A Remark on Harmonic Quasiconformal Mappings

By

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$K$-quasiconformal mappings of Riemann surfaces was investigated by P. J. Kiernan in [2]. One of his interesting results is that harmonic $K$-quasiconformal mappings of certain Riemann surfaces are distance-decreasing. We shall discuss here harmonic $K$-quasiconformal mappings of $n$-dimensional Riemannian manifolds and generalize the above Kiernan's theorem to this case. In section 1 we review the theory of harmonic forms as found in [4]. Section 2 is devoted to get some lemmas which are used to prove our theorem in the last section. Concerning quasiconformal mappings, we use the fact given by H. Wu in [5].

§ 1. Vector bundle valued harmonic forms

Let $M$ be an $n$-dimensional Riemannian manifold and $E$ a vector bundle over $M$ with a metric along fibres and covariant differentiation $D_X$ compatible with the metric for any vector field $X$. $C^p(E)$ is the real vector space of all $E$-valued differential $p$-forms on $M$. Next, an operator $\partial: C^p(E) \to C^{p+1}(E)$ is defined by

$$(\partial \theta)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} D_{X_i} \big( \theta(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \big)$$

$$+ \sum_{i<j}^{p+1} (-1)^{i+j} \theta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}),$$

where $X_i$'s denote vector fields on $M$. The covariant derivative $D_X \theta$ of $\theta \in C^p(E)$ is an $E$-valued $p$-form given by

$$(D_X \theta)(X_1, \ldots, X_p) = D_X(\theta(X_1, \ldots, X_p)) - \sum_{i=1}^{p} \theta(X_1, \ldots, \nabla_X X_i, \ldots, X_p),$$

where $\nabla_X X_i$ represents the covariant derivative of the vector field $X_i$ in the Riemannian manifold $M$. An operator $\partial^*$ is defined as follows. Let $x \in M$ and $u_1, \ldots, u_{p-1}$ be any tangent vectors at $x$. We define
\[(\partial^* \theta)_x(u_1, \ldots, u_{p-1}) = -\sum_{k=1}^n (D_{e_k} \theta)_x(e_k, u_1, \ldots, u_{p-1}),\]

where \(\{e_1, \ldots, e_n\}\) is an orthonormal base of the tangent space \(T_x(M)\) at \(x\) and \(\theta \in C^\infty(E)\). The Laplacian \(\Box\) for \(E\)-valued differential forms is given by \(\Box = \partial \partial^* + \partial^* \partial\). The scalar product of two \(E\)-valued \(p\)-forms \(\theta\) and \(\eta\) is given by

\[<\theta, \eta>(x) = \sum_{i_1, \ldots, i_p=1}^n <\theta(e_{i_1}, \ldots, e_{i_p}), \eta(e_{i_1}, \ldots, e_{i_p})>\]

Now we have

**Theorem (Matsushima).** Let \(\theta\) be an \(E\)-valued 1-form. Then

\[<\Box \theta, \theta> = \frac{1}{2} \Delta <\theta, \theta> + <D\theta, D\theta> + A,
\]

where \(\Delta\) denotes the Laplacian of the Riemannian manifold \(M\) and \(A\) denotes a smooth function in \(M\) defined as follows;

\[A(x) = \sum_{i,j} <\tilde{R}(e_i, e_j) \theta(e_i), \theta(e_j)> + \sum_i <\theta(S(e_i)), \theta(e_i)>,\]

where \(\tilde{R}\) is the curvature tensor of \(D\) and \(S\) denotes the endomorphism of \(T_x(M)\) defined by Ricci tensor \(S\) of \(M\), i.e. \(S(e_i) = \sum_k S_{ki} e_k\).

