Almost Complex Projective Connections

By

Yoshihiro Ichijyo

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The theory of real projective connections standing on the view-point of the theory of vector bundles was investigated perfectly by T. Otsuki [2]. Its extension to generalized spaces was done by the present author and K. Eguchi [1]. On the other hand, T. Ishihara [4] studied complex projective structures and found the close relation between the complex projective structures and H-projective connections which had been introduced by T. Tashiro [5] and developed by S. Ishihara [3] and many other authors [6].

In the present paper, our main purpose is to study manifolds endowed with almost complex projective structures. First of all, in §§1 and 2, we consider an almost complex projective vector bundle introduced in [4] and several distributions which define an almost complex projective connection. This connection, however, can be determined also by an usual differential geometric method. In §3, we find a certain distribution $\mathcal{D}$ which is intrinsic to the almost complex projective connection. A mapping $\sigma$ which gives a correspondence between any two fibers in the almost complex projective vector bundle is treated in §4. And we show that the mapping $\sigma$ preserves the complex projective structure in each fiber invariant if the distribution $\mathcal{D}$ is integrable. The last section is devoted to study a manifold where the given distribution $\mathcal{D}$ is integrable, for example, we obtain that if $\mathcal{D}$ is integrable then the manifold is an H-projectively flat complex manifold.

§1. Almost complex projective vector bundles

Let $M$ be a $2n$-dimensional differentiable manifold of class $C^\infty$ and let there be given an almost complex structure $f$. Let $P$ be an almost complex projective vector bundle [4], that is, $P$ is a vector bundle over $M$ which has $R^{2n+2}$ as a standard fiber, and is constructed as follows;  
1) The structure group $G$ is formed by all elements of the type
\[
\begin{pmatrix}
g_j^i \\
0 \\
\vdots \\
0 \\
* \\
1 \\
0 \\
1 
\end{pmatrix}, \text{ where } (g_j^i) \in GL(2n, R).^{1)}
\]

Obviously $G$ is a subgroup of $GL(2n+2, R)$.

2) Let $U$ and $V$ be two coordinate neighbourhoods of $M$ such that $U \cap V \neq \emptyset$ and $(x^1, x^2, \cdots, x^{2n}), (\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^{2n})$ be the local coordinate systems valid on $U$ and $V$ respectively. We define the transition function $G_{UV}: U \cap V \to G$ by

\[
\begin{pmatrix}
\frac{\partial x^i}{\partial \bar{x}^j} \\
\alpha \bar{x}^i + \beta f_j^i \\
\frac{\partial \bar{x}^j}{\partial x^i} \\
\frac{\partial \bar{x}^j}{\partial \bar{x}^i} \\
\alpha \bar{x}^j + \beta \bar{x}^i \\
\beta \alpha \\
\beta \alpha \\
\end{pmatrix}, \text{ where } \alpha = \det |\partial_{i}x^{j}|, b \text{ and } c \text{ are constants, and } f_j^i \text{ is the given almost complex structure.}
\]

Let $\pi$ be the projection of bundle $P (\pi: P \to M)$. If we write $(x^i, w^\lambda)$ and $(\bar{x}^i, \bar{w}^\lambda)$ for the canonical coordinate systems valid on $\pi^{-1}(U)$ and $\pi^{-1}(V)$ respectively, due to the bundle structure, the $w^\lambda$ and $\bar{w}^\lambda$ are related by

\[
\begin{align*}
uc^i &= \frac{\partial x^i}{\partial \bar{x}^i} u^i, \\
v^i &= (b \partial_{i} \log \Delta + cf_j^i \partial_m \log \Delta)u^i + v^i, \\
v^\bar{i} &= (c \partial_{i} \log \Delta - bf_j^i \partial_m \log \Delta)u^i + v^\bar{i},
\end{align*}
\]

where we put $(w^\lambda) = (u^i, v^i, v^\bar{i})$.

Now, we define a mapping $C^*$, which is fiber-preserving, by

\[
C^*(w^\lambda) = \begin{pmatrix}
\alpha \delta^i_j + \beta f_j^i \\
0 \\
\vdots \\
0 \\
\alpha \delta^i_i + \beta f_j^i \\
0 \\
\alpha \\
\beta \\
\end{pmatrix} \begin{pmatrix}
u^i \\
v^i \\
\vdots \\
v^i \\
u^i \\
v^i \\
\alpha v^i + \beta v^\bar{i}
\end{pmatrix} = \begin{pmatrix}
\alpha u^i + \beta f_j^i u^i \\
v^i \\
\vdots \\
v^i \\
\alpha v^i + \beta v^\bar{i}
\end{pmatrix},
\]

where $\alpha^2 + \beta^2 \neq 0$.

---

1) Throughout the paper, the Roman indices $h, i, j, k \text{ etc. run over the range } 1, \cdots, 2n \text{ and the Greek indices } \alpha, \beta, \tau \text{ etc. run over the range } 1, \cdots, 2n, 2n+1, 2n+2, \text{ and } 1^* \text{ stands for } 2n+1 \text{ and } 2^* \text{ for } 2n+2.$
The relation $G_{UV}C^*_U = C^*_V.G_{UV}$ follows at once, hence the mapping $C^*$ is well-defined on each fiber.

In the case where $\alpha = 0$ and $\beta = 1$, we denote $C^*$ by $F$, then

$$F = \begin{pmatrix} f^j_i & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

satisfies $F^\alpha = -E$ and $\frac{\partial F^\alpha}{\partial \omega^\alpha} = 0$, that is, $F$ gives a complex structure on each fiber.

