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Adjacent Topologies

By

Pushpa AGASHE and Norman LEVINE

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The concepts of immediate predecessor and of immediate successor in the lattice of uniformities and consequently that of adjacent uniformities were introduced by N. Levine and L. Nachman in [3]. In the same paper, the authors have also shown that every uniformity whose topology is not discrete has an immediate successor and that every non trivial uniformity has an immediate predecessor.

In this paper we introduce similar concepts of immediate predecessor and immediate successor in the lattice of topologies and consequently that of adjacent topologies. It is then a natural question to ask what types of topologies have immediate predecessors or immediate successors or both. We have attempted to give a solution to this problem.

§ I deals with immediate successors and § II deals with immediate predecessors. In § III we discuss some properties of adjacent topologies.

§ I **Immediate Successors**

I.1 **Introduction**

Throughout this paper $\mathrm{T}(X)$ will denote the lattice of topologies on the set X. Elements of $\mathrm{T}(X)$ will be represented by $\mathscr{T}$, $\mathscr{U}$, $\mathscr{V}$ etc. Also, we make the assumption that $|X|>1$ where $|X|$ denotes the cardinal number of X.

I.1.1 DEFINITION. *Let $\mathscr{T}\in\mathrm{T}(X)$ and suppose that $A\notin\mathscr{T}$, $A\subseteq X$. Then $\mathscr{T}(A)$ is the collection of sets of the form $O_1\cup(O_2\cap A)$, $O_i\in\mathscr{T}$, and is called the simple extension of $\mathscr{T}$ determined by A* (see[4]).

I. 1. 2 THEOREM. *Let $\mathscr{T}\in\mathrm{T}(X)$ and suppose that $\mathscr{T}(A)$ is a simple extension of $\mathscr{T}$. Then*

(1) *$\mathscr{T}(A)\in\mathrm{T}(X)$ and*

(2) *$\mathscr{T}(A)=\mathscr{T}\vee\mathscr{U}$ where $\mathscr{U}=\{\phi, A, X\}$ and $\vee$ denotes supremum.*

See [4] for elementary properties of $\mathscr{T}(A)$.

I. 1. 3 DEFINITION. *Let* $\mathscr{T}$, $\mathscr{U}$ *be elements of* $\mathrm{T}(X)$. *Then* $\mathscr{T}$ *imp* $\mathscr{U}$ *iff* (i) $\mathscr{T} \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{U}$ *and* (ii) *if* $\mathscr{T} \subseteq \mathscr{V} \subseteq \mathscr{U}$ *and* $\mathscr{V} \in \mathrm{T}(X)$ *then* $\mathscr{T} = \mathscr{V}$ or $\mathscr{V} = \mathscr{U}$. *We call* $\mathscr{U}$ *and* $\mathscr{V}$ *adjacent iff* $\mathscr{U}$ *imp* $\mathscr{V}$ *or* $\mathscr{V}$ *imp* $\mathscr{U}$. *Also if* $\mathscr{U}$ *imp* $\mathscr{V}$ *we say that* $\mathscr{U}$ *is an immediate predecessor of* $\mathscr{V}$ *and* $\mathscr{V}$ *is an immediate successor of* $\mathscr{U}$.

We shall make frequent use of the following important lemma.

I. 1. 4 LEMMA. *Let* $\mathscr{T}$ *imp* $\mathscr{U}$ *in* $\mathrm{T}(X)$. *Then* $\mathscr{U}$ *is a simple extension of* $\mathscr{T}$.

PROOF: Let $A \in \mathscr{U} - \mathscr{T}$. Then $\mathscr{T} \subseteq \mathscr{T}(A) \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{T}(A)$ and hence $\mathscr{T}(A) = \mathscr{U}$.

The converse of the above lemma is false as is seen in

I. 1. 5 EXAMPLE. Let $X = \{a, b, c\}$ and $\mathscr{T} = \{\phi, \{a\}, X\}$. Then $\mathscr{T}$ imp $\mathscr{T}(\{b\})$ is false since $\mathscr{T} \subseteq \{\phi, \{a\}, \{a, b\}, X\} \subseteq \mathscr{T}(\{b\})$, the inclusions being proper.

The next theorem is simple but useful.

I. 1. 6 THEOREM. *If* $|\mathscr{T}|$ *is finite, then* $\mathscr{T}$ *has both an immediate predecessor and an immediate successor iff* $\mathscr{T}$ *is non trivial* (in this paper non trivial means both non indiscrete and non discrete).

We use the following lemma many times in this paper.

I. 1. 7 LEMMA. *Let* $\mathscr{T}, \mathscr{U} \in \mathrm{T}(X)$ *and* $A \notin \mathscr{T}$, $\mathscr{T} \subseteq \mathscr{U} \subseteq \mathscr{T}(A)$. *Then* $\mathscr{U} = \mathscr{T}(A)$ *iff* $A \in \mathscr{U}$.

PROOF: If $\mathscr{U} = \mathscr{T}(A)$, then $A \in \mathscr{U}$ since $A \in \mathscr{T}(A)$. Conversely, suppose $A \in \mathscr{U}$. Then $\mathscr{T} \vee \{\phi, A, X\} \subseteq \mathscr{U}$ and thus $\mathscr{T}(A) \subseteq \mathscr{U}$. Hence $\mathscr{T}(A) = \mathscr{U}$.

