A Specific \((B, f)\) Bordism Group \(\mathcal{Q}_b(\hat{B}, f)\)

By

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In this note, we construct an example, \(\mathcal{Q}_b(\hat{B}, f)\), of \((B, f)\) bordism group \([2], [3]\) which may be of some interest in connection with the unitary bordism group \(\mathcal{Q}_u^b\). This bordism group is formed by the \((\hat{B}, f)\) manifolds where \(f_r: \hat{B}_r \to BSO_r\) is the fibration induced by a suitable map \(h_r: BSO_r \to X\) from the standard path fibration \(p: PX \to X\) over \(X = \prod_{k \geq 2} K(Z_2, 2^{k} - 1)\), with \(K(Z_2, n)\) being the Eilenberg-MacLane space. In this bordism group \(\mathcal{Q}_p(\hat{B}, f)\), all the odd dimensional Stiefel-Whitney classes of every \((\hat{B}, f)\) manifold \((M^n, \nu)\) vanish, i.e., \(w_{2i+1}(M) = 0\) for all \(i \geq 0\).

Let \(BSO_r = \lim_{n \to \infty} \bar{G}_{r,n}\) be the limit of Grassmannians of oriented \(r\)-planes with universal oriented \(r\)-plane bundle \(\varphi = \lim_{n \to \infty} \bar{G}_{r,n}\) where \(\bar{G}_{r,n}\) is the Grassmann manifold of oriented \(r\)-planes in the \((n + r)\)-dimensional euclidean space \(R^{n+r}\) and \(\bar{G}_{r,n}\) is the universal oriented \(r\)-plane bundle over \(\bar{G}_{r,n}\).

Let \(f_r: B_r \to BSO_r\) be a fibration. If \(\xi\) is an \(r\)-plane bundle over a \(CW\) complex \(X\) classified by the map \(\xi: X \to BSO_r\), then a \((B_r, f_r)\) structure \(\mathcal{V}\), on \(\xi\) is a homotopy class of lifting \(\xi\) to \(B_r\) of the map \(\xi\).

Suppose one is given a sequence \((B, f)\) of fibrations \(f_r: B_r \to BSO_r\), maps \(g_r: B_r \to B_{r+1}\) and the usual inclusions \(j_r: BSO_r \to BSO_{r+1}\) such that the diagram

\[
\begin{array}{ccc}
B_r & \xrightarrow{g_r} & B_{r+1} \\
\downarrow f_r & & \downarrow f_{r+1} \\
BSO_r & \xrightarrow{j_r} & BSO_{r+1}
\end{array}
\]

commutes. Let \(M^n\) be a compact oriented differentiable \(C^\infty\)-manifold (with or without boundary). Embed \(M^n\) in \(R^{n+r}\), \(r \geq n + 2\). Let \(\mathcal{V} = \{\mathcal{V}_r\}\) be a sequence of \((B_r, f_r)\) structures on the normal bundles \(\nu\) to \(M\). Two sequences of \((B_r, f_r)\) structures are equivalent if they agree for sufficiently large \(r\).

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A $(B, f)$ structure on $M^n$ is an equivalence class of a sequence $\mathcal{F}$ of $(B_\tau, f_\tau)$ structures on the normal bundles of $M$. A $(B, f)$ manifold is a pair consisting of a manifold $M^n$ and a $(B, f)$ structure on $M$, we denote it by $(M^n, \varpi)$ where $\varpi$ is a lifting to $B_\tau$ of the classifying map $\nu: M^n \to BSO_\tau$.

The bordism category $\{\mathcal{D}, \partial, i\}$ of $(B, f)$ manifolds is the category whose objects are oriented compact differentiable manifolds with $(B, f)$ structure and whose maps are the boundary preserving differentiable imbedding with trivialized normal bundle such that the $(B, f)$ structure induced by the map coincides with the $(B, f)$ structure on the domain manifold. The functor $\partial$ applied to a $(B, f)$ manifold $M$ is the manifold $\partial M$ with $(B, f)$ structure induced by the inner normal trivialization, and $\partial$ on maps is restriction. The natural transformation $i$ is the inclusion of the boundary with inner normal trivialization.

