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A Specific (B, f) Bordism Group $\Omega_*(\hat{B}, f)$

By

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In this note, we construct an example, $\Omega_*(\hat{B}, f)$, of (B, f) bordism group [2], [3] which may be of some interest in connection with the unitary bordism group Ω_*^U. This bordism group is formed by the $(\hat{B}, f)$ manifolds where $f_r\colon \hat{B}_r \to BSO_r$ is the fibration induced by a suitable map $h_r\colon BSO_r \to X$ from the standard path fibration $p\colon PX \to X$ over $X = \prod_{k\geqslant 2} K(Z_2, 2^k-1)$, with $K(Z_2, n)$ being the Eilenberg-MacLane space. In this bordism group $\Omega_n(\hat{B}, f)$, all the odd dimensional Stiefel-Whitney classes of every $(\hat{B}, f)$ manifold $(M^n, \tilde{\nu})$ vanish, i.e., $w_{2i+1}(M)=0$ for all $i\geqslant 0$.

Let $BSO_r = \lim_{n\to\infty} \tilde{G}_{r,n}$ be the limit of Grassmannians of oriented r-planes with universal oriented r-plane bundle $\gamma^r = \lim_{n\to\infty} \gamma_n^r$ where $\tilde{G}_{r,n}$ is the Grassmann manifold of oriented r-planes in the $(n+r)$-dimensional euclidean space R^{n+r} and γ_n^r is the universal oriented r-plane bundle over $\tilde{G}_{r,n}$.

Let $f_r\colon B_r \to BSO_r$ be a fibration. If ξ is an r-plane bundle over a CW complex X classified by the map $\xi\colon X \to BSO_r$, then a (B_r, f_r) structure Ψ_r on ξ is a homotopy class of lifting $\hat{\xi}$ to B_r of the map ξ.

Suppose one is given a sequence (B, f) of fibrations $f_r\colon B_r \to BSO_r$, maps $g_r\colon B_r \to B_{r+1}$ and the usual inclusions $j_r\colon BSO_r \to BSO_{r+1}$ such that the diagram

$$\begin{array}{ccc} B_r & \xrightarrow{g_r} & B_{r+1} \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_{r+1}} \\ BSO_r & \xrightarrow{j_r} & BSO_{r+1} \end{array}$$

commutes. Let M^n be a compact oriented differentiable C^∞-manifold (with or without boundary). Embed M^n in R^{n+r}, $r\geqslant n+2$. Let $\Psi=\{\Psi_r\}$ be a sequence of (B_r, f_r) structures on the normal bundles ν to M. Two sequences of (B_r, f_r) structures are equivalent if they agree for sufficiently large r.

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A (B, f) structure on M^n is an equivalence class of a sequence Ψ of (B_r, f_r) structures on the normal bundles of M. A (B, f) manifold is a pair consisting of a manifold M^n and a (B, f) structure on M, we denote it by $(M^n, \tilde{\nu})$ where $\tilde{\nu}$ is a lifting to B_r of the classifying map $\nu: M^n \to BSO_r$.

The bordism category $\{\mathfrak{D}, \partial, i\}$ of (B, f) manifolds is the category whose objects are oriented compact differentiable manifolds with (B, f) structure and whose maps are the boundary preserving differentiable imbedding with trivialized normal bundle such that the (B, f) structure induced by the map coincides with the (B, f) structure on the domain manifold. The functor ∂ applied to a (B, f) manifold M is the manifold ∂M with (B, f) structure induced by the inner normal trivialization, and ∂ on maps is restriction. The natural transformation i is the inclusion of the boundary with inner normal trivialization.