In this section we will first show that "almost" every non-T_1 topology has an immediate successor (Theorem I. 2. 3). In I. 3, we give a necessary condition on A for $\mathscr{T}$ *imp* $\mathscr{T}(A)$ whenever $\mathscr{T}$ is a T_1 topology. In I.4, we show that no metric topology has an immediate successor.

I. 2 Non-T_1 spaces

I. 2. 1 DEFINITION. *A set* $A \subseteq X$ *is said to be generalized closed (written henceforth as g-closed) iff* $cA \subseteq O$ *whenever* $A \subseteq O$ *and* O *is open* (see [2]). (c *denotes the closure operator relative to* $\mathscr{T}$)

Throughout the rest of the paper $c\{x\}$ denotes closure of the singleton set $\{x\}$.

I. 2. 2 LEMMA. *Let $(X, \mathscr{T})$ be a space and $x \in X$. Then $\{x\}$ is g-closed iff $y \in c\{x\}$ implies that $x \in c\{y\}$* (see [2]).

I. 2. 3 THEOREM. *Let $\{x\}$ be g-closed, but not closed relative to $\mathscr{T}$. Then $\mathscr{T}$ imp $\mathscr{T}(\mathscr{C}\{x\})$.*

PROOF: Clearly $\mathscr{C}\{x\} \notin \mathscr{T}$. Thus, $\mathscr{T} \subseteq \mathscr{T}(\mathscr{C}\{x\})$, $\mathscr{T} \neq \mathscr{T}(\mathscr{C}\{x\})$. Suppose $\mathscr{T} \subseteq \mathscr{U} \subseteq \mathscr{T}(\mathscr{C}\{x\})$, $\mathscr{T} \neq \mathscr{U}$. Let $U = O_1 \cup (O_2 \cap \mathscr{C}\{x\}) \in \mathscr{U} - \mathscr{T}$ where O_1, $O_2 \in \mathscr{T}$. If $x \notin O_2$ or if $x \in O_1$, then $U = O_1 \cup O_2 \in \mathscr{T}$, a contradiction. Thus $x \in O_2$ and $x \notin O_1$. But then $c\{x\} \subseteq O_2$ and (1) $c\{x\} \cap \mathscr{C}\{x\} \subseteq O_2 \cap \mathscr{C}\{x\} \subseteq U$. Also, we have (2) $U \subseteq \mathscr{C}\{x\}$. It follows then that

$$\begin{aligned} \mathscr{C}\{x\} &= (\mathscr{C}\{x\} \cap c\{x\}) \cup (\mathscr{C}\{x\} \cap \mathscr{C}c\{x\}) \\ &\subseteq U \cup \operatorname{Int}\mathscr{C}\{x\} \quad \text{by (1)} \\ &\subseteq \mathscr{C}\{x\} \qquad\qquad\quad \text{by (2).} \end{aligned}$$

Thus, $\mathscr{C}\{x\} = U \cup \operatorname{Int}\mathscr{C}\{x\} \in \mathscr{U}$ and $\mathscr{T}(\mathscr{C}\{x\}) = \mathscr{U}$ by Lemma I. 1. 7. Hence $\mathscr{T}$ *imp* $\mathscr{T}(\mathscr{C}\{x\})$ and the proof is complete.

The following example indicates that the converse of Theorem I. 2. 3 is false.

I. 2. 4 EXAMPLE. Let $x \neq y$ in a set X. Define $\mathscr{T} = \{O \subseteq X \mid x \in O$ implies $y \in O\}$. Then $\mathscr{T} \in \mathrm{T}(X)$ and $\{y\}$ is neither closed nor g-closed. We prove that $\mathscr{T}$ *imp* $\mathscr{T}(\mathscr{C}\{y\})$. Suppse $\mathscr{T} \subseteq \mathscr{U} \subseteq \mathscr{T}(\mathscr{C}\{y\})$; $\mathscr{T} \neq \mathscr{U}$. Let $U \in \mathscr{U} - \mathscr{T}$. Then $x \in U$ and $y \notin U$. But then $\mathscr{C}\{y\} = (\mathscr{C}\{y\} - \{x\}) \cup (\{x, y\} \cap U) \in \mathscr{U}$. Thus $\mathscr{U} = \mathscr{T}(\mathscr{C}\{y\})$ and $\mathscr{T}$ *imp* $\mathscr{T}(\mathscr{C}\{y\})$.

I. 2. 5 REMARK. $\mathscr{T}(\mathscr{C}\{y\})$ in Example I. 2. 4 is the discrete topology on X.

I. 2. 6 REMARK. $\mathscr{T}(\mathscr{C}\{x\})$ in Theorem I. 2. 3 is not T_1. Let $y \neq x$ and $y \in c\{x\}$. By I. 2. 2 $\mathscr{C}\{y\} \notin \mathscr{T}$. We claim that $c\{y\} \notin \mathscr{T}(\mathscr{C}\{x\})$. So suppose $\mathscr{C}\{y\} \in \mathscr{T}(\mathscr{C}\{x\})$. Then $\mathscr{C}\{y\} = O_1 \cup (O_2 \cap \mathscr{C}\{x\})$ for some O_1, $O_2 \in \mathscr{T}$. Then $x \in \mathscr{C}\{y\}$ implies $x \in O_1$. Therefore $c\{x\} \subseteq O_1$ and $y \in O_1 \subseteq \mathscr{C}\{y\}$, which is not true. Hence $\mathscr{T}(\mathscr{C}\{x\})$ is not T_1.