Now, we say that two points $(w^\alpha)$ and $(\bar{w}^\alpha)$ on a fiber are mutually complex-projective if $(w^\alpha)$ and $(\bar{w}^\alpha)$ are related by $\bar{w}^\alpha = c^\alpha w^\alpha$ for certain constants $\alpha$ and $\beta$. Denoting this relation by $w^\alpha \sim \bar{w}^\alpha$, we can easily see that the relation $\sim$ satisfies an equivalence relation. Hence, if we remove the origine from $\mathbb{R}^{2m+2}$ and identify all points which are mutually complex-projective, we obtain an $n$-dimensional complex projective space $P^n(C)$. In an almost complex projective vector bundle $P$, if we convert each fiber into a complex projective space by means of the above mentioned method, the fiber bundle is called an almost complex projective bundle over $M$ and is denoted by $\tilde{P}$ hereafter.

§ 2. Almost complex projective connections

Let $P$ be an almost complex projective vector bundle over an almost complex manifold $M$. Let $P_p$ be the tangent space of $P$ at $p \in P$. If we assign to each $p \in P$ a subspace $\Phi_p^V$ of $P_p$ formed by all vectors tangent to the fiber through $p$, then we have a $(2n+2)$-dimensional distribution on $P$. We call it the vertical distribution and denote it by $\Phi^V$. A distribution $\Phi^h$ which is complementary to the $\Phi^V$ and satisfies the following conditions

1) $P_p = \Phi_p^h \oplus \Phi_p^V$,
2) $\partial \varepsilon^* \Phi_p^h = \Phi^V_{\varepsilon^* p}$,
3) $\Phi_p^h$ is a linear space with respect to $w^\alpha$

is called the horizontal distribution, where $\varepsilon^*$ is the mapping $P \rightarrow P$ which maps each $p = (x^i, w^\alpha)$ into $\varepsilon^* p = (x^i, c^\alpha w^\alpha)$.

If we denote by $p^V$ and $p^h$ the projection operator of vector fields on $P$ to $\Phi^V$ and $\Phi^h$ respectively, then from the condition 1) these projection operators should have the following forms

$$p^V = \begin{pmatrix} 0 & 0 \\ 0 & \delta^i_j \end{pmatrix}, \quad p^h = \begin{pmatrix} \delta^i_j & 0 \\ -\delta^i_j & 0 \end{pmatrix}.$$
When we put

\begin{equation}
X_i = \frac{\partial}{\partial x^i} - \theta_i^j(x, w) \frac{\partial}{\partial \tilde{\omega}^j} = \frac{\partial}{\partial x^i} - \theta_i^1(x, w) \frac{\partial}{\partial u^1} - \theta_i^2(x, w) \frac{\partial}{\partial \tilde{\omega}^2} = 0,
\end{equation}

then \(\{X_i\}\) forms a local basis on \(\Phi^b\). Calculating the condition 2) with respect to the basis \(\{X_i\}\), we have

\begin{align*}
\theta_i^1(x, e^*w) &= \alpha \theta_i^1(x, w) + \beta f^1 \theta_i^m(x, w) - \beta \partial_\tilde{w} f^1 u^m, \\
\theta_i^2(x, e^*w) &= \alpha \theta_i^2(x, w) - \beta \partial_\tilde{w} u^2, \\
\theta_i^3(x, e^*w) &= \alpha \theta_i^3(x, w) + \beta \partial_\tilde{w} u^3.
\end{align*}

On the other hand, from the condition 3), \(\theta_i^3\) can be written as

\begin{align*}
\theta_i^1(x, w) &= \varphi_i^m(x) u^m + p_i(x) v^1 + q_i(x) v^2, \\
\theta_i^2(x, w) &= p_i(x) u^m + q_i(x) v^1 + r_i(x) v^2, \\
\theta_i^3(x, w) &= \bar{p}_i(x) u^m + q_i(x) v^1 + r_i(x) v^2.
\end{align*}

Thus we obtain

\begin{align*}
\partial_\tilde{w} f^1 &= f^1 \varphi_i^m + \varphi_i^m f^1 = 0, \\
p_i &= f_i^m p_i^m, \\
\bar{p}_i &= -p_i f_i^m, \\
q_i &= -r_i, \\
r_i &= q_i.
\end{align*}

Hence \(\theta_i^3\) have the form

\begin{align*}
\theta_i^1(x, w) &= \varphi_i^m(x) u^m + p_i(x) v^1 + f_i^m(x) p_i^m(x) v^2, \\
\theta_i^2(x, w) &= p_i(x) u^m + q_i(x) v^1 + r_i(x) v^2, \\
\theta_i^3(x, w) &= -p_i(x) f_i^m(x) u^m - r_i(x) v^1 + q_i(x) v^2,
\end{align*}

where \(\varphi_i^m\) satisfy the condition

\begin{equation}
\partial_\tilde{w} f^1 - f^1 \varphi_i^m + \varphi_i^m f^1 = 0.
\end{equation}

Since the projection operator \(p^\nu\) must be a (1-1)-tensor, the condition \(\bar{p}^\nu f = J p^\nu\) is necessarily satisfied, where \(J\) is the Jacobian matrix of the coordinate transformation \((x^i, \tilde{\omega}^\lambda) \rightarrow (x^i, \bar{\omega}^\lambda)\) on \(P\). So the law of transformation of the \(\theta_i^3\) under the coordinate transformation of \(M\) is
\[
\begin{align*}
\vartheta_i^m \frac{\partial x^m}{\partial x^k} + \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^m} u^m &= \frac{\partial \tilde{x}^i}{\partial x^m} \theta_k^i, \\
\vartheta_i^m \frac{\partial \tilde{x}^m}{\partial x^k} &= \theta_i^m + b \theta_k^m \partial_m \log \Delta + c \theta_i^m f_i^m \partial_i \log \Delta \\
-(b \partial_i \partial_j \log \Delta + c \partial_k f_i^m \partial_m \log \Delta + c f_i^m \partial_k \partial_m \log \Delta)u^i, \\
\vartheta_i^m \frac{\partial \tilde{x}^m}{\partial x^k} &= \theta_i^m + c \theta_i^m \partial_m \log \Delta - b \theta_i^m f_i^m \partial_i \log \Delta \\
-(c \partial_i \partial_j \log \Delta - b \partial_k f_i^m \partial_m \log \Delta - b f_i^m \partial_k \partial_m \log \Delta)u^i. 
\end{align*}
\]
(2.5)