A closed manifold $(M^n, \tilde{\nu}) \in \mathfrak{D}$ bords if there exists a manifold $(W^{n+1}, \tilde{\nu}')$ $\in \mathfrak{D}$ with ∂W diffeomorphic to M via an orientation preserving diffeomorphism and with $\tilde{\nu}'|\partial W$ homotopic to $\tilde{\nu}$. Two closed manifolds $(M_0^n, \tilde{\nu}_0)$ and $(M_1^n, \tilde{\nu}_1)$ in $\mathfrak{D}$ are bordant if the disjoint union $(M_0 \cup -M_1, \tilde{\nu}_0 \cup \tilde{\nu}_1)$ bords. Two (B, f) manifolds $(M_0^n, \tilde{\nu}_0)$ and $(M_1^n, \tilde{\nu}_1)$ are equivalent provided there exists an orientation preserving diffeomorphism $\varphi: M_0^n \to M_1^n$ such that $\tilde{\nu}_1 \circ \varphi$ is homotopic to $\tilde{\nu}_0$. It is then clear that if two (B, f) manifolds $(M_0^n, \tilde{\nu}_0)$ and $(M_1^n, \tilde{\nu}_1)$ with $\partial(M_0^n, \tilde{\nu}_0)$ and $\partial(M_1^n, \tilde{\nu}_1)$ being equivalent, the sum along the boundary, $(M_0^n \bigcup_\varphi -M_1^n, \tilde{\nu}_0 \bigcup_\varphi \tilde{\nu}_1)$ is also a (B, f) manifold where $\varphi: \partial M_0^n \to \partial M_1^n$ is an orientation preserving diffeomorphism. This bordism relation is an equivalence relation on the class of closed n-manifolds in $\mathfrak{D}$. The resulting set $\Omega_n(B, f)$ of equivalence classes is an abelian group with addition induced by disjoint union.

Consider now a path-connected topological space X with base-point x. Let PX be the paths in X starting at x and let ΩX be the loops in X based on x. We then see that PX is contractible and the fibre of the standard fibration $p: PX \to X$ is ΩX. We also see that

$$\partial: \pi_i(X) \approx \pi_{i-1}(\Omega X)$$

by the homotopy exact sequence for a fibration. If X is an Eilenberg-MacLane space of type $K(\pi, n)$, then this shows that ΩX is an Eilenberg-MacLane space of type $K(\pi, n-1)$. Let $X = \prod_{i \geqslant 2} K(Z_2, 2^i - 1)$ and let $h_r: BSO_r \to X$ be a map which will be defined below. Let $f_r: \hat{B}_r \to BSO_r$ be the fibration induced from the standard fibration over X by h_r. Consider the following commutative diagram

$$\begin{array}{ccc} \hat{B}_r & \longrightarrow & PX \\ \big\downarrow f_r & & \big\downarrow p \\ BSO_r & \xrightarrow{h_r} & X \end{array}$$

where the vertical maps are fibrations with fibre $\Omega X = \prod_{i \geqslant 2} K(Z_2, 2^i - 2)$. Recall that $H^*(BSO_r; Z_2) = Z_2[w_2, \cdots, w_r]$, a polynomial algebra with generators $w_2, \cdots, w_r$, where $w_k \in H^k(BSO_r; Z_2)$, $2 \leqslant k \leqslant r$, are the universal Stiefel-Whitney classes. Let $u_n \in H^n(K(Z_2, n); Z_2) \approx Z_2$ be the fundamental class. The map h_r is to be defined by

$$h_r^*(u_{2^i-1}) = \begin{cases} w_{2^i-1} & \text{for } 2^i - 1 \leqq r, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_r^*(w_{2j+1}) = 0$ for all $2j+1 \leqq r$. For by the Wu's Formula

$$Sq^i w_j = \sum_{t=0}^{i} \binom{j-i+t-1}{t} w_{i-t} \cdot w_{j+t},$$

we have $Sq^2 w_3 = w_2 w_3 + w_1 w_4 + w_5$, $Sq^2 w_7 = w_2 w_7 + w_1 w_8 + w_9$, etc., and $f_r^* h_r^*(u_1) = f_r^*(w_1) = 0$, $f_r^* h_r^*(u_3) = f_r^*(w_3) = 0$, $f_r^*(w_5) = f_r^*(Sq^2 w_3 + w_2 w_3 + w_1 w_4) = 0$, $f_r^* h_r^*(u_7) = f_r^*(w_7) = 0$, $f_r^*(w_9) = f_r^*(Sq^2 w_7 + w_2 w_7 + w_1 w_8) = 0$, etc. In general, it is verfied by induction as follows.