We can say even more (Theorem I. 2. 7 below), namely, that any other immediate successor of $\mathscr{T}$ in I. 2. 3 must be a non-T_1 topology.

I. 2. 7 THEOREM. *In a space X let $\{x\}$ be g-closed, but not closed with respect to $\mathscr{T}$. If $\mathscr{T}$ imp $\mathscr{U}$, then $\mathscr{U}$ is not* T_1.

PROOF: Assume $\mathscr{U}$ is T_1. Then $\mathscr{C}\{x\}$, $\mathscr{C}\{y\} \in \mathscr{U}$ where $y \in c\{x\}$, $y \neq x$ and thus, $\mathscr{T}(\mathscr{C}\{x\}) \subseteq \mathscr{U}$. But a similar argument as in the Remark I. 2. 6 would prove that $\mathscr{C}\{y\} \notin \mathscr{T}(\mathscr{C}\{x\})$. Hence $\mathscr{T} \subseteq \mathscr{T}(\mathscr{C}\{x\}) \subseteq \mathscr{U}$, the inclusions being proper, a contradiction.

The subsequent lemma will be helpful in general.

I. 2. 8 LEMMA. *Let X be a regular space. Then singletons are g-closed.*

PROOF: Let $x \in X$. We show that $\{x\}$ is g-closed as follows: Suppose $x \in O \in \mathscr{T}$. Since $\mathscr{T}$ is regular, there exists $O_1 \in \mathscr{T}$ such that $x \in O_1 \subseteq cO_1 \subseteq O$. But then $c\{x\} \subseteq cO_1 \subseteq O$ and thus $\{x\}$ is g-closed.

I. 2. 9 THEOREM. *Let $\mathscr{T}$ be a regular, but a non-T_1 topology. Then $\mathscr{T}$ has an immediate successor.*

PROOF: The result will be an immediate consequence of I. 2. 3, if we show that there exists $\{x\}$ such that $\{x\}$ is g-closed, but not closed. Since $\mathscr{T}$ is not T_1, there exists $\{x\}$ such that $\{x\}$ is not closed. But by Lemma I. 2. 8 $\{x\}$ is also g-closed.

Since a completely regular space is regular we have the following corollary:

I. 2. 10 COROLLARY. *Let $\mathscr{T}$ be a completely regular, but a non-T_1 topology. Then $\mathscr{T}$ has an immediate successor.*

The result that follows indicates that unlike the metric topologies, pseudometric topologies which are not metric topologies always possess immediate successors.

I. 2. 11 THEOREM. *Every pseudometric topology which is not a metric topology has an immediate successor.*

PROOF: A pseudometric topology is regular, and since it is not a metric topology, it is not T_1. Hence the theorem follows from Theorem I. 2. 9.

I. 3 T_1 spaces

I. 3. 1 THEOREM. *Let $\mathscr{T} \in T(X)$ and $A \notin \mathscr{T}$. If $\mathscr{T}$ imp $\mathscr{T}(A)$, then a, $b \in A - \mathrm{Int}A$ implies $c\{a\} = c\{b\}$, c, Int denoting the closure and the interior operators respectively, relative to $\mathscr{T}$.*

PROOF: If we show that $a \in c\{b\}$ and $b \in c\{a\}$, it will automatically follow that $c\{a\} = c\{b\}$. Suppose $a \notin c\{b\}$. If $a \in \mathrm{Int}(A - c\{b\})$, then $a \in \mathrm{Int}A$, a contradiction. Thus $a \notin \mathrm{Int}(A-c\{b\})$ and $A-c\{b\} \notin \mathscr{T}$. Hence,

$$(1) \qquad \mathscr{T} \subseteq \mathscr{T}(A-c\{b\}), \quad \mathscr{T} \neq \mathscr{T}(A-c\{b\}).$$

Also, since $A-c\{b\} \in \mathscr{T}(A)$, $\mathscr{T}(A-c\{b\}) \subseteq \mathscr{T}(A)$. We assert that $A \notin \mathscr{T}(A-c\{b\})$. So suppose that $A \in \mathscr{T}(A-c\{b\})$. Then we can write $A = O_1 \cup (O_2 \cap (A-c\{b\}))$, O_1, $O_2 \in \mathscr{T}$. Now, $b \in A$ and $b \notin A-c\{b\}$. Hence $b \in O_1$. But then $b \in O_1 \subseteq A$, which means $b \in \mathrm{Int}A$, a contradiction. Hence $A \notin \mathscr{T}(A-c\{b\})$ and

$$(2) \qquad \mathscr{T}(A-c\{b\}) \subseteq \mathscr{T}(A), \quad \mathscr{T}(A-c\{b\}) \neq \mathscr{T}(A).$$

From (1) and (2) we get, $\mathscr{T} \subseteq \mathscr{T}(A-c\{b\}) \subseteq \mathscr{T}(A)$, the inclusions being proper. This is a contradiction, since we assumed $\mathscr{T}$ *imp* $\mathscr{T}(A)$. Thus $a \in c\{b\}$. Similarly we can prove that $b \in c\{a\}$ and the proof is complete.