§ 2. Some preliminaries

We now list our notations. \(M\) and \(N\) denote \(n\)-dimensional Riemannian manifolds with metric connections \(\nabla\) and \(\nabla'\) respectively. \(f\) is \(C^\infty\)-mapping of \(M\) to \(N\). \(E\) is the bundle induced by \(f\) from \(T(N)\). Then \(E\) has the covariant differential operator \(D\) compatible with the metric deduced naturally from the Riemannian metric of \(N\). Let \(s = (s_i)\) be a frame of \(T(N)\) over \(V\) where \(V\) is an open set in \(N\). There exist 1-forms \(\theta_{ij}\) on \(V\) such that \(\nabla'_{s_j} = \sum_{j=1}^n \theta_{ij} s_j\). If \(U\) is an open set in \(M\) with \(f(U) \subset V\), then \(D\) is given as \(D(s_i(f(x))) = \sum_{j=1}^n f^*(\theta_{ij}) s_j(f(x))\).

Obviously, the differential \(f_*\) of \(f\) is regarded as an \(E\)-valued 1-form on \(M\). Let \(x \in M\), we take orthonormal bases \(\{e_1, \ldots, e_n\}\) and \(\{f_1, \ldots, f_n\}\) of \(T_x(M)\) and \(T_{f(x)}(N)\). For the curvature tensor \(R\) of \(\nabla\) and \(R'\) of \(\nabla'\), we set

\[R'_{abcd} = R(f_a, f_b, f_c, f_d), S_{ij} = \sum_{k=1}^n R(e_k, e_i, e_k, e_j).\]
In terms of the base \{\{f_a\}\} and the dual base \{\{e^i\}\} of \{\{e_i\}\} the \(E\)-valued 1-form \(f_\star\) is represented as

\[
(f_\star)_a = \sum_{a,i} \phi_i^a(x) e^i \otimes f_a.
\]

In this situation, we get the following lemma directly from the theorem in Section 1.

**Lemma 1.** (Ells and Sampson, Kiernan)

(1) \[
\langle \Box f_\star, f_\star \rangle(x) = \langle Df_\star, Df_\star \rangle(x) + \frac{1}{2} \partial_x \langle f, f_\star \rangle - \sum R'_{abcd} \phi_i^a \phi_j^b \phi_i^c \phi_j^d + \sum S_{ij} \phi_i^a \phi_j^b.
\]

Next, we shall explain some properties of \(K\)-quasiconformal mappings according to [5] of H. Wu. Suppose that \(f\) is \(K\)-quasiconformal. Let \(\lambda_1 \leq \ldots \leq \lambda_n\) be eigenvalues at \(x\) of the symmetric matrix \(G(x) = (G_{ij}(x))\), where \(G_{ij}(x) = \sum_{a=1}^{n} \phi_i^a(x) \phi_j^a(x)\), by definition. Then we have \((\frac{\lambda_n}{\lambda_1})^2 \leq K\) at each point \(x \in M\). This implies \(\text{trace } G \leq nK^2 (\det G)^{\frac{1}{n}}\). Evidently, \(\det G(x) = |\phi_i^a(x)|^2\) and \(\text{trace } G(x) = \sum_{a,i} (\phi_i^a(x))^2 = \langle f_\star, f_\star \rangle(x) = \|f_\star\|_2^2\). Summing up, we have

**Lemma 2.**

(2) \[
\|f_\star\|^2 \leq nK^2 J^2,
\]

where \(J = \left| |\phi_i^a| \right|\).

We shall show another inequality about \(\det G\) without the assumption that \(f\) is \(K\)-quasiconformal. If \(\phi_i = (\phi_i^1, \ldots, \phi_i^n)\), it is evident that

(3) \[
\begin{vmatrix}
G_{ij} G_{ij}
\end{vmatrix} = \|\phi_i\|^2 ||\phi_j||^2 - (\langle \phi_i, \phi_j \rangle)^2.
\]

In addition we put \(H = \sum_{\substack{i<j}} \begin{vmatrix} G_{ij} G_{ij} \end{vmatrix}\), then

**Lemma 3.**

(4) \[
nJ^4 \leq 2H.
\]