Suppose $f_r^*(w_{2j+1}) = 0$ for all $j \leqslant k-1$. We shall show that this is also true for $j = k$. We shall use the fact that every odd number $2k+1 \neq 2^j - 1 = 2^{j-1} + 2^{j-2} + \cdots + 2 + 1$ can be written in the form

$$2k+1 = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_s} + 2^{\iota} + 2^{\iota-1} + \cdots + 2 + 1$$

where $i_1 > i_2 > \cdots > i_s > \iota + 1$. For $2^j - 1$, we see that $f_r^*(w_{2^j-1}) = 0$ by the definition of h_r. For $2k+1 \neq 2^j - 1$, we have, by Wu's Formula,

$$Sq^{2^{i_s-1}} w_{2k+1-2^{i_s-1}} = \sum_{t=0}^{2^{i_s-1}} \binom{2k-2^{i_s}+t}{t} w_{2^{i_s-1}-t} \cdot w_{2k+1-2^{i_s-1}+t}$$

$$= \sum_{t=0}^{2^{i_s-1}-1} \binom{2k-2^{i_s}+t}{t} w_{2^{i_s-1}-t} \cdot w_{2k+1-2^{i_s-1}+t} + \binom{2k-2^{i_s}+2^{i_s-1}}{2^{i_s-1}} w_{2k+1}$$

$$= \sum_{t=0}^{2^{i_s-1}-1} \binom{2k-2^{i_s}+t}{t} w_{2^{i_s-1}-t} \cdot w_{2k+1-2^{i_s-1}+t}$$

$$+ \binom{(2^{i_1} + \cdots + 2^{i_{(s-1)}}) + 2^{i_s-1} + (2^{\iota} + 2^{\iota-1} + \cdots + 2)}{2^{i_s-1}} w_{2k+1}$$

$$=\sum_{t=0}^{2^{i_s-1}-1}\binom{2k-2^{i_s}+t}{t}w_{2^{i_s-1}-t}\cdot w_{2k+1-2^{i_s-1}+t}+w_{2k+1}\quad (\mathrm{mod}\ 2).$$

Hence

$$w_{2k+1}=Sq^{2^{i_s-1}}w_{2k+1-2^{i_s-1}}+\sum_{t=0}^{2^{i_s-1}-1}\binom{2k-2^{i_s}+t}{t}w_{2^{i\ -1}-t}\cdot w_{2k+1-2^{i\ -1}+t}$$

where each term $w_{2^{i_s-1}-t}\cdot w_{2k+1-2^{i_s-1}+t}$ has odd and even subscripts alternately. We then see that $f_r^*(w_{2k+1})=0$ by the hypothesis of induction. Hence $f_r^*(w_{2j+1})=0$ for all $2j+1\leqslant r$.

Consider next the commutative diagram

$$\begin{array}{ccccc} E(\nu(M^n)) & \longrightarrow & E(\xi^r) & \longrightarrow & E(\gamma^r) \\ \downarrow & & \downarrow & & \downarrow \\ M^n & \xrightarrow{\tilde{\nu}} & \hat{B}_r & \xrightarrow{f_r} & BSO_r \end{array}$$

where $\nu(M^n)$ is the normal bundle to M^n in R^{n+r} which is classified by the map $\nu: M^n\to BSO_r$, and $\xi^r=f_r^*\gamma^r$ is the r-plane bundle induced from the universal bundle γ^r by the fibration f_r. Then for any $(\hat{B}, f)$ manifold $(M^n, \tilde{\nu})$ we have $w_{2i+1}(\nu(M))=0$ for all $i\geqslant 0$. But this implies $w_{2i+1}(M)=0$ for all $i\geqslant 0$. Hence we obtain the following.

THEOREM. *In the $(\hat{B}, f)$ bordism group $\Omega_n(\hat{B}, f)$, every $(\hat{B}, f)$ manifold $(M^n, \tilde{\nu})$ satisfies $w_{2i+1}(M)=0$ for all $i\geqslant 0$.*

REMARK. For the $\hat{B}_r$ constructed above, we see that

$$\varprojlim H^*(\hat{B}_r;\ Z_2)=Z_2[w_2, w_4, \cdots].$$

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