As promised earlier, we now present a necessary condition on a set A for $\mathscr{T}$ *imp* $\mathscr{T}(A)$, whenever $\mathscr{T}$ is T_1.

I. 3. 2 THEOREM. *Let $\mathscr{T}$ be a T_1 topology and $A \notin \mathscr{T}$. If $\mathscr{T}$ imp $\mathscr{T}(A)$ then $A-\mathrm{Int}A=\{a\}$,* Int *being the interior operator relative to $\mathscr{T}$.*

PROOF: Let a, $b \in A-\mathrm{Int}A$. Using Theorem I. 3. 1 and the fact $\mathscr{T}$ is T_1, we have $\{a\}=c\{a\}=c\{b\}=\{b\}$. Thus $A-\mathrm{Int}A=\{a\}$.

To see that the converse of I. 3. 2 is not true, the reader is referred to Example I. 4. 2 below.

I. 4 Topologies without Immediate Successors

I. 4. 1 THEOREM. *Let $(X, \mathscr{T})$ be a topological space. Suppose for every set $A \notin \mathscr{T}$, we have $\mathscr{C}A=O_1 \cup O_2$ where $A \cup O_i \notin \mathscr{T}$ and $O_i \in \mathscr{T}$, $i=1, 2$. Then $\mathscr{T}$ dose not have an immediate successor.*

PROOF: To prove that $\mathscr{T}$ does not have an immediate successor it suffices to prove that for any $A \notin \mathscr{T}$, $\mathscr{T}(A)$ is not an immediate successor of $\mathscr{T}$. So suppose $A \notin \mathscr{T}$ and let $x \in A-\mathrm{Int}A$, where Int denotes the interior operator with respect to $\mathscr{T}$. Then $x \in c\mathscr{C}A=cO_1 \cup cO_2$. Assume $x \in cO_1$. It is obvious that

$$(1) \qquad \mathscr{T} \subseteq \mathscr{T}(A \cup O_1), \quad \mathscr{T} \neq \mathscr{T}(A \cup O_1)$$

and that $\mathscr{T}(A \cup O_1) \subseteq \mathscr{T}(A)$. We show that $A \notin \mathscr{T}(A \cup O_1)$ by denial. Suppose then that $A \in \mathscr{T}(A \cup O_1)$. We can write $A=O_3 \cup (O_4 \cap (A \cup O_1))$ for some O_3, $O_4 \in \mathscr{T}$. But then $x \notin O_3$, for otherwise $x \in O_3 \subseteq A$, implying that $x \in \mathrm{Int}A$, a contradiction. Thus, $x \in O_4$ and $O_4 \cap O_1 \neq \phi$, since $x \in cO_1$. But this not true, for we have $O_4 \cap O_1 = A \cap O_1 = \phi$. Hence $A \notin \mathscr{T}(A \cup O_1)$ and

(2) $$\mathscr{T}(A\cup O_1)\subseteq\mathscr{T}(A),\quad \mathscr{T}(A\cup O_1)\neq\mathscr{T}(A).$$

From (1) and (2) we have $\mathscr{T}\subseteq\mathscr{T}(A\cup O_1)\subseteq\mathscr{T}(A)$, the inclusions being proper, and thus $\mathscr{T}(A)$ is not an immediate successor of $\mathscr{T}$.

The case $x\in cO_2$ is treated similarly.

I. 4. 2 EXAMPLE. Let $X=[1, 2]$ and let $\mathscr{T}=\{O\subseteq X\mid 1\notin O$ or $1\in O$ and $\mathscr{C}O$ is finite$\}$. Then $(X, \mathscr{T})$ is a compact Hausdorff space and hence completely regular. In [3], Norman Levine and Louis Nachman have proved that if $\mathscr{U}$ is completely regular topology such that $\mathscr{T}\subseteq\mathscr{U}$, $\mathscr{T}\neq\mathscr{U}$, then there exists a completely regular topology $\mathscr{V}$ such that $\mathscr{T}\subseteq\mathscr{V}\subseteq\mathscr{U}$, inclusions being proper. In fact, their argument proves even more, namely, that $\mathscr{T}$ does not have an immediate successor. But we shall illustrate this by using Theorem I. 4. 1.

It suffices to show that, if $A\notin\mathscr{T}$, then we can write $\mathscr{C}A=O_1\cup O_2$ such that $A\cup O_i\notin\mathscr{T}$ and $O_i\in\mathscr{T}$, $i=1, 2$. But if $A\notin\mathscr{T}$, then $1\in A$ and $\mathscr{C}A$ is not finite. Hence we can write $\mathscr{C}A=O_1\cup O_2$, where $O_1\cap O_2=\phi$ and O_1 and O_2 are both infinite. Clearly O_1, $O_2\in\mathscr{T}$. Also, since $1\in A\cup O_1$, $A\cup O_2$ and since O_1, O_2 are infinite $A\cup O_1$, $A\cup O_2\notin\mathscr{T}$.

Now we turn to a slightly different and perhaps a larger class of topologies which do not possess immediate successors.

