \xrightarrow{h_n} D^n_{p-1,n-1},$$

it holds $D^n_{p+n,-1}=D^n_{p-1,n-1}=0$ by (P$_5$). Hence $E^n_{p,n-1}=0$ for each p.

(P$_7$) *If* $n \geqq 3$ *and if* $E^{n-1}_{p,n}=0$ *for some* p, *then* $E^{n-1}_{p+n,1}=0$.

For, consider the following exact diagram:

$$\begin{array}{ccccccccc} \cdots \longrightarrow D^{n-1}_{p+2n-1,-n+2} & \xrightarrow{g_{n-1}} & E^{n-1}_{p+n,1} & \xrightarrow{h_{n-1}} & D^{n-1}_{p+n-1,1} & \xrightarrow{g_{n-1}} & E^{n-1}_{p,n} & \longrightarrow \cdots \\ & & & & \uparrow f_{n-1} & & & \\ & & & & D^{n-1}_{p+n-2,2} & & & \end{array}$$

Since $D^{n-1}_{p+n-2,2}=0$ by (P_5), the exactness from the bottom to the right hand side implies that $D^{n-1}_{p+n-1,1}=0$ by virtue of our assumption. Now the exactness of the left hand side shows that $E^{n-1}_{p+n,1}=0$ since $D^{n-1}_{p+2n-1,-n+2}$ is zero by the same property (P_5).

In the following, we restrict our consideration when the embedding dimension n is 3 and calculate the Betti numbers B_p $(=\dim_k \operatorname{Tor}^R_p(k, k))$ by using the spectral sequences.

THEOREM 1. *Let R be a local ring of embedding dimension* 3. *Then, we have*

$$\begin{aligned} B_{p+5} = {} & \dim_k E^3_{p+1,3} + \dim_k E^2_{p+2,2} - \dim_k E^2_{p+3,1} \\ & + \dim_k \operatorname{Ker}(E^1_{p+3,1} \to E^1_{p+1,2}) \qquad \text{for } p \geqq 0. \end{aligned}$$

PROOF. Since $E^r_{p+5,0}=\operatorname{Ker}(E^{r-1}_{p+5,0} \to E^{r-1}_{p+5-r,r-1})$ for $r>1$, the following three sequences are exact:

$$0 \longrightarrow E^2_{p+5,0} \longrightarrow E^1_{p+5,0} \longrightarrow \operatorname{Im}(E^1_{p+5,0} \longrightarrow E^1_{p+3,1}) \longrightarrow 0,$$

$$0 \longrightarrow E^3_{p+5,0} \longrightarrow E^2_{p+5,0} \longrightarrow \operatorname{Im}(E^2_{p+5,0} \longrightarrow E^2_{p+2,2}) \longrightarrow 0,$$

$$0 \longrightarrow E^4_{p+5,0} \longrightarrow E^3_{p+5,0} \longrightarrow \operatorname{Im}(E^3_{p+5,0} \longrightarrow E^3_{p+1,3}) \longrightarrow 0.$$

Hence, we have

$$\begin{aligned} (*) \quad B_{p+5} = \dim_k E^1_{p+5,0} = {} & \dim_k E^4_{p+5,0} + \dim_k \operatorname{Im}(E^1_{p+5,0} \longrightarrow E^1_{p+3,1}) \\ & + \dim_k \operatorname{Im}(E^2_{p+5,0} \longrightarrow E^2_{p+2,2}) + \dim_k \operatorname{Im}(E^3_{p+5,0} \longrightarrow E^3_{p+1,3}). \end{aligned}$$

Clearly, $E^4_{p+5,0}=0$ by (P_3). Now, since $E^2_{p+3,1}=\operatorname{Ker}(E^1_{p+3,1} \to E^1_{p+1,2})/\operatorname{Im}(E^1_{p+5,0} \to E^1_{p+3,1})$, we obtain the following exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Im}(E^1_{p+5,0} \longrightarrow E^1_{p+3,1}) \longrightarrow \operatorname{Ker}(E^1_{p+3,1} \longrightarrow E^1_{p+1,2}) \\ \longrightarrow E^2_{p+3,1} \longrightarrow 0, \end{aligned}$$

so that the second term of the right hand side of (*) equals to $\dim_k \mathrm{Ker}(E^1_{p+3,1} \to E^1_{p+1,2}) - \dim_k E^2_{p+3,1}$.

