Direct Limits of Finitary Relation Spaces

By

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In [1] the direct limit of a sequence of semigroups is defined as follows: Consider a sequence \( \{D_i: i=1, 2, \cdots \} \) of semigroups with isomorphisms \( \varphi_{ji} \) of \( D_i \) into \( D_j \), \( i \leq j \), such that for \( i \leq j \leq k \), \( \varphi_{ki}(x) = \varphi_{kj} \cdot \varphi_{ji}(x) \) and \( \varphi_{ii}(x) = x \). The semigroup \( D \) of the set union \( \bigcup_{i=1}^{\infty} D_i \) which, for every \( i, j \) and \( x \) such that \( i \leq j \) and \( x \in D_i \), identifies \( x \) with \( \varphi_{ji}(x) \) is called the direct limit of \( \{D_i: i=1, 2, \cdots \} \) with respect to the isomorphism family \( F = \{\varphi_{ji}: i=1, 2, \cdots, j=1, 2, \cdots; i \leq j\} \) and is denoted by \( D = \lim_{i \to \infty} (D_i; F) \).

Further in [1] the problem is asked to describe the isomorphism condition for \( S \) and \( S' \) in terms of \( S_i, S'_i, \varphi_{ji}, \varphi'_{ji} \), if \( S_i \) and \( S'_i \) are positive integer semigroups and \( S = \lim_{i \to \infty} (S_i; \varphi_{ji}), S' = \lim_{i \to \infty} (S'_i; \varphi'_{ji}) \).

In this paper this concrete problem is not solved, but some conditions are given for a more general case.

The definition of the direct limit of the sequence of semigroups given in [1] can be generalized to the sequence of finitary relation spaces, if the word “semigroup” in it is substituted by the words “finitary relation space”. The relation space is a set on which some relations are given. If all of those relations are finitary, this relation space is called finitary.

We may adapt this definition by such a way that \( S \) is defined as such a set that for any \( i \) an isomorphism \( \varphi_i \) of \( S_i \) into \( S \) exists and any element of \( S \) is an image of some element of \( S_i \) in \( \varphi_i \) for some \( i \) and \( \varphi_j \varphi_{ji} = \varphi_i \) for \( j \geq i \).

Now we shall prove a lemma.

**Lemma.** Let \( M \) and \( M' \) be two finitary relation spaces. Let \( \alpha_i \) for each positive integer \( i \) be an isomorphism from \( M \) into \( M' \). For \( i \leq j \) let the definition domain of \( \alpha_i \) be included in the definition domain of \( \alpha_j \). Let \( \alpha_i(x) = \alpha_j(x) \) for any \( x \) from the intersection of definition domains of \( \alpha_i \) and \( \alpha_j \). Let any element of \( M \) be in the definition domain of some \( \alpha_i \). Then there exists an isomorphism of \( M \) into \( M' \).

**Proof.** Let us define the mapping \( \alpha \) so that \( \alpha(x) = \alpha_i(x) \) for such an \( i \) that \( x \) is in the definition domain of \( \alpha_i \). The element \( \alpha(x) \) is determined uniquely,
because $\alpha_i(x) = \alpha_j(x)$ for $x$ from the intersection of definition domains of $\alpha_i$ and $\alpha_j$. Let some elements $x_1, \ldots, x_n$ be in an $n$-ary relation on the relation space $M$. Let $x_i (i = 1, \ldots, n)$ be in the definition domain of $\alpha_{k(i)}$. Let $N = \max \{k(i) : i = 1, \ldots, n\}$. Then $x_i$ is in the definition domain of $\alpha_N$ for $i = 1, \ldots, n$ and the elements $\alpha_N(x_1), \ldots, \alpha_N(x_n)$ are in the corresponding relation, because $\alpha_N$ is an isomorphism. But $\alpha(x_i) = \alpha_N(x_i)$ for $i = 1, \ldots, n$ and so $\alpha(x_1), \ldots, \alpha(x_n)$ are in the corresponding relation. This can be made for any $n$-tuple which is in some relation on $M$ and so we have proved that $\alpha$ is an isomorphism.

The assumption that $M$ is a finitary relation space was made, because in the contrary case $N$ should not always have to exist.

All groups, semigroups, lattices, rings, fields, graphs etc. are finitary relation spaces.

Now we shall prove a theorem.