**Proof.** As \(J^2 = \det G\), it is sufficient to show that
\[(\det G)^2 \left( \frac{n}{2} \right)^n \leq H^n.\]

As it is well known, the positive definite symmetric matrix \(G\) is written as

\[
G = \begin{pmatrix}
p_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
p_{n1} & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix}
= \begin{pmatrix}
p_{11} \\
p_{21} \\
p_{31} \\
\vdots \\
p_{n1}
\end{pmatrix}
\begin{pmatrix}
p_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
p_{n1} & \cdots & 0
\end{pmatrix}
= PP^t,
\]

that is, \(G_{ij} = \sum_{k=1}^{n} p_{ik} p_{jk}(i \leq j)\). Hence, we have

\[
\det G = (\det P)^2 = \prod_{i=1}^{n} (p_{ii})^2.
\]

On the other hand, it holds

\[
\begin{vmatrix}
G_{ii} & G_{ij} \\
G_{ij} & G_{jj}
\end{vmatrix}
= (p_{i1}^2 + \cdots + p_{ii}^2)(p_{j1}^2 + \cdots + p_{jj}^2)
- (p_{i1} p_{j1} + \cdots + p_{ii} p_{jj})^2
\geq p_{i1}^2 p_{jj}^2.
\]

This implies

\[
H^n \geq (\sum_{i<j} p_{i1}^2 p_{j1}^2)^n.
\]

The problem is reduced to comparing \(II(p_{ii})^4\) with the right hand side of the last inequality. Now define \(A_k\) by

\[
(x_1 x_2 + \cdots + x_{k-1} x_k)^k = A_k x_1^2 x_2^2 \cdots x_k^2 + \cdots.
\]

Then \(A_k\) is determined inductively by

\[
A_k = k(k-1)A_{k-1} + \frac{k(k-1)^2}{2} A_{k-2}(A_2 = 1, A_3 = 6)
\]

and it satisfies

\[
A_k \geq \left( \frac{k}{2} \right)^k \quad (k \geq 2).
\]

Thus we have

\[
H^n \geq (\sum_{i<j} p_{i1}^2 p_{j1}^2)^n \geq A_n II(p_{ii})^4 \geq \left( \frac{n}{2} \right)^n (\det G)^2.
\]
§3. Harmonic $K$-quasiconformal mappings

We are now in position to prove the following theorem which is a generalization of the Kiernan's result ([2], Theorem 6).

**Theorem.** Let $M$ and $N$ be $n$-dimensional Riemannian manifolds, and let $f$ be a harmonic $K$-quasiconformal mapping of $M$ to $N$ such that the function $||f_*||^2 = <f_*, f_*>$ has its maximum on $M$. Suppose that the sectional curvature of $M$ is everywhere $\geq -A$ and that the sectional curvature of $N$ is everywhere $\leq -B$, where $A$ and $B$ are positive constants, then

\begin{equation}
||f_*(X)||^2 \leq n(n-1) \frac{A}{B} K^4 ||X||^2
\end{equation}

for every vector $X \in T(M)$.

**Proof.** Firstly we have $\Box f^* = 0$. Let $p$ be a maximum point of $||f_*||^2$. Using $\Delta p ||f_*||^2 \geq 0$, (1) yields

\begin{equation}
- \sum_{a, b, c, d} R'_{abcd} \phi^a_i \phi^b_j \phi^c_i \phi^d_j \leq \sum_{a, i, j} S_{ij} \phi^a_i \phi^a_j.
\end{equation}

By assumption, for a vector $X \in T_x(M)$

\[-(n-1)A||X||^2 \leq \sum_{i,k} S_{ik} \xi^i \xi^k,
\]

where $X = \sum \xi^i e_i$ with respect to an orthonormal base $\{e_i\}$ of $T_x(M)$. Making use of this, we get

\begin{equation}
-(n-1)A \sum_{a, i} ||\phi^a_i||^2 \leq (n-1)A \sum_{a, i} ||\phi^a_i||^2 \leq (n-1)A||f_*||^2.
\end{equation}