By substituting (2.3) into (2.5), and (2.5), we obtain the law of transformation under the coordinate transformation of \( M \) with respect to the components \( \varphi_{i}^j, p_j^i, q_j^k, q, \) and \( r \) as follows:

\[
\begin{align*}
\varphi_{i}^j &\frac{\partial x^m}{\partial x^k} \frac{\partial \tilde{x}^l}{\partial x^j} + \frac{\partial^2 \tilde{x}^l}{\partial x^k \partial x^j} = \frac{\partial \tilde{x}^l}{\partial x^m} \varphi_{i}^m - \frac{\partial \tilde{x}^l}{\partial x^m} P_j^m (b \partial_j \log \Delta + c f_i^m \partial_i \log \Delta) \\
- \frac{\partial \tilde{x}^l}{\partial x^m} f_i^m P_j^m (c \partial_j \log \Delta - b f_i^m \partial_i \log \Delta), \\
\tilde{p}_m \frac{\partial x^m}{\partial x^k} = \tilde{x}^l \theta_k^m, \\
\tilde{p}_m \frac{\partial x^m}{\partial x^k} + \tilde{q}_m \frac{\partial x^m}{\partial x^k} (b \partial_j \log \Delta + c f_i^m \partial_i \log \Delta) + \tilde{r}_m \frac{\partial x^m}{\partial x^k} (c \partial_j \log \Delta - b f_i^m \partial_i \log \Delta) \\
= p_k + b \varphi_{i}^m \partial_m \log \Delta + c \varphi_{i}^m \partial_i \log \Delta \\
- (b \partial_k \partial_j \log \Delta + c \partial_k f_i^m \partial_m \log \Delta + c f_i^m \partial_k \partial_m \log \Delta), \\
\tilde{q}_m \frac{\partial x^m}{\partial x^k} = q_k + b \varphi_{i}^m \partial_m \log \Delta + c \varphi_{i}^m \partial_i \log \Delta \\
- (b \partial_k \partial_j \log \Delta - b \partial_k f_i^m \partial_m \log \Delta - b f_i^m \partial_k \partial_m \log \Delta), \\
\tilde{r}_m \frac{\partial x^m}{\partial x^k} = r_k + b \varphi_{i}^m \partial_m \log \Delta - c \varphi_{i}^m \partial_i \log \Delta.
\end{align*}
\]
(2.6)

In analogy with the real projective connections [1], we assume that

\[
(2.7) \quad p_j^i = \delta_j^i, \quad r_k = f_i^m q_m.
\]

It is evident from the law of transformation (2.6) that this assumption is independent of the choice of the canonical coordinate system. At this stage, for the components of \( \theta_i^j \), we obtain

\[
\begin{align*}
\theta^i(x, w) &= \varphi_{i}^i(x) u^i + \delta^i_j v^j + f_i^j(x) e^j, \\
\theta_i^j(x, w) &= p_k i(x) u^i + q_k(x) e^i + q_m(x) f_i^m(x) e^m, \\
\theta^i(x, w) &= -p_k m(x) f_i^m(x) u^i - q_m(x) f_i^m(x) e^i + q_k(x) e^i,
\end{align*}
\]
(2.8)

where \( \varphi_{i}^j \) satisfies the condition
(2.9) \[ \partial_k f^j_k - f^j_m q^m_n + \varphi^l_n f^m_n = 0. \]

Again, the law of transformation for the components of \( \theta^i_j \) is given by

\[
\left\{
\begin{align*}
\varphi^m_i \frac{\partial x^m}{\partial x^i} &= \frac{\partial x^i}{\partial x^m} \varphi^m_j - \frac{\partial x^i}{\partial x^m} \left( b \partial_j \log \Delta + cf^j \partial_r \log \Delta \right) \\
\partial_m f^m_i &= \partial^j_i \log \Delta - bf^j \partial_r \log \Delta,
\end{align*}
\right.
\] (2.10)

Conversely, after some calculation, we can easily verify that if the \( \theta^i_j \) given by (2.8) satisfies (2.9) and (2.10), then \( \theta^i_j \) satisfies (2.5)\(_{1,2,3}\). Hence we call the \( \theta^i_j \) an almost complex projective connection. Thus we obtain

**Proposition 2.1.** In an almost complex projective vector bundle over an almost complex manifold, if the quantity \( \theta^i_j \) is given by (2.8) and satisfies (2.9) and (2.10), then the \( \theta^i_j \) defines an almost complex projective connection.

**Remark.** Henceforth we shall restrict ourselves to the case where almost complex projective connections are given by (2.8) together with (2.10) and satisfy (2.9).