I. 4. 3 THEOREM. *Let $\mathscr{T}$ be a T_1, completely normal topology satisfying the first axiom of countability. Then $\mathscr{T}$ does not have an immediate successor.*

PROOF: We prove the result by showing that for any $A\notin\mathscr{T}$, $\mathscr{T}(A)$ is not an immediate successor of $\mathscr{T}$. Let then $A\notin\mathscr{T}$ and choose $a\in c\mathscr{C}A-\mathscr{C}A$. There exists a sequence $\{x_n\}$ of distinct elements in $\mathscr{C}A$ such that $\{x_n\}\to a$. Let $B_1=\{x_i\mid i \text{ odd}\}$ and $B_2=\{x_j\mid j \text{ even}\}$. Clearly $B_1, B_2\subseteq\mathscr{C}A$. Also $B_1\cap cB_2=\phi$ and $cB_1\cap B_2=\phi$. But then because of complete normality, there exist O_1, $O_2\in\mathscr{T}$ such that $B_1\subseteq O_1$, $B_2\subseteq O_2$ and $O_1\cap O_2=\phi$. We assert that $\mathscr{T}\subseteq\mathscr{T}(A\cup O_1)\subseteq\mathscr{T}(A)$, the inclusions being proper, or that (1) $A\cup O_1\notin\mathscr{T}$ and (2) $A\notin\mathscr{T}(A\cup O_1)$.

To prove (1) assume $A\cup O_1\in\mathscr{T}$. There exists $O_a\in\mathscr{T}$ such that $a\in O_a\subseteq A\cup O_1$. Because $a\in cB_2$, $O_a\cap B_2\neq\phi$. Choose $x_j\in O_a\cap B_2$. But then $x_j\in A\cup O_1$, which is not true and (1) is proved.

Likewise, to prove (2) assume $A\in\mathscr{T}(A\cup O_1)$. Then we can write $A=O_3\cup(O_4\cap(A\cup O_1))$, where O_3, $O_4\in\mathscr{T}$. If $a\in O_3$, then $a\in\mathrm{Int}A$, a contradiction. Therefore $a\in O_4\cap(A\cup O_1)$. Again, since $a\in cB_1$, $O_4\cap B_1\neq\phi$. Let $x_i\in O_4\cap B_1$. This implies that $x_i\in O_4\cap(A\cup O_1)\subseteq A$, meaning $x_i\in A$, which cannot be true. Thus (2) is proved and the proof is complete.

I. 4. 4 Example. Let $X=[O, \Omega)$, the space of all ordinals less than the first uncountable ordinal Ω and let $\mathscr{T}$ be the order topology on $[O, \Omega)$. We know that $\mathscr{T}$ is T_1, completely normal and also satisfies the first axiom of countability. Hence by I. 4. 3 it does not have an immediate successor.

I. 4. 5 Example. Let $(X, \mathscr{T})$ be the real line with the half-open interval topology. Again it is known that $(X, \mathscr{T})$ is T_1, completely normal and first countable, and thus does not have an immediate successor.

We return to Example I. 4. 2 to see that first axiom is not a necessary condition for a T_1, completely normal topology not to have an immediate successor. The topology in this example is T_1, completely normal, but not first countable and it does not have an immediate successor.

Since T_1, completely normal and first axiom are hereditary properties, as a direct consequence of I. 4. 3 we have the ensuing corollary.

I. 4. 6 Corollary. *Let $(X, \mathscr{T})$ be a T_1, completely normal and first countable topological space. If $Y \subseteq X$, then $(Y, Y \cap \mathscr{T})$ has no immediate successor, where $Y \cap \mathscr{T}$ denotes the subspace topology induced by $\mathscr{T}$.*

Finally, as mentioned earlier in the introduction, we obtain the following:

I. 4. 7 Theorem. *No metric topology has an immediate successor.*

Proof: A metric topology is T_1, completely normal and saitisfies the first axiom of countability and thus does not have an immediate successor by I. 4. 3.

§ II **Immediate Predecessors**

In this section we shall study the immediate predecessors of a topology $\mathscr{T}$. First we shall show that "almost" every non D-topology (see Definition II. 1. 1) has an immediate predecessor (Theorem II. 1. 6). Consequently, we can conclude that every T_1 topology which is not a D-topology and every non indiscrete regular topology have immediate predecessors.

II. 1 **Non D-topologies**

II. 1. 1 Definition. *A topology for a set X is called a* D-*topology (and $(X, \mathscr{T})$ a* D-*space) whenever every non empty open set is dense in X* (see [1]).

We borrow the following theorem from [1].

II. 1. 2 THEOREM. *In a topological space* $(X, \mathscr{T})$, *the following are equivalent*: (i) $(X, \mathscr{T})$ *is a* D-*space*, (ii) *every pair of non empty open sets has a non empty intersection, and* (iii) *every open set in* X *is connected.*

The next three lemmas will be quite useful.

II. 1. 3 LEMMA. *Let* $(X, \mathscr{T})$ *be a topological space and* $x \neq y$. *Define* $\mathscr{U} = \{O \in \mathscr{T} \mid x \in O \rightarrow y \in O\}$. *Then* $\mathscr{U} \in \mathrm{T}(X)$ *and* $\mathscr{U} \subseteq \mathscr{T}$.