The left hand side of the inequality (6) can be written as

\[- \sum_{a, b, c, d} R'_{abcd} \phi^a_i \phi^b_j \phi^c_i \phi^d_j = - \sum_{i, j} R'(p; \phi_i, \phi_j)||\phi_i \wedge \phi_j||^2.
\]

Here $||\phi_i \wedge \phi_j||$ denotes the area of the parallelogram spanned by the vectors $\phi_i$ and $\phi_j$ and $R'(p; \phi_i, \phi_j)$ denotes the sectional curvature of $N$ at $f(p)$ along the section spanned by $\phi_i$ and $\phi_j$. By taking account of $||\phi_i \wedge \phi_j||^2 = ||\phi_i||^2||\phi_j||^2 - (\langle \phi_i, \phi_j \rangle)^2$, we obtain

\begin{equation}
- \sum_{a, b, c, d} R'_{abcd} \phi^a_i \phi^b_j \phi^c_i \phi^d_j \geq 2B \sum_{i, j} ||\phi_i \wedge \phi_j||^2 \geq 2BH.
\end{equation}
Using (7) and (8), (6) gives

\[ 2H \leq (n-1) \frac{A}{B} ||f_*||^2. \]

Combining (2) and (4) with this, we get

\[ nJ^2_n \leq (n-1) \frac{A}{B} ||f_*||^2 \leq n(n-1) \frac{A}{B} K^2 J^2_n. \]

Hence we have

\[ J^2_n \leq (n-1) \frac{A}{B} K^2. \]

Thus we find

\[ \langle f_*, f_* \rangle (p) = ||f_*||^2 \leq nK^2J^2 \leq n(n-1) \frac{A}{B} K^4. \]  

Now, let \( X \) be an arbitrary vector at some point \( q \in M \). Then it follows

\[ ||f_*(X)||^2 = \Sigma G_{ij}(q) \zeta^i \zeta^j, \]

where \( X = \Sigma \zeta^i e_i \) with respect to an orthonormal base \( \{e_i\} \) of \( T_q(M) \). By an orthonormal transformation \( T \), the symmetric matrix \( G(q) = (G_{ij}(q)) \) is reduced to a diagonal matrix, that is,

\[ 'TGT = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \cdots & \alpha_n \end{pmatrix}. \]

If we put \( TX = Y = \Sigma \zeta^i e_i \), then we get \( ||Y|| = ||X|| \) and

\[ ||f_*(X)||^2 = \alpha_1^2(\zeta^1)^2 + \cdots + \alpha_n^2(\zeta^n)^2 \leq \text{trace } G(q) ||X||^2. \]

As \( \text{trace } G(q) = ||f_*||^2 \), this implies

\[ ||f_*(X)||^2 \leq ||f_*||^2 ||X||^2 \leq ||f_*||^2 ||X||^2. \]

The desired result follows from this and (9).

**Corollary.** In addition to the assumption of the theorem, we suppose that \( M \) is connected. If \( d_M \) and \( d_N \) denote the metrics induced from the Riemannian metrics respectively. Then it follows
\[ d_N(f(p_1), f(p_2)) \leq \sqrt{n(n-1)} \frac{A}{B} K^4 d_M(p_1, p_2) \]

for every \( p_1, p_2 \in M \).

**Proof.** For any positive constant \( \epsilon \), there exists a curve \( \tau(t) (0 \leq t \leq 1) \) on \( M \) which satisfies \( \tau(0) = p_1, \tau(1) = p_2 \) and

\[ d_M(p_1, p_2) + \epsilon > \int_0^1 \left\| \frac{d\tau}{dt} \right\| \, dt. \]

Therefore we get

\[ \sqrt{n(n-1)} \frac{A}{B} K^4 (d_M(p_1, p_2) + \epsilon) > \int_0^1 \left\| f_\sigma \left( \frac{d\tau}{dt} \right) \right\| \, dt \geq d_N(f(p_1), f(p_2)). \]

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**References**