**Theorem 2.1.** When an almost complex projective connection is given in an almost complex projective vector bundle over an almost complex manifold \( M \), then

(2.11) \[ \Pi^j_f = \varphi^j_f + q_m (\delta^j_f \delta^m_h - f^j_f f^m_f) \]

gives an affine connection on \( M \) and satisfies

(2.12) \[ \nabla_k f^j_f = \partial_k f^j_f - f^j_m \Pi^m_f + \Pi^i_m f^j_i = 0. \]

Moreover

(2.13) \[ \Pi^i_m = \partial_m q_i - q_i \varphi^m_i \]

is a tensor field on \( M \).
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PROOF. From the definition, it follows directly that $\Pi^i_j$ is a function of $x^i$ and is independent of $\omega^k$. For the law of transformation, by virtue of (2.10) and (2.11), it is easily seen that

$$\Pi^i_m \frac{\partial x^m}{\partial x^i} \frac{\partial x^m}{\partial x^k} = \Phi^i_m \frac{\partial x^m}{\partial x^i} \frac{\partial x^m}{\partial x^k} + \Phi^i_m \frac{\partial x^m}{\partial x^i} \frac{\partial x^m}{\partial x^k} - \Phi^i_f \frac{\partial x^m}{\partial x^i} \frac{\partial x^m}{\partial x^k}$$

$$= - \Phi^i_m \frac{\partial x^m}{\partial x^i} \frac{\partial x^m}{\partial x^k} [\Phi^r_j + q_h \delta^r_j - q_r f^r_i f^r_j].$$

Hence we obtain $\Pi^i_m \frac{\partial x^m}{\partial x^i} \frac{\partial x^m}{\partial x^k} + \Phi^i_m \frac{\partial x^m}{\partial x^i} \frac{\partial x^m}{\partial x^k} = \frac{\partial x^i}{\partial x^m} \Pi^m_j$, which implies $\Pi^i_j$ is an affine connection on $\mathcal{M}$. Moreover it follows that

$$\nabla_k f^i_j = \partial_k f^i_j - f^i_j (\Phi^r_i + q_h (\delta^r_j - f^r_j f^r_i)) + [\Phi^r_i + q_h (\delta^r_j - f^r_j f^r_i)] f^i_j$$

$$= \partial_k f^i_j - f^i_j (\Phi^r_i + q_h f^r_i f^r_j).$$

Thus (2.9) leads us to $\nabla_k f^i_j = 0$.

For $\Pi_{lm}^{ij}$, due to (2.10) and (2.13), we see

$$\Pi_{lm} \frac{\partial x^m}{\partial x^i} \frac{\partial x^l}{\partial x^j} = [\Phi_{ml} - q_l \Phi^m_i + \Phi_m^i] \frac{\partial x^m}{\partial x^i} \frac{\partial x^l}{\partial x^j}$$

$$= q_{i} - (b \partial_k \partial_j \log \Delta + c \partial_k f^m_j \partial_m \log \Delta + c f^m_j \partial_k \partial_m \log \Delta)$$

$$+ \frac{\partial}{\partial x^k} [q_j + b \partial_j \log \Delta + c f^r_i \partial_r \log \Delta] - q_r \Phi^r_j$$

$$= q_{i} - q_r \Phi^r_j + \partial_k q_j.$$

Hence $\Pi_{ij}$ is a tensor field on $\mathcal{M}$.

Besides, direct calculation shows us that the tensor $\Pi_{lm}$ takes the form of

$$\Pi_{lm} = \Phi_{lm} + \partial_i q_m - q_i \Phi_{lm} + q_l q_m - q_l f^r_i q_r f^r_m.$$  

REMARK. An almost complex manifold $\mathcal{M}$ admits an almost complex projective connection if and only if $\mathcal{M}$ admits a tensor field $\Pi_{lm}$, an affinen connection $\Pi^i_j$ satisfying $\nabla f = 0$ and a quantity $q_h$ whose law of transformation is given by

$$q_m \frac{\partial x^m}{\partial x^k} = q_h + b \partial_k \log \Delta + c f^m_j \partial_m \log \Delta.$$

§ 3. The distribution $\mathcal{D}$

In §2 we take $\{X_i\}$, which is defined by (2.2), as a local basis of the distribution $\Phi^i$. In our case, the $X_i$ are expressed as
\begin{equation}
X_i = \frac{\partial}{\partial x^i} - \left( \rho_{im}^k u^m + \delta_i^i v^i + f_{i}^m v^m \right) \frac{\partial}{\partial u^k} - \left( \rho_{im}^i u^m + q_i v^i + q_m f_{i}^m v^m \right) \frac{\partial}{\partial v^i} - \left( \rho_{im}^m u^r - q_i f_{i}^r v^r + q_i v^r \right) \frac{\partial}{\partial v^r}.
\end{equation}

Now we put
\begin{equation}
\begin{aligned}
Y_i &= \frac{\partial}{\partial u^i} + q_i \frac{\partial}{\partial v^i} - q_i f_{i}^i \frac{\partial}{\partial v^i}, \\
Z_{*} &= \frac{\partial}{\partial v^i}, \\
Z_{*} &= \frac{\partial}{\partial v^i}.
\end{aligned}
\end{equation}

From (2.1), the laws of transformation with respect to these, under the coordinate transformation, are given as
\begin{equation}
X_h = \frac{\partial x^m}{\partial x^h} X_m, \quad Y_h = \frac{\partial x^m}{\partial x^h} Y_m, \quad Z_{*} = Z_{*}, \quad Z_{*} = Z_{*}.
\end{equation}

And obviously the set \( \{Y_i, Z_{*}, Z_{*}\} \) forms a local basis of \( \Phi^\nu \).

If \( V = v^i \frac{\partial}{\partial x^i} \) is a vector field on \( M \), (3.3) shows us that \( V^\langle h \rangle = v^i X_i \) and \( V^\langle v \rangle = v^i Y_i \) are vector fields on \( P \). Hence \( V^\langle h \rangle \) and \( V^\langle v \rangle \) are called a horizontal lift and a vertical lift respectively.

Next we put
\begin{equation}
\begin{aligned}
Y &= u^k \frac{\partial}{\partial u^k} + v^i \frac{\partial}{\partial v^i} + v^r \frac{\partial}{\partial v^r}, \\
Y' &= F(Y) = f_{i}^k u^m \frac{\partial}{\partial u^k} - v^r \frac{\partial}{\partial v^r} + v^i \frac{\partial}{\partial v^i}.
\end{aligned}
\end{equation}

Then the \( Y \) and \( Y' \) are also vector fields on \( P \).