II. 1. 4 LEMMA. *Let* $\{\mathscr{T}_\alpha \mid \alpha \in \Delta\}$ *be a chain of topologies for a set* X *and* $O \subseteq X$. *Also, let* $\mathscr{T} = \vee\{\mathscr{T}_\alpha \mid \alpha \in \Delta\}$ *where* $\vee$ *denotes the supremum. Then* $O \in \mathscr{T}$ *iff for each* $x \in O$, *there exists an* $O_\alpha \in \mathscr{T}_\alpha$ *for some* $\alpha \in \Delta$ *such that* $x \in O_\alpha \subseteq O$.

PROOF: The sufficiency follows from the definition of $\mathscr{T}$. To show the necessity, let $x \in O \in \mathscr{T}$. There exist $\alpha_1, \ldots, \alpha_n$ in Δ and $O_i \in \mathscr{T}_{\alpha_i}$ such that $x \in O_1 \cap \cdots \cap O_n \subseteq O$. Suppose that the notation is chosen so that $\mathscr{T}_{\alpha_1} \subseteq \mathscr{T}_{\alpha_2} \subseteq \cdots \subseteq \mathscr{T}_{\alpha_n}$. Then clearly $O_1 \cap \cdots \cap O_n$ is in $\mathscr{T}_{\alpha_n}$.

II. 1. 5 LEMMA. *Let* $x \neq y$ *in* X. *Let* $\{\mathscr{T}_\alpha \mid \alpha \in \Delta\}$ *be a chain of topologies for* X *such that* x, y *cannot be separated by the open sets in* $\mathscr{T}_\alpha$ *for every* $\alpha \in \Delta$. *Then* x, y *cannot be separated by the open sets in* $\mathscr{T} = \vee\{\mathscr{T}_\alpha \mid \alpha \in \Delta\}$.

PROOF: Suppose there exist $O_x, O_y \in \mathscr{T}$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \phi$. But then by Lemma II. 1. 4 there exist $O_\alpha \in \mathscr{T}_\alpha$ and $O_\beta \in \mathscr{T}_\beta$ for some α, β in Δ such that $x \in O_\alpha \subseteq O_x$ and $y \in O_\beta \subseteq O_y$. Suppose the notation is chosen so that $\mathscr{T}_\alpha \subseteq \mathscr{T}_\beta$. Then $O_\alpha, O_\beta \in \mathscr{T}_\beta$. Also, $O_\alpha \cap O_\beta = \phi$. This means x, y can be separated by the open sets in $\mathscr{T}_\beta$, a contradiction. Hence the result follows.

II. 1. 6 THEOREM. *Suppose that* $\mathscr{T}$ *is not a* D-*topology and that the singletons are g-closed. Then* $\mathscr{T}$ *has an immediate predecessor.*

PROOF: Let O_1, O_2 be the non empty disjoint open sets in $\mathscr{T}$ (see Theorem II. 1. 2). Suppose $x \in O_1$. Choose $y \in O_2$. Then $x \neq y$. Define $\mathscr{U} = \{O \in \mathscr{T} \mid x \in O \rightarrow y \in O\}$. Then, by Lemma II. 1. 3, $\mathscr{U} \in \mathrm{T}(X)$ and $\mathscr{U} \subseteq \mathscr{T}$. Also, x, y cannot be separated by the open sets in $\mathscr{U}$. Let $\mathscr{S} = \{\mathscr{V} \in \mathrm{T}(X) \mid \mathscr{U} \subseteq \mathscr{V} \subseteq \mathscr{T}$ and x, y cannot be separated by the open sets in $\mathscr{V}\}$. $\mathscr{S} \neq \phi$, since $\mathscr{U} \in \mathscr{S}$. Let $\{\mathscr{V}_\alpha \mid \alpha \in \Delta\}$ be a chain in $\mathscr{S}$. Then $\vee\{\mathscr{V}_\alpha \mid \alpha \in \Delta\} \supseteq \mathscr{U}$ and by Lemma II. 1. 5 x, y cannot be separated by the open sets in $\vee\{\mathscr{V}_\alpha | \alpha\in \Delta\}$. Hence, $\vee\{\mathscr{V}_\alpha|\alpha\in\Delta\}$ is an upper bound for the chain $\{\mathscr{V}_\alpha|\alpha\in\Delta\}$. Therefore, by the Zorn's lemma $\mathscr{S}$ has a maximal element, say $\mathscr{V}$. We have $\mathscr{U}\subseteq\mathscr{V}\subseteq\mathscr{T}$, $\mathscr{V}\neq\mathscr{T}$. To complete the proof it suffices to show that $\mathscr{V}$ *imp* $\mathscr{T}$. So let $\mathscr{V}\subseteq\mathscr{W}\subseteq\mathscr{T}$, $\mathscr{V}\neq\mathscr{W}$. Then, because of the maximality of $\mathscr{V}$, x, y can be separated by the open sets in $\mathscr{W}$, e.g., by W_x, W_y. Now we show that $\mathscr{W}=\mathscr{T}$ as follows: Let $O\in\mathscr{T}$. If either $x\notin O$ or x, $y\in O$, then $O\in\mathscr{U}\subseteq\mathscr{W}$. Assume therefore that $x\in O$ and that $y\notin O$. Then,

$$\begin{aligned} O&=(O\cap c\{x\})\cup(O\cap \mathscr{C}c\{x\})\\ &\subseteq(O\cap W_x)\cup(O\cap\mathscr{C}c\{x\}) \quad \text{since } \{x\} \text{ is } g\text{-closed}\\ &=((O\cup W_y)\cap W_x)\cup(O\cap\mathscr{C}c\{x\}) \quad \text{since } W_x\cap W_y=\phi\\ &\subseteq O \end{aligned}$$

which means $O=((O\cup W_y)\cap W_x)\cup(O\cap\mathscr{C}c\{x\})$. But, $O\cap\mathscr{C}c\{x\}, O\cup W_y\in\mathscr{U}\subseteq\mathscr{W}$. Hence, $O\in\mathscr{W}$ and $\mathscr{W}=\mathscr{T}$. This completes the proof.