**Theorem 1.** Let $\{S_i : i = 1, 2, \cdots\}$, $\{S'_i : i = 1, 2, \cdots\}$ be two infinite sequences of finitary relation spaces, let $S = \lim \rightarrow (S_i ; \varphi_{ji})$, $S' = \lim \rightarrow (S'_i ; \varphi'_{ji})$, where $\varphi_{ji}$, $\varphi'_{ji}$ are corresponding isomorphisms (see the definition of the direct limit). The relation spaces $S$ and $S'$ are isomorphic to each other, if and only if there exists an infinite sequence $\{T_i : i = 1, 2, \cdots\}$, whose terms are finitary relation spaces or empty sets, and the isomorphisms $\psi_i$, $\psi'_i$, $\tau_{ji}$ for $i \leq j$ so that $\psi_i$ is an isomorphism of $T_i$ into $S_i$, $\psi'_i$ is an isomorphism of $T_i$ into $S'_i$ and $\tau_{ji}$ is an isomorphism of $T_i$ into $T_j$ so that the following conditions are satisfied:

(A) $\tau_{kj}(x) = \tau_{kj}(x) \cdot \tau_{ji}(x)$, $\tau_{ij}(x) = x$ for $i \leq j \leq k$,

(B) $\psi_j \tau_{ji}(x) = \varphi_{ji} \psi_i(x)$

(C) $\psi'_j \tau_{ji}(x) = \varphi'_{ji} \psi'_i(x)$

for any $i$.

**Remark.** The conjunction of the conditions (B) and (C) is equivalent to the following condition:

$$\bigcap_{i=1}^{\infty} (S_i - \psi_i(T_i)) = \bigcap_{i=1}^{\infty} (S'_i - \psi'_i(T_i)) = \emptyset.$$ 

**Proof.** Let there exist the sequence $\{T_i : i = 1, 2, \cdots\}$ and the isomorphisms $\psi_i$, $\psi'_i$ and $\tau_{ji}$ with the above described properties. For each $i$ consider the isomorphism $\eta_i = \varphi_i \psi_i$ of $T_i$ into $S$ and the isomorphism $\eta'_i = \varphi'_i \psi'_i$ of $T_i$ into $S'$. Let $y \in S$. We have $y = \varphi_i(x)$ for some positive integer $k$ and some
According to (B) there exists such a positive integer \( N \) that for \( n > N \) we have \( \psi_{n}(x) \in \psi_{n}(T_{n}) \). But \( \psi_{n}(x) = \psi_{n}^{-1}(y) \). Thus \( \psi_{n}^{-1}(y) \in \psi_{n}(T_{n}) \) and this means that there exists \( z \in T_{n} \) such that \( \psi_{n}(z) = \psi_{n}^{-1}(y) \), which implies \( z = \psi_{n}^{-1}(y) = \eta_{n}^{-1}(y) \). We have proved that for any \( y \in S \) there exists a positive integer \( N \) such that for \( n > N \) the element \( y \) is in the definition domain of \( \eta_{n}^{-1} \). Analogously we can prove that for any \( y' \in S' \) there exists a positive integer \( N' \) such that for \( n > N' \) the element \( y' \) is in the definition domain of \( \eta_{n}^{-1} \).

Now let us consider the mappings \( \omega_{n} = \eta_{n} \eta_{n}^{-1} = \psi_{n} \psi_{n}^{-1} \psi_{n} \psi_{n}^{-1} \) for positive integers \( n \); they are isomorphisms from \( S \) into \( S' \) and they can be eventually empty, i.e. defined for no element (if \( T_{n} = 0 \)). Let us study interrelations between \( \omega_{n} \) for \( m < n \). We have \( \psi_{m}(x) = \psi_{n}(\psi_{nm}(x)) \), \( \psi_{m}(x) = \psi_{n}^{-1}(\psi_{nm}(x)) \) for any \( x \) for which it is defined, therefore

\[
\omega_{m} = \psi_{m} \psi_{n}^{-1} \psi_{m} \psi_{n}^{-1} = \psi_{n} \psi_{nm} \psi_{m} \psi_{n}^{-1} \psi_{n}^{-1} \psi_{n}.
\]