Further, using the properties (P_4) and (P_6), we can deduce that $0 = E^3_{p+2,2} = E^2_{p+2,2}/\mathrm{Im}(E^2_{p+5,0} \to E^2_{p+2,2})$. Therefore we have $\mathrm{Im}(E^2_{p+5,0} \to E^2_{p+2,2}) = E^2_{p+2,2}$. Also, by the similar argument, we have $\mathrm{Im}(E^3_{p+5,0} \to E^3_{p+1,3}) = E^3_{p+1,3}$ by virtue of $E^4_{p+1,3} = 0$. This completes the proof.

2. As an application of the preceding result, we give the recurrence relation which holds between the Betti numbers B_p under the additional assumption on the multiplication of the homology algebra $H(K)$.

LEMMA 1. If $H_1(K)^2 = 0$, then

$$\dim_k \mathrm{Ker}(E^1_{p,1} \to E^1_{p-2,2}) = \varepsilon_1 B_p \qquad \text{for } p \geqq 0.$$

PROOF. Since the homology algebra $H(K)$ is a k-vector space, $E^1_{p,1}$ is isomorphic to the k-module $\mathrm{Tor}^R_p(k, k) \otimes H_1(K)$. By the property ($P_1$), the map $g_1 h_1 : E^1_{p,1} \to E^1_{p-2,2}$ is $H(K)$-linear, hence the image of $E^1_{p,1}$ in $E^1_{p-2,2}$ is the submodule of $\mathrm{Tor}_{p-2}(k, H_1(K)) \otimes H_1(K)$ which is zero by our hypothesis. Therefore, we have

$$\dim_k \mathrm{Ker}(E^1_{p,1} \to E^1_{p-2,2}) = \dim_k E^1_{p,1} = B_p \cdot \dim_k H_1(K).$$

Whence, the lemma follows since $\varepsilon_1 = \dim_k H_1(K)$.

REMARK. It can be shown that the lemma above holds for an arbitrary embedding dimension n.

LEMMA 2. *Suppose* $H_1(K)^2 = 0$ *and* $H_3(K) = H_1(K) H_2(K)$. *Then* $E^2_{p,1} = 0$ *implies* $E^2_{p,3} = 0$ *where* p *is any integer.*

PROOF. By the definition, $E^2_{p,3} = \mathrm{Ker}(E^1_{p,3} \to E^1_{p-2,4})/\mathrm{Im}(E^1_{p+2,2} \to E^1_{p,3})$. Obviously, we have $\mathrm{Ker}(E^1_{p,3} \to E^1_{p-2,4}) = E^1_{p,3}$. Now, we see that the map $g_1 h_1 : E^1_{p+2,0} \to E^1_{p,1}$ is an epimorphism. In fact, by considering the complex

$$\cdots \longrightarrow E^1_{p+2,0} \xrightarrow{g_1 h_1} E^1_{p,1} \xrightarrow{g_1 h_1} E^1_{p-2,2} \longrightarrow \cdots,$$

we obtain from $E^2_{p,1} = 0$ that $\mathrm{Im}(E^1_{p+2,0} \to E^1_{p,1}) = \mathrm{Ker}(E^1_{p,1} \to E^1_{p-2,2})$. Whence, from lemma 1, we have $\dim_k \mathrm{Im}(E^1_{p+2,0} \to E^1_{p,1}) = \varepsilon_1 B_p$ which implies that the above map $g_1 h_1 : E^1_{p+2,0} \to E^1_{p,1}$ is an epimorphism.

It then follows, by linearity, that the image of $E^1_{p+2,2}$ in $E^1_{p,3}$ is isomorphic to the k-module $\mathrm{Tor}_p(k, k) \otimes H_1(K) H_2(K)$. Therefore,

$$\dim_k E^2_{p,3} = \dim_k E^1_{p,3} - \dim_k \operatorname{Im}(E^1_{p+2,2} \to E^1_{p,3})$$
$$= B_p \cdot \dim_k H_3(K) - B_p \cdot \dim_k H_1(K) H_2(K)$$

which is 0 by our assumption.

COROLLARY. *Under the same hypothesis as in lemma 2, we have*

$$E^2_{p,1} = 0 \quad \textit{and} \quad E^2_{p,3} = 0 \qquad \textit{for each } p.$$

PROOF. From (P_4), we have $E^2_{i,1} = 0$ for $i = 0, 1, 2$. Hence it follows that $E^2_{i,3} = 0$ for $i = 0, 1, 2$ by lemma 2. Applying the property (P_7) to the fact $E^2_{i,3} = 0$ ($i = 0, 1, 2$), we obtain $E^2_{j,1} = 0$ for $j = 3, 4, 5$. Thus, we can proceed this argument indefinitely.