If $\mathscr{T}$ is a T_1 topology, then the singletons are necessarily g-closed. Therefore, the next theorem results directly from II. 1. 6.

II. 1. 7 THEOREM. *If $\mathscr{T}$ is a T_1 but a non D-topology, then $\mathscr{T}$ has an immediate predecessor.*

II. 1. 8 THEOREM. *Let $\mathscr{T}$ be Hausdorff. Then $\mathscr{T}$ has an immediate predecessor.*

PROOF: Let $x\neq y$. Then there exist $O_1, O_2\in\mathscr{T}$ such that $x\in O_1$, $y\in O_2$ and $O_1\cap O_2=\phi$. Thus $\mathscr{T}$ is not a D-topology and the result follows from II. 1. 7.

As an immediate consequence of II. 1. 8 we obtain the following:

II. 1. 9 THEOREM. *Every metric topology has an immediate predecessor.*

II. 1. 10 THEOREM. *Let $\mathscr{T}$ be regular and not indiscrete. Then $\mathscr{T}$ has an immediate predecessor.*

PROOF: By Lemma I. 2. 8 the singletons are g-closed. Hence, to complete the proof it suffices to show that $\mathscr{T}$ is not a D-topology (see Theorem II. 1. 6). Let $\phi\neq O\subseteq X$, $O\neq X$ and $O\in\mathscr{T}$. Pick $x\in O$. By regularity there exists $O_1\in\mathscr{T}$ such that $x\in O_1\subseteq cO_1\subseteq O$. But then $\mathscr{C}cO_1\supseteq\mathscr{C}O\neq\phi$. Thus, O_1, $\mathscr{C}cO_1$ are non empty disjoint open subsets and $\mathscr{T}$ is not a D-topology.

The subsequent corollary and the theorem after it follow directly from II. 1. 10.

II. 1. 11 COROLLARY. *If $\mathscr{T}$ is a non indiscrete completely regular topology, then $\mathscr{T}$ has an immediate predecessor.*

II. 1. 12 THEOREM. *Let $\mathscr{T}$ be a non indiscrete pseudometric topology. Then $\mathscr{T}$ has an immediate predecessor.*

Before we go to § III, let us pause for a moment to compare and contrast our results on the immediate successors in § I and our results on the immediate predecessors discussed so far in this section. We may combine Theorem I. 2. 3 and Theorem II. 1. 6 into the following theorem.

II. 1. 13 THEOREM. *Suppose that $\mathscr{T}$ is not a* D*-topology and that the singletons are g-closed. If further, $\mathscr{T}$ is not* T_1*, then $\mathscr{T}$ has both an immediate predecessor and an immediate successor.*

Every non trivial, non T_1 regular or completely regular topology has both an immediate predecessor and an immediate successor (see Theorem I. 2. 9 and Theorem II.1.10, Corollary I. 2. 10 and Corollary II.1.11). Also, every non trivial pseudometric topology which is not a metric topology has both an immediate predecessor and an immediate successor (Theorem I.2.11 and Theorem II.1.12); but strangely enough a metric topology always has an immediate predecessor (Theorem II. 1. 9) and never has an immediate successor (Theorem I.4.7).

§ III **Properties of Adjacent Topologies**

So far we concerned ourselves with the existence of adjacent topologies. Now, assuming the existence of adjacent topologies we will further investigate the properties of adjacent topologies. Weak and quotient topologies induced by adjacent topologies are explored in III. 1. III. 2 treats the cartesian products of adjacent topologies and III. 3 treats the topological sums of adjacent topologies.

III. 1 **Weak and Quotient Topologies**

III. 1. 1 NOTATION. If $f\colon X\rightarrow Y$ is an onto function and $\mathscr{T}\in \mathrm{T}(X)$, we let $f\mathscr{T}$ denote the quotient topology for Y, that is, $f\mathscr{T}=\{U\subseteq Y \mid f^{-1}[U]\in\mathscr{T}\}$ is the largest topology on Y such that f is continuous.

If $f\colon X\rightarrow Y$ and $\mathscr{U}\in \mathrm{T}(Y)$, we let $f^{-1}\mathscr{U}$ denote the weak topology on X, that is, $f^{-1}\mathscr{U}=\{f^{-1}[U] \mid U\in\mathscr{U}\}$ is the smallest topology on X such that f is continuous.

We first examine the quotient topologies.

III. 1. 2 Lemma. *Let* $f\colon X \to Y$ *be onto and* $\mathscr{T}$, $\mathscr{T}' \in \mathrm{T}(X)$. *If* $\mathscr{T} \subseteq \mathscr{T}'$, *then* $f\mathscr{T} \subseteq f\mathscr{T}'$.