Further \( \psi_{nm} \psi_{m} = \psi_{n} \psi_{nm} \), according to (A), and thus \( \psi_{n} \psi_{nm}^{-1} = \tau_{nm} \psi_{n}^{-1} \). According to (A) also \( \psi_{nm} \psi_{m} = \psi_{n} \tau_{nm} \). Thus we have \( \omega_{m}(y) = \psi_{n} \psi_{nm}^{-1} \psi_{n}^{-1} \psi_{n} \psi_{nm} \psi_{m} \psi_{n}^{-1} \psi_{n}^{-1} \psi_{n}^{-1}(y) = \psi_{n} \psi_{nm} \psi_{m} \psi_{n}^{-1} \psi_{n}^{-1} \psi_{n}^{-1}(y) = \omega_{n}(y) \) for all \( y \) for which \( \omega_{m}(y) \) is defined; therefore \( \omega_{n} \) is an extension of \( \omega_{m} \) for \( n > m \). To each \( y \in S \) there exists a positive integer \( N \) such that for \( n > N \) the mapping \( \eta_{n}^{-1} \) is defined in \( y \); therefore also \( \omega_{n} = \eta_{n} \eta_{n}^{-1} \) is defined in \( y \), because \( \eta_{n}^{-1}(y) \in T_{n} \) and \( \eta_{n}^{-1} \) is the mapping of \( T_{n} \) into \( S' \). Therefore let us define the mapping \( \omega \) of \( S \) into \( S' \) so that \( \omega(y) = \omega_{n}(y) \) for such a positive integer \( n \) that \( \omega_{n} \) is defined in \( y \); according to the above proved \( \omega(y) \) is determined uniquely. The mapping \( \omega \) is an extension of \( \omega_{n} \) for any positive integer \( n \). According to Lemma \( \omega \) is an isomorphism of \( S \) into \( S' \). Now we shall prove that \( \omega \) is even an isomorphism of \( S \) onto \( S' \). We have \( \omega_{n}^{-1} = \eta_{n} \eta_{n}^{-1} \) for any \( n \). The mapping \( \omega_{n}^{-1} \) is defined in all elements of \( S' \) in which \( \eta_{n}^{-1} \) is defined. We have proved that for each \( y' \in S' \) there exists a positive integer \( N \) such that for \( n > N \) the element \( y' \) is in the definition domain of \( \eta_{n}^{-1} \). We define the mapping \( \omega' \) of \( S' \) into \( S \) so that \( \omega'(y') = \omega_{n}^{-1}(y') \) for such a positive integer \( n \) that \( \omega_{n}^{-1} \) is defined in \( y' \); the mapping \( \omega' \) is determined uniquely. Now let \( y = \omega'(y') \), \( y \in S \), \( y' \in S' \). Therefore there exists a positive integer \( m \) such that \( y = \omega_{m}^{-1}(y') \); thus \( y' = \omega_{m}(y) \) and \( y' = \omega(y) \). So \( \omega' = \omega^{-1} \); as \( \omega' \) is defined in all elements of \( S' \), \( \omega \) must be an isomorphism onto \( S' \). Therefore \( S \) and \( S' \) are isomorphic.

Now suppose that \( S \) and \( S' \) are isomorphic. Let \( \eta \) be an isomorphism of \( S \) onto \( S' \). Put \( T_{n} = \eta \psi_{n}(S_{n}) \cap \psi_{n}^{-1}(S_{n}) \) for each positive integer \( n \). Further put \( \psi_{n} = \phi_{n}^{-1} \eta_{n}^{-1} \), \( \psi_{n} = \phi_{n}^{-1} \) for every positive integer \( n \), \( \tau_{nm}(x) = x \) for \( x \in T_{m} \), \( m \) and \( n \) positive integers, \( m < n \). The condition (A) is fulfilled, because \( \psi_{n} \tau_{ji}(x) = \phi_{ji}^{-1} \eta_{n}^{-1}(x) \) for \( x \in T_{i} \), \( \phi_{ji} \psi_{n}(x) = \phi_{ji} \phi_{n}^{-1} \eta_{n}^{-1}(x) \). Thus \( \phi_{ji} \phi_{n}^{-1} \eta_{n}^{-1}(x) \) for \( x \in T_{i} \), therefore \( \psi_{i} \tau_{ji}(x) = \psi_{ji}(x) \). Further \( \psi_{i} \tau_{ji}(x) = \psi_{ji}^{-1}(x) \) for \( x \in T_{i} \), \( \phi_{ji} \phi_{i}^{-1}(x) = \phi_{ji} \phi_{i}^{-1}(x) \).
for \( x \in T_i \), therefore \( \psi_{i}^j \tau_{ji} = q_{ji}^i \psi_{i}^j \). We shall verify the condition (B). Let \( x \in S_k \) and consider the element \( q_{ab}(x) \in S \). As \( \eta \) is an isomorphism of \( S \) onto \( S' \), there exists an element \( z = \eta q_{ab}(x) \in S' \). This element is equal to \( \phi_{ab}^m(y) \) for some \( m \) and some \( y \in S_m \). Let \( N = \max \{k, m\} \). For \( n > N \) we have \( z = \eta q_{ab}^n q_{ab}(x) = \eta q_{ab}^n q_{ab}(y) \), therefore \( z = \eta q_{ab}^n(S) \cap q_{ab}^n(S') = T_n \). Then \( x = q_{ab}^{-1}(z) \) and \( q_{ab}^n(x) = q_{ab}^n q_{ab}^{-1}(z) = q_{ab}^{-1}(z) = \phi_{ab}^n(x) \in S_n(T_n) \). Analogously we verify the condition (C). The proof is ready.