Moreover we put
\begin{equation}
\mathcal{O}(\theta) = \Phi^h \oplus Y \oplus Y'.
\end{equation}

Then \( \mathcal{O}(\theta) \) is a \((2n+2)\)-dimensional distribution in \( P \). Of course, the distribution \( \Phi^h \) is dependent upon the connection \( \theta_j \), so that the distribution \( \mathcal{O}(\theta) \) is also dependent upon the \( \theta_j \).

**Proposition 3.1.** The vector fields \( Y \) and \( Y' \) are invariant under the mapping \( C^* \) and so is the distribution \( \mathcal{O}(\theta) \).

**Proof.** We find, from the definition of the mapping \( C^* \),
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\[ d\mathcal{C}^\ast Y_p = u'(\alpha \delta^i + \beta f^i) \frac{\partial}{\partial u^i} + v'(\alpha \frac{\partial}{\partial \psi^i} + \beta \frac{\partial}{\partial \psi^j} + \beta \frac{\partial}{\partial \psi^k}) + v^2 \left( -\beta \frac{\partial}{\partial \psi^i} + \alpha \frac{\partial}{\partial \psi^j} \right) \]
\[ = \bar{u}' \frac{\partial}{\partial \bar{u}^i} + \psi^i \frac{\partial}{\partial \psi^i} + \psi^j \frac{\partial}{\partial \psi^j} \]
\[ = Y_{\mathcal{C}^\ast <p>}. \]

Similarly we find \( d\mathcal{C}^\ast Y_p = Y_{\mathcal{C}^\ast <P>}. \) Thus we have

(3.6) \[ d\mathcal{C}^\ast Y = Y, \quad d\mathcal{C}^\ast Y' = Y'. \]

By the definition, we also have \( d\mathcal{C}^\ast \Phi = \Phi_{\mathcal{C}^\ast <p>}. \) Thus we obtain

\[ d\mathcal{C}^\ast \Phi(\theta) = \Phi(\theta). \]

**Remark.** The Proposition 3.1 shows us that the distribution \( \Phi(\theta) \) is well-defined not only on \( P \) but also on \( \bar{P} \).

Let there be given two almost complex projective connections \( \theta \) and \( \hat{\theta} \). Then we have two distributions \( \Phi(\theta) \) and \( \Phi(\hat{\theta}) \). For these, it is apparent that a relation \( \Phi(\theta) = \Phi(\hat{\theta}) \) holds if and only if \( X_i = X_i - P_i Y - Q_i Y' \) for certain covector fields \( P_i \) and \( Q_i \). This condition is rewritten as

\[
\begin{align*}
\hat{\phi}_{i j} & = \phi_{i j} + P_i \delta_j^i + Q_i f_i^j, \\
\hat{\bar{P}}_{i j} & = P_{i j}, \\
\hat{q}_i & = q_i + P_i \\
\hat{f}_m^i & = q_m f^i_m - Q_i.
\end{align*}
\]

Hence we have \( Q_i = -P_m f^m_i \), which leads us to

\[
\begin{align*}
\hat{\phi}_{i j} & = \phi_{i j} + P_i \delta_j^i - P_m f^m_i f_j^i, \\
\hat{q}_i & = q_i + P_i, \\
\hat{P}_{i j} & = P_{i j}.
\end{align*}
\]

For \( \Pi_{i j} \) and \( \Pi_{l m} \), we have

\[
\begin{align*}
\hat{\Pi}_{j k} & = \phi_{j k} + q_m (\delta^i_j \delta^m_k - f_i^j f^m_k) + P_j \delta_k^i + P_k \delta_j^i \\
& - P_m f^m_i f_k^i f_j, \\
\hat{\Pi}_{l m} & = P_{l m} - q_i \phi_{i m} + \delta_i q_m - q_m P_i + q_m f^m_i P f_j. \\
& - P_i \phi_{i m} - P_m P_i + P f_i P f_m + \delta_i P_m.
\end{align*}
\]

Consequently we obtain

**Theorem 3.1.** In order that any two almost complex projective connections \( \theta \) and \( \hat{\theta} \) have the distribution \( \Phi(\theta) \) in common, it is necessary and sufficient that the
almost complex projective connections $\theta$ and $\hat{\theta}$ are related by

$$
\begin{align*}
\phi^i_j &= \phi^i_j + P_i \delta^i_j - P_m f^m_j f^i_j, \\
\hat{q}_i &= q_i + P_i, \\
\hat{P}_{ij} &= P_{ij},
\end{align*}
$$

(3.7)

for a certain covector field $P_i$. Further, in this case, the following holds

$$
\begin{align*}
\hat{\Pi}_{jk} &= P_{jk} + P_j \delta^i_k - P_k \delta^i_j - P_m f^m_j f^i_j - P_m f^m_k f^i_k, \\
\hat{\Pi}_{im} &= \Pi_{im} - P_i \Pi_{jm} - P_j \Pi_{im} + \delta_i P_m - \delta_j P_m + f^i_j P_m f^m_k.
\end{align*}
$$

(3.8)

In this theorem, of course, the relation $\hat{\nabla}_k f^i_j = -\nabla_k f^i_j = 0$ is true. Because

$$
\hat{\nabla}_k f^i_j = \partial_k f^i_j - f^i_j (\phi^i_j + P_i \delta^i_j - P_m f^m_j f^i_j) + (\phi^i_k + P_k \delta^i_j - P_m f^m_k f^i_k) f^i_j = \nabla_k f^i_j.
$$

Any two almost complex projective connections $\theta$ and $\hat{\theta}$ satisfying $\mathcal{O}(\theta) = \mathcal{O}(\hat{\theta})$ are called mutually complex-projectively related. Prof. Y. Tashiro called the linear connections satisfying (3.8), HP-related [5]. The HP-related connections are linear connections which have all HP-curves in common. Thus

**Corollary.** If linear connections $\Pi_{jk}$ and $\hat{\Pi}_{jk}$ are determined respectively by two almost complex projective connections which are mutually complex-projectively related, then they are mutually HP-related.