LEMMA 3. *Under the same hypothesis as in lemma 2, we have*

$$\dim_k E^2_{p+2,2} = \varepsilon_2 B_{p+2} - (\dim_k H_1(K) H_2(K)) \cdot B_p \qquad \textit{for } p \geqq 0.$$

PROOF. The argument similar to the proof of lemma 1 shows that $\operatorname{Im}(E^1_{p+4,1} \to E^1_{p+2,2}) = 0$ since $H_1(K)^2 = 0$. Hence we have

$$\dim_k E^2_{p+2,2} = \dim_k \operatorname{Ker}(E^1_{p+2,2} \to E^1_{p,3}).$$

On one hand, by the same argument as in the proof of lemma 2, we have that $\operatorname{Im}(E^1_{p+2,2} \to E^1_{p,3}) \simeq \operatorname{Tor}_p(k, k) \otimes H_1(K) H_2(K)$. Therefore, we obtain

$$\dim_k E^2_{p+2,2} = \dim_k E^1_{p+2,2} - \dim_k \operatorname{Im}(E^1_{p+2,2} \to E^1_{p,3})$$
$$= B_{p+2} \cdot \dim_k H_2(K) - B_p \cdot \dim_k H_1(K) H_2(K).$$

Now the formula follows from the fact $\varepsilon_2 = \dim_k(H_2(K)/H_1(K)^2)$.

By making use of these lemmas, we can give the alternating proof of a theorem due to Wiebe [9, Satz 9] in some extended form.

THEOREM 2. *Let R be a local ring of embedding dimension 3. Suppose that $H_1(K)^2 = 0$ and $H_3(K) = H_1(K) H_2(K)$ where K is the Koszul complex of R. Then the Betti series of R has the following form*

$$\mathscr{B}(R) = \frac{(1+Z)^3}{1 - \varepsilon_1 Z^2 - \varepsilon_2 Z^3 - \left(\varepsilon_3 - \binom{\varepsilon_1}{2}\right) Z^4 - (\varepsilon_4 - \varepsilon_1 \varepsilon_2) Z^5}.$$

PROOF. We can easily see that the modules $E^3_{p+1,3}$ and $E^2_{p+3,1}$ which appeared in theorem 1 are zero by using the corollary of lemma 2. Therefore we obtain the following recurrence relation of Betti numbers using above lemmas:

$$B_{p+5}=\varepsilon_1 B_{p+3}+\varepsilon_2 B_{p+2}-(\dim_k H_1(K)H_2(K))\cdot B_p \qquad (p\geqq 0).$$

Now the Betti series $\mathscr{B}(R)$ follows from this by using the relation, $\dim_k H_1(K)H_2(K)=\varepsilon_1\varepsilon_2-\varepsilon_4$ [6, Lemma 4, or Theorem 3 bellow].

Remark. When the embedding dimension n is 2, we can easily deduce the following formula by the similar argument as in the theorem 1:

$$B_{p+3}=\dim_k E^2_{p,2}+\dim_k \operatorname{Ker}(E^1_{p+1,1}\to E^1_{p-1,2})-\dim_k E^2_{p+1,1} \qquad \text{for } p\geqq 0.$$

Now if $H_1(K)^2=0$, we can prove that

$$\dim_k E^2_{p,2}=\dim_k E^1_{p,2}=\varepsilon_2 B_p.$$

On one hand, from the remark of lemma 1 and the property (P_6), we see that the dimension of $\operatorname{Ker}(E^1_{p+1,1}\to E^1_{p-1,2})$ is equal to $\varepsilon_1 B_{p+1}$ and that the module $E^2_{p+1,1}$ is zero.

Therefore, in this case, we have the next recurrence relation of Betti numbers:

$$B_{p+3}=\varepsilon_1 B_{p+1}+\varepsilon_2 B_p \qquad (p\geqq 0),$$

so that we get

$$\mathscr{B}(R)=\frac{(1+Z)^2}{1-\varepsilon_1 Z^2-\varepsilon_2 Z^3}$$

(see [7], [5] and [9]).