Now if we put \( \xi_i = \psi_i^j \psi_{i}^{-1} \), the mapping \( \xi_i \) is an isomorphism from \( S_i \) into \( S_i' \). Further \( \xi_j q_{ji}(x) = \psi_j^i \psi_i^j q_{ji}(x) = \psi_j^i \psi_i^j \psi_{ji}^i \psi_i^j(x) = q_{ji}^i \psi_i^j(x) \) for any \( x \) for which both \( \xi_j q_{ji} \) and \( q_{ji}^i \) are defined; this follows from the conditions (A). From the conditions (B) and (C) we can obtain the result that for any positive integer \( k \) and \( x \in S_k \) there exists a positive integer \( N \) such that for \( n > N \) the element \( q_{ab}^n(x) \) is in the definition domain of \( \xi_n \) and for any positive integer \( k \) and \( x' \in S_k' \) there exists a positive integer \( n' \) such that for \( n > n' \) the element \( q_{ab}^n(x') \) is in the set of values of \( \xi_n \).

On the other hand, if such mappings \( \xi_i \) are defined, we may obtain the sets \( T_i \) and the mappings \( \psi_i, \psi_{i}^j, \tau_{ji} \). For each \( i \) we define \( T_i \) as the subset of \( S_i \) consisting of all elements \( x \) for which \( \xi_i(x) \) is defined. Then \( \psi_i(x) = x, \psi_{i}^j(x) = \xi_j(x), \tau_{ji}(x) = q_{ji}(x) \) for \( x \in T_i, j > i \). We can easily verify all the conditions for these sets and mappings.

Therefore we can express a theorem which is a simplification of Theorem 1; we do not need the sets \( T_i \) in it.

**Theorem 2.** Let \( \{S_i; i = 1, 2, \cdots\}, \{S_i'; i = 1, 2, \cdots\} \) be two infinite sequences of finitary relation spaces, let \( S = \lim (S_i; q_{ji}), S' = \lim (S_i'; q_{ji}) \), where \( q_{ji}, q_{ji}^i \) are corresponding isomorphisms. The relation spaces \( S \) and \( S' \) are isomorphic to each other, if and only if there exists a family \( \Xi = \{\xi_i; i = 1, 2, \cdots\} \) of isomorphisms such that \( \xi_i \) is an isomorphism from \( S_i \) into \( S_i' \) (some of these isomorphisms may be empty) so that the following conditions are satisfied:

(A) \( \xi_j q_{ji}(x) = q_{ji}^i(x) \xi_j(x) \)

for any \( x \) for which both \( \xi_j q_{ji} \) and \( q_{ji}^i \xi_j \) are defined.

(B) To each positive integer \( k \) and to each element \( x \in S_k \) there exists such a positive integer \( N \) that for each integer \( n > N \) the element \( q_{ab}^n(x) \) is contained in the definition domain of \( \xi_n \).

(C) To each positive integer \( k \) and to each element \( x' \in S_k' \) there exists such a positive integer \( N' \) that for each integer \( n > N' \) the element \( q_{ab}^n(x') \) is contained in the set of values of \( \xi_n \).

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