It is obvious that (3.8) is a generalization of a law of change of frames in infinite hyperplane in a theory of projectively connected real manifold.

**§ 4 Holonomy mappings**

Let us consider a $C^1$-curve $C: x^i = x^i(t)$ on $M$. A curve $C$ on $P$ is called the horizontal lift of $C$ if it satisfies following two conditions, that is (1) $\pi(C) = C$ and (2) the tangent vector to $C$ at each point of $C$ belongs to the horizontal distribution $\Phi^h$. The condition (2) can be restated as follows: the tangent vector field along $C$ is given by $\frac{dx^i}{dt} X_i$. And so, a curve $C: x^i = x^i(t), u^i = u^i(t), v^i = v^i(t), \dot{v}^i = \dot{v}^i(t)$ on $P$ is the horizontal lift of $C: x^i = x^i(t)$ on $M$ if and only if the condition

$$
\begin{align*}
\frac{du^b}{dt} + q^b_{mi} \frac{dx^m}{dt} u^i + f^b_{m} \frac{dx^m}{dt} v^i = 0, \\
\frac{dv^i}{dt} + P^i_{mi} \frac{dx^m}{dt} u^i + q_m \frac{dx^m}{dt} v^i + q_m f^r_{mr} \frac{dx^r}{dt} v^i = 0, \\
\frac{dv^2}{dt} - P^i_{mr} \frac{dx^m}{dt} f^i_r u^r - q_m f^r_{mr} \frac{dx^r}{dt} v^i + q_m \frac{dx^m}{dt} v^i = 0
\end{align*}
$$

(4.1)
is satisfied.

Let there be given a curve $C$ joining two points $A$ and $B$ in $M$. As the equation (4.1) is a simultaneous linear homogeneous differential equation with respect to $(u^i, v^i, \psi^i)$, the horizontal lift of $C$ which passes through a point $P$ in $\pi^{-1}(A)$ is uniquely determined. Further, $\pi^{-1}(B) \cap C$ becomes an only point $Q$. Then we can define a mapping $\sigma_e : \pi^{-1}(A) \rightarrow \pi^{-1}(B)$ by $\sigma_e(P) = Q$. It is obvious that $\sigma_e$ is a diffeomorphism from $\pi^{-1}(A)$ to $\pi^{-1}(B)$. Thus we call the mapping $\sigma_e$ a holonomy mapping corresponding to a curve $C$.

**Theorem 4.1.** By a holonomy mapping $\sigma_e$ corresponding to a curve $C$ joining two points $A$ and $B$ in a manifold $M$ endowed with an almost complex projective connection, mutually complex-projective points in $\pi^{-1}(A)$ are mapped into mutually complex-projective points in $\pi^{-1}(B)$. That is to say, if $P \not\equiv Q$ in $\pi^{-1}(A)$, then $\sigma_e(P) \not\equiv \sigma_e(Q)$ in $\pi^{-1}(B)$.

**Proof.** With respect to a local canonical coordinate system, the equation of a horizontal lift $\dot{C}$ of $C$ which passes through a point $P$ in $\pi^{-1}(A)$ can be represented by $(x^i(t), u^i(t), \psi^i(t))$. Any point on $\dot{C}$ satisfies the equation (4.1). On the other hand, by the definition of $C^*$, we find

$$C^*P = (x^i(t), \alpha u^i(t) + \beta f^i_j u^j(t), \alpha \psi^i(t) - \beta \psi^i(t), \beta \psi^i(t) + \alpha \psi^i(t)).$$

Direct calculation will show us that these points satisfy also the equation (4.1), for example, the left hand of the first term of (4.1) is reduced to

$$\frac{d}{dt} [\alpha u^i + \beta f^i_j u^j] + \phi^i_{m1} \frac{dx^m}{dt} (\alpha u^i + \beta f^i_j u^j) + \phi^i_{n1} \frac{dx^n}{dt} (\alpha \psi^i - \beta \psi^i) + \phi^i_{m} \frac{dx^m}{dt} (\beta \psi^i + \alpha \psi^i)$$

$$= \beta^i_0 \frac{dx^m}{dt} u^i \left[ \partial_m f^i_1 - f^i_1 \phi^i_{m1} + \phi^i_{mr} f^r_1 \right]$$

$$= 0.$$  

The rest terms are verified by similar calculation. Hence the Theorem 4.1 is proved.

From the Theorem 4.1, the holonomy mapping $\sigma_e$ can be induced in the almost complex projective bundle $\hat{P}$. We denote it by $\hat{\sigma}_e$. Then it follows that the mapping $\sigma_e$ maps each point of the complex projective space $\pi^{-1}(A)$ into $\pi^{-1}(B)$ holomorphically.

Now, let us define a 2-dimensional distribution $\mathcal{D}$ in $P$ by $\mathcal{D} = Y \oplus Y'$. Then

**Lemma** The distribution $\mathcal{D}$ is always integrable, and its integral manifold $D$
passing through a point \( P_0 \) is a set of all points which are complex-projective to \( P_0 \) in a fiber.