A closed manifold $(M^n, \varpi) \in \mathcal{D}$ bords if there exists a manifold $(W^{n+1}, \varpi') \in \mathcal{D}$ with $\partial W$ diffeomorphic to $M$ via an orientation preserving diffeomorphism and with $\varpi' | \partial W$ homotopic to $\varpi$. Two closed manifolds $(M^n_0, \varpi_0)$ and $(M^n_1, \varpi_1)$ in $\mathcal{D}$ are bordant if the disjoint union $(M^n_0 \cup -M_1, \varpi_0 \cup \varpi_1)$ bords. Two $(B, f)$ manifolds $(M^n_0, \varpi_0)$ and $(M^n_1, \varpi_1)$ are equivalent provided there exists an orientation preserving diffeomorphism $\varphi: M^n_0 \to M^n_1$ such that $\varpi_1 \circ \varphi$ is homotopic to $\varpi_0$. It is then clear that if two $(B, f)$ manifolds $(M^n_0, \varpi_0)$ and $(M^n_1, \varpi_1)$ with $\partial(M^n_0, \varpi_0)$ and $\partial(M^n_1, \varpi_1)$ being equivalent, the sum along the boundary, $(M^n_0 \cup -M^n_1, \varpi_0 \cup \varpi_1)$ is also a $(B, f)$ manifold where $\varphi: \partial M^n_0 \to \partial M^n_1$ is an orientation preserving diffeomorphism. This bordism relation is an equivalence relation on the class of closed $n$-manifolds in $\mathcal{D}$. The resulting set $\mathcal{Q}(B, f)$ of equivalence classes is an abelian group with addition induced by disjoint union.

Consider now a path-connected topological space $X$ with base-point $x$. Let $PX$ be the paths in $X$ starting at $x$ and let $\mathcal{Q}X$ be the loops in $X$ based on $x$. We then see that $PX$ is contractible and the fibre of the standard fibration $p: PX \to X$ is $\mathcal{Q}X$. We also see that

$$\partial: \pi_1(X) \simeq \pi_1(\mathcal{Q}X)$$

by the homotopy exact sequence for a fibration. If $X$ is an Eilenberg-MacLane space of type $K(\pi, n)$, then this shows that $\mathcal{Q}X$ is an Eilenberg-MacLane space of type $K(\pi, n-1)$. Let $X = \bigvee_{i \neq 2} K(\mathbb{Z}_2, 2^i - 1)$ and let $h_\tau: BSO_\tau \to X$ be a map which will be defined below. Let $f_\tau: B_\tau \to BSO_\tau$ be the fibration induced from the standard fibration over $X$ by $h_\tau$. Consider the following commutative diagram
where the vertical maps are fibrations with fibre $\mathcal{O}_X = \prod_{i \geq 2} K(Z_2, 2^i-2)$. Recall that $H^*(BSO_r; Z_2) = Z_2[\omega_2, \ldots, \omega_r]$, a polynomial algebra with generators $\omega_2, \ldots, \omega_r$, where $\omega_k \in H^*(BSO_r; Z_2)$, $2 \leq k \leq r$, are the universal Stiefel-Whitney classes. Let $u_n \in H^n(K(Z_2, n); Z_2) \cong Z_2$ be the fundamental class. The map $h_r$ is to be defined by

$$h_r^*(u_{2^{i-1}}) = \begin{cases} w_{2^{i-1}} & \text{for } 2^{i-1} \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_r^*(w_{2j+1}) = 0$ for all $2j+1 \leq r$. For by the Wu’s Formula

$$Sq^i w_j = \sum_{t=0}^i \binom{j-i-t}{t} w_{j-t} w_{j+t},$$

we have $Sq^2 w_3 = w_2 w_3 + w_1 w_4 + w_5$, $Sq^2 w_7 = w_2 w_7 + w_1 w_6 + w_9$, etc., and $f_r^* h_r^*(u_1) = f_r^* h_r^*(u_3) = f_r^*(w_3) = 0$, $f_r^* h_r^*(u_7) = f_r^*(w_7) = 0$, $f_r^* h_r^*(u_9) = f_r^*(w_9) = 0$, etc. In general, it is verified by induction as follows.