3. As the second application of theorem 1, we can compute the fourth deflection ε_4 in some restricted case.

Theorem 3. *Let R be a local ring of embedding dimension $n\leqq 3$. Suppose $H_3(K)=H_1(K)H_2(K)$, then we have*

$$\varepsilon_4=\varepsilon_1\varepsilon_2-\dim_k H_1(K)H_2(K)+\dim_k H_1(K)^3.$$

Proof. By theorem 1, we have

$$B_5=\dim_k E^3_{1,3}+\dim_k E^2_{2,2}-\dim_k E^2_{3,1}+\dim_k \mathrm{Ker}(E^1_{3,1}\to E^1_{1,2}).$$

First we show that the modules $E^2_{3,1}$ and $E^3_{1,3}$ are zero. For, in the complex

$$0\longrightarrow E^2_{3,1}\xrightarrow{g_2h_2}E^2_{0,3}\longrightarrow\cdots,$$

the map $g_2h_2: E^2_{3,1}\to E^2_{0,3}$ is zero [3, proof of theorem 4.4.3], and also the module $E^3_{3,1}=\mathrm{Ker}(E^2_{3,1}\to E^2_{0,3})$ is zero by (P_4). This proves that $E^2_{3,1}=0$.

To see $E^3_{1,3}=0$, it is enough to show $E^2_{1,3}=0$. Now, we consider the module $E^2_{1,3}=E^1_{1,3}/\mathrm{Im}(E^1_{3,2}\to E^1_{1,3})$. Since the map $g_1h_1: E^1_{3,0}\to E^1_{1,1}$ is an epimorphism in virtue of $E^2_{1,1}=0$, the linearity of g_1h_1 gives the image of $E^1_{3,2}$ in $E^1_{1,3}$ is isomorphic to the k-module $\mathrm{Tor}_1(k, k)\otimes H_1(K)H_2(K)$. So we have that $E^2_{1,3}\cong\mathrm{Tor}_1(k, k)\otimes(H_3(K)/H_1(K)H_2(K))$ which is zero by our assumption.

Next we compute the dimension of the module $\mathrm{Ker}(E^1_{3,1}\to E^1_{1,2})$. By the similar argument as above, we see that $\mathrm{Im}(E^1_{3,1}\to E^1_{1,2})\cong\mathrm{Tor}_1(k, k)\otimes H_1(K)^2$. From this, we deduce that

$$\begin{aligned}\dim_k \mathrm{Ker}(E^1_{3,1}\to E^1_{1,2})&=\dim_k E^1_{3,1}-\dim_k \mathrm{Im}(E^1_{3,1}\to E^1_{1,2})\\&=B_3\varepsilon_1-B_1\cdot\dim_k H_1(K)^2.\end{aligned}$$

Finally, we compute the dimension of the module $E^2_{2,2}$. In the complex

$$\cdots\longrightarrow E^1_{4,1}\longrightarrow E^1_{2,2}\longrightarrow E^1_{0,3}\longrightarrow 0,$$

the image of $E^1_{2,2}$ in $E^1_{0,3}$ is isomorphic to the k-module $H_1(K)H_2(K)$, since the map $g_1h_1: E^1_{2,0}\to E^1_{0,1}$ is an epimorphism. Therefore, we get

$$\dim_k \mathrm{Ker}(E^1_{2,2}\to E^1_{0,3})=B_2\cdot\dim_k H_2(K)-\dim_k H_1(K)H_2(K).$$

Now, from the fact $E^2_{2,1}=0$, we have $\mathrm{Im}(E^1_{4,0}\to E^1_{2,1})=\mathrm{Ker}(E^1_{2,1}\to E^1_{0,2})$ whose dimension is equal to $B_2\cdot\dim_k H_1(K)-\dim_k H_1(K)^2$ since $\mathrm{Im}(E^1_{2,1}\to E^1_{0,2})\cong H_1(K)^2$. Hence, by linearity, we have

$$\dim_k \mathrm{Im}(E^1_{4,1}\to E^1_{2,2})=B_2\cdot\dim_k H_1(K)^2-\dim_k H_1(K)^3.$$

Therefore, we obtain

$$\begin{aligned}\dim_k E^2_{2,2}&=\dim_k \mathrm{Ker}(E^1_{2,2}\to E^1_{0,3})-\dim_k \mathrm{Im}(E^1_{4,1}\to E^1_{2,2})\\&=B_2\varepsilon_2-\dim_k H_1(K)H_2(K)+\dim_k H_1(K)^3.\end{aligned}$$

Putting all the information together, we get the expression of B_5:

$$B_5 = B_2\varepsilon_2 - \dim_k H_1(K)H_2(K) + \dim_k H_1(K)^3 + B_3\varepsilon_1 - B_1 \cdot \dim_k H_1(K)^2.$$

On one hand, we know that

$$B_5 = 4\varepsilon_1 + 3\binom{\varepsilon_1}{2} + 3\varepsilon_2 + \varepsilon_1\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4,$$

where $\varepsilon_3 = \binom{\varepsilon_1}{2} - \dim_k H_1(K)^2$ [5, Theorem 2 and 6, Theorem 1], so that our assertion follows.

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