**Proof.** Indeed, from (3.4), it follows directly that

\[
[Y, Y'] = 0.
\]

Hence the distribution \( \mathcal{D} \) is integrable. Therefore, each point of the integral manifold \( D \) of the distribution \( \mathcal{D} \), which passes through a point \( P_0(x_0^i, u_0^j, v_0^k, \bar{v}_0^l) \), is written, by solving its differential equations, as

\[
\begin{align*}
  x^k &= x_0^k, \\
  u^k &= u_0^k(e^t \cos s) + u_0^m f_m^k(e^t \cos s), \\
  v^l &= v_0^l(e^t \cos s) - v_0^m(e^t \sin s), \\
  \bar{v}^l &= \bar{v}_0^l(e^t \sin s) + \bar{v}_0^m(e^t \cos s),
\end{align*}
\]

(4.4)

where \( t \) and \( s \) are parameters. It is clear that each point given by (4.4) is complex-projective to \( P_0 \).

**Theorem 4.2.** Let \( C \) be a \( C^1 \)-curve joining two points \( A \) and \( B \) in a manifold \( M \) endowed with an almost complex projective connection \( \theta \) and \( \varphi \), be a holonomy mapping corresponding to the curve \( C \) and the connection \( \theta \), that is \( \sigma_{(\theta)}: \pi^{-1}(A) \to \pi^{-1}(B) \). If \( \hat{\theta} \) is another almost complex projective connection admitting the distribution \( \mathcal{D}'(\theta) \) in common with \( \theta \), then \( \sigma_{(\hat{\theta})}(P) \) and \( \sigma_{(\theta)}(P) \) are mutually complex-projective for each point \( P \) in \( \pi^{-1}(A) \).

**Proof.** The local basis \( \{X_i\} \) of the horizontal distribution \( \Phi^h(\theta) \) is given by (3.1). If \( \{\hat{X}_i\} \) is the local basis of the distribution \( \Phi^h(\hat{\theta}) \) with respect to the \( \hat{\theta} \), the relations (3.7) and (3.4) give us

\[
\hat{X}_i = X_i - P_i Y + P_{m} f_m^i Y'.
\]

(4.5)

On the other hand (3.1) and (3.4) lead us to

\[
[X_i, Y] = 0 \quad \text{and} \quad [X_i, Y'] = 0,
\]

(4.6)

by virtue of (2.9). In the manifold \( M \), however, we may consider only the points on the given curve \( C \). Hence we denote by \( P(C) \) a submanifold of \( P \) whose projection to \( M \) is the given curve \( C \). Put \( X = \frac{dX^i}{dt} X_i \) and \( \mathcal{H} = X + Y + Y' \) in the \( P(C) \), then we have that \( X \) is a vector field on \( P(C) \) and \( \mathcal{H} \) is a 3-dimensional distribution on \( P(C) \). And we have that \( \mathcal{H} \) is integrable because of (4.3) and (4.6). Hence we denote by \( H \) an integral manifold of \( \mathcal{H} \) passing through a point \( P \) which exists in \( \pi^{-1}(A) \). Since the vector field \( X \) is horizontal, \( Y \) and
\( Y' \) are vertical, \( H \cap \pi^{-1}(B) \) coincides with a 2-dimensional submanifold \( D \) which is given by the preceding Lemma and is defined in \( \pi^{-1}(B) \).

Now it is clear that \( \sigma_{\xi^*}(P) \) exists in the \( D \). The relation (4.5) shows us that \( \sigma_{\xi^*}(P) \) exists also in the same \( D \). Hence the proof is complete from the Lemma.

**Theorem 4.3.** Let \( M \) be a manifold endowed with an almost complex projective connection, \( A \) and \( B \) be two points in \( M \), \( C_1 \) and \( C_2 \) be arbitrary \( C^1 \)-curves joining \( A \) and \( B \) in \( M \). If the distribution \( \mathcal{D}(\theta) \) is integrable on the almost complex projective vector bundle over \( M \), then holonomy mappings \( \sigma_{\xi_1} \) and \( \sigma_{\xi_2} \) map any point \( P_0 \) in \( \pi^{-1}(A) \) into the points which are mutually complex-projective in \( \pi^{-1}(B) \), that is to say, \( \sigma_{\xi_1}(P_0) \wedge \sigma_{\xi_2}(P_0) \) for each point \( P_0 \) in \( \pi^{-1}(A) \).

**Proof.** Since the distribution \( \mathcal{D}(\theta) \) includes the horizontal distribution \( \Phi^h_{\theta} \), and is integrable, a horizontal lift of any curve joining \( A \) and \( B \) is included in an integral manifold \( S_0 \) of \( \mathcal{D}(\theta) \) passing through the point \( P_0 \). Hence, we have \( \sigma_{\xi_1}(P_0) \in S_0 \cap \pi^{-1}(B) \) and \( \sigma_{\xi_2}(P_0) \in S_0 \cap \pi^{-1}(B) \). On the other hand \( S_0 \cap \pi^{-1}(B) \subseteq D \) where \( D \) is a certain integral manifold of the distribution \( \mathfrak{D} = Y \oplus Y' \) which is a vertical part of \( \mathcal{D}(\theta) \). Thus the lemma shows us that

\[
\sigma_{\xi_1}(P_0) \wedge \sigma_{\xi_2}(P_0).
\]

§ 5. An algebraic condition for the distribution \( \mathcal{D}(\theta) \) to be integrable