Suppose $f_r^*(w_{2j+1}) = 0$ for all $j \leq k-1$. This shall show that this is also true for $j = k$. We shall use the fact that every odd number $2k+1 \neq 2^i - 1 = 2^{i-1} + 2^{i-2} + \cdots + 2 + 1$ can be written in the form

$$2k+1 = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_s} + 2 + 1$$

where $i_1 > i_2 > \cdots > i_s > i + 1$. For $2^i - 1$, we see that $f_r^*(w_{2^i-1}) = 0$ by the definition of $h_r$. For $2k+1 \neq 2^i - 1$, we have, by Wu’s Formula,

$$Sq^{2^{i-1}} w_{2k+1 - 2^{i-1}} = \sum_{t=0}^{2^{i-1}-1} \binom{2k-2^{i-1}+t}{t} w_{2^{i-1}-1-t} w_{2k+1 - 2^{i-1}+t}$$

$$= \sum_{t=0}^{2^{i-1}-1} \binom{2k-2^{i-1}+t}{t} w_{2^{i-1}-1-t} w_{2k+1 - 2^{i-1}+t} + \binom{2k-2^{i-1}+2^{i-1}}{2^{i-1}-1} w_{2k+1}$$

$$= \sum_{t=0}^{2^{i-1}-1} \binom{2k-2^{i-1}+t}{t} w_{2^{i-1}-1-t} w_{2k+1 - 2^{i-1}+t}$$

$$+ \binom{2^{i-1} + \cdots + 2^{i-1}}{2^{i-1}-1} w_{2k+1}$$
\[ w_{2k+1} = \sum_{t} (2^{\nu_{t+1} - 1} t^{-2^t + t}) w_{2^{\nu_{t+1}} - 1 - t} w_{2^{\nu_{t+1}} + 1 - 2^{t+1} + t} + w_{2k+1} \quad (\text{mod } 2). \]

Hence

\[ w_{2k+1} = Sq^{2^{\nu_{t+1} - 1}} w_{2^{\nu_{t+1}} - 1 - t} w_{2^{\nu_{t+1}} + 1 - 2^{t+1} + t} \]

where each term \( w_{2^{\nu_{t+1}} - 1 - t} w_{2^{\nu_{t+1}} + 1 - 2^{t+1} + t} \) has odd and even subscripts alternately.

We then see that \( f_{\gamma'}^*(w_{2k+1}) = 0 \) by the hypothesis of induction. Hence \( f_{\gamma'}^*(w_{2j+1}) = 0 \) for all \( 2j+1 \leq r \).

Consider next the commutative diagram

\[
\begin{array}{ccc}
E(\nu(M^n)) & \longrightarrow & E(\xi') \\
\downarrow & & \downarrow \\
M^n & \xrightarrow{\nu} & B_r \\
\end{array}
\]

where \( \nu(M^n) \) is the normal bundle to \( M^n \) in \( R^{n+r} \) which is classified by the map \( \nu : M^n \to BSO_r \), and \( \xi' = f_{\gamma'}^* \gamma' \) is the \( r \)-plane bundle induced from the universal bundle \( \gamma' \) by the fibration \( f_{\gamma} \). Then for any \( (B, f) \) manifold \( (M^n, \nu) \) we have \( w_{2i+1}(\nu(M)) = 0 \) for all \( i \geq 0 \). But this implies \( w_{2i+1}(M) = 0 \) for all \( i \geq 0 \). Hence we obtain the following.

**Theorem.** In the \( (B, f) \) bordism group \( \Omega_n(B, f) \), every \( (B, f) \) manifold \( (M^n, \nu) \) satisfies \( w_{2i+1}(M) = 0 \) for all \( i \geq 0 \).

**Remark.** For the \( B_r \) constructed above, we see that

\[ \lim H^*(B_r; Z_2) = Z_2[w_2, w_4, \ldots]. \]

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**References**