Let us consider a condition for the distribution \( \mathcal{D}(\theta) \) to be integrable on an almost complex projective vector bundle over the manifold \( M \) endowed with an almost complex projective connection. The definition (3.5) and the relation (4.3) and (4.6) show us that a condition for the \( \mathcal{D}(\theta) \) to be integrable is \([X_i, X_j] \in \mathcal{D}(\theta)\). For \([X_i, X_j]\), after some complicated calculations, we obtain

\[
[X_i, X_j] = \{R_{ijh} + \delta_i^h \Pi_{jk} - \delta_i^h \Pi_{hk} + f_{i}^h f_{j}^m \Pi_{jm} - f_{j}^m f_{i}^h \Pi_{im} + \delta_i^h \delta_j^h - f_{i}^m f_{j}^m g_{ml} \} u^k \frac{\partial}{\partial u^h} + (\Pi_{ij} - \Pi_{ji})(\delta_i^h \delta_j^m - f_{i}^m f_{j}^m) g_{ml} \} u^h \frac{\partial}{\partial u^h} + (\Pi_{ij} - \Pi_{ji}) \pi^i \frac{\partial}{\partial \psi^j} + f_{j}^m (\Pi_{ij} - \Pi_{ji}) \psi^j \frac{\partial}{\partial u^h} + [\nabla_j \Pi_{ik} - \nabla_i \Pi_{jk} - \Pi_{mk} (\Pi_{ij}^m - \Pi_{ij}^n) + \Pi_{im} \Pi_{jk} - \Pi_{jm} \Pi_{ik} + f_{i}^m f_{j}^m \Pi_{jm} - f_{j}^m f_{i}^m \Pi_{im} + \delta_i^j (\Pi_{ij} - \Pi_{ji}) - f_{j}^m (\Pi_{im} f_{j}^m - \Pi_{jm} f_{i}^m) \} u^h \frac{\partial}{\partial \psi^i}.
\]
(5.1) 
\[-\{(\Pi_{ij} - \Pi_{ji}) + q_r(\Pi_{ij} - \Pi_{ji})\} v^i \frac{\partial}{\partial v^i} \]
\[-\{(\Pi_{im} f_j^m - \Pi_{jm} f_i^m) + q_r f_m^i (\Pi_{ij} - \Pi_{ji})\} v^m \frac{\partial}{\partial v^i} \]
\[-f^i_j [\nabla_j \Pi_{ij} - \nabla_i \Pi_{ji} - \Pi_{im} (\Pi_{mj} - \Pi_{mj}) + q_m q_i (\Pi_{ij} - \Pi_{ji})] \]
\[-q_r f_m^i q_i f_i^m (\Pi_{ij} - \Pi_{ji}) \]
\[+ q_r [R_{jst} - \delta^s_i \Pi_{jt} + \delta^s_j \Pi_{it} + f^s_i f^m_j \Pi_{jm} - f^s_i f^m_i \Pi_{im} \]
\[+ \delta^s_i (\Pi_{ij} - \Pi_{ji}) - f^s_i (\Pi_{im} f_j^m - \Pi_{jm} f_i^m)] u^k \frac{\partial}{\partial v^s} \]
\[+ \{(\Pi_{im} f_j^m - \Pi_{jm} f_i^m) + q_r f_m^i (\Pi_{ij} - \Pi_{ji})\} v^i \frac{\partial}{\partial v^m} \]
\[-\{(\Pi_{ij} - \Pi_{ji}) + q_r (\Pi_{ij} - \Pi_{ji})\} v^s \frac{\partial}{\partial v^s}, \]

where

(5.2) 
\[R_{j św} = \partial_j \Pi_{i wk} - \partial_i \Pi_{j wk} + \Pi_{j wk}^m \Pi_{im} - \Pi_{im} \Pi_{j wk}, \]

and $\nabla$ implies a covariant derivative with respect to the linear connection $\Pi_{ij}$. The relation (5.1) implies that $[X_i, X_j] \in \Phi^Y$. Hence the condition $[X_i, X_j] \in \Phi(\theta)$ is equivalent to $[X_i, X_j] = \alpha_{ij} Y + \beta_{ij} Y'$ for certain $\alpha_{ij}$ and $\beta_{ij}$. This is rewritten as

(5.3) 
\[[X_i, X_j] = (\alpha_{ij} \delta^s_i + \beta_{ij} f^i_s) u^k \frac{\partial}{\partial u^k} + (\alpha_{ij} v^i - \beta_{ij} \delta^i_s) \frac{\partial}{\partial v^i} + (\alpha_{ij} v^s + \beta_{ij} \delta^s_s) \frac{\partial}{\partial v^s}. \]

Hence, comparing (5.3) with (5.1), we get $\Pi_{ij} = F_{ij}$, $\alpha_{ij} = \Pi_{ij} - \Pi_{ji}$ and $\beta_{ij} = \Pi_{im} f^m_j - \Pi_{jm} f^m_i$. From these, we obtain

**Theorem 5.1.** Let $M$ be a manifold endowed with an almost complex projective connection. The distribution $\Phi(\theta)$ defined by (3.5) is integrable on the almost complex projective vector bundle over $M$ if and only if the following conditions for the connection are satisfied

\[
\begin{align*}
\Pi_{ij} &= \Pi_{ji}, \\
\nabla_j \Pi_{jk} &= \nabla_i \Pi_{jk}, \\
R_{j św} &= \delta^s_j \Pi_{i wk} - \delta^s_k \Pi_{j wk} + f^s_i f^m_j \Pi_{jm} - f^s_i f^m_i \Pi_{im} \\
&\quad + \delta^s_i (\Pi_{ij} - \Pi_{ji}) - f^s_i (\Pi_{im} f^m_j - \Pi_{jm} f^m_i) = 0.
\end{align*}
\]

From the Theorem 5.1, Theorem 2.1 together with the theorem of Tashiro [5], we obtain directly
Theorem 5.2. Let $M$ be a manifold stated in the Theorem 5.1. If the distribution $\mathcal{D}(\theta)$ is integrable, then the manifold $M$ is a complex manifold and is $HP$-flat, and the connection $\Pi^i_j$, is a symmetric $f$-connection.

College of General Education  Tokushima University

References