The reader can readily produce an example to see that $\mathscr{T}$ *imp* $\mathscr{T}'$ need not imply $f\mathscr{T}$ *imp* $f\mathscr{T}'$.

We now turn to weak topologies. We first give an example to show that $\mathscr{U}$ *imp* $\mathscr{U}'$ where $\mathscr{U}$, $\mathscr{U}' \in \mathrm{T}(Y)$ need not imply $f^{-1}\mathscr{U}$ *imp* $f^{-1}\mathscr{U}'$.

III. 1. 3 Example. Let $X = Y = \{a, b, c\}$, $\mathscr{U} = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$ and $\mathscr{U}' = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. Then $\mathscr{U}$ *imp* $\mathscr{U}'$. Define $f\colon X \to Y$ as follows: $f(a) = f(b) = a$, $f(c) = c$. Then $f^{-1}\mathscr{U} = f^{-1}\mathscr{U}' = \{\phi, \{a, b\}, X\}$.

III. 1. 4 Lemma. *Let* $f\colon X \to Y$ *and* $\mathscr{U}$, $\mathscr{U}' \in \mathrm{T}(Y)$. *If* $\mathscr{U} \subseteq \mathscr{U}'$ *then* $f^{-1}\mathscr{U} \subseteq f^{-1}\mathscr{U}'$.

Proof: Let $f^{-1}[U] \in f^{-1}\mathscr{U}$. Then $U \in \mathscr{U} \subseteq \mathscr{U}'$ implies that $f^{-1}[U] \in f^{-1}\mathscr{U}'$. Hence, $f^{-1}\mathscr{U} \subseteq f^{-1}\mathscr{U}'$.

III. 1. 5 Theorem. *Let* $f\colon X \to Y$ *and* $\mathscr{U}$, $\mathscr{U}' \in \mathrm{T}(Y)$. *If* $\mathscr{U}$ *imp* $\mathscr{U}'$, *then either* $f^{-1}\mathscr{U} = f^{-1}\mathscr{U}'$ *or* $f^{-1}\mathscr{U}$ *imp* $f^{-1}\mathscr{U}'$.

Proof: By Lemma III. 1. 4 $f^{-1}\mathscr{U} \subseteq f^{-1}\mathscr{U}'$. Suppose $f^{-1}\mathscr{U} \neq f^{-1}\mathscr{U}'$. We assert $f^{-1}\mathscr{U}$ *imp* $f^{-1}\mathscr{U}'$. So let $\mathscr{V} \in \mathrm{T}(X)$ such that $f^{-1}\mathscr{U} \subseteq \mathscr{V} \subseteq f^{-1}\mathscr{U}'$, $\mathscr{V} \neq f^{-1}\mathscr{U}'$. To prove $f^{-1}\mathscr{U}$ *imp* $f^{-1}\mathscr{U}'$ it suffices to show that $\mathscr{V} = f^{-1}\mathscr{U}$. If $U \in \mathscr{U}$, then $f^{-1}[U] \in f^{-1}\mathscr{U} \subseteq \mathscr{V}$. This means $U \in f\mathscr{V}$ and thus, $\mathscr{U} \subseteq f\mathscr{V}$. Now, choose $f^{-1}[U_1'] \in f^{-1}\mathscr{U}' - \mathscr{V}$. But then $U_1' \notin f\mathscr{V}$, otherwise, $f^{-1}[U_1'] \in \mathscr{V}$, a contradiction. Thus, $U_1' \notin f\mathscr{V} \cap \mathscr{U}'$ and we have $\mathscr{U} \subseteq f\mathscr{V} \cap \mathscr{U}' \subseteq \mathscr{U}'$, $f\mathscr{V} \cap \mathscr{U}' \neq \mathscr{U}'$. This implies $\mathscr{U} = f\mathscr{V} \cap \mathscr{U}'$. Now we show $\mathscr{V} = f^{-1}\mathscr{U}$ as follows: Let $V = f^{-1}[U_2'] \in \mathscr{V}$ for some $U_2' \in \mathscr{U}'$. Then $U_2' \in f\mathscr{V} \cap \mathscr{U}'$ which means $U_2' \in \mathscr{U}$. This implies that $f^{-1}[U_2'] \in f^{-1}\mathscr{U}$ or that $V \in f^{-1}\mathscr{U}$. Hence, $\mathscr{V} = f^{-1}\mathscr{U}$ and the proof is complete.

As a direct consequence of III. 1. 5 we have the following.

III. 1. 6 Corollary. *Let* $Y \subseteq X$ *and* $\mathscr{T}$, $\mathscr{T}' \in \mathrm{T}(X)$. *If* $\mathscr{T}$ *imp* $\mathscr{T}'$, *then either* $\mathscr{T} \cap Y = \mathscr{T}' \cap Y$ *or* $\mathscr{T} \cap Y$ *imp* $\mathscr{T}' \cap Y$.

III. 2 Cartesian Products

Let $\{X_\alpha \mid \alpha \in \Delta\}$ be a non empty family of non empty sets and $\mathscr{T}_\alpha$, $\mathscr{U}_\alpha \in T(X_\alpha)$ for all $\alpha \in \Delta$. Let $X = \times\{X_\alpha \mid \alpha \in \Delta\}$, $\mathscr{T} = \times\{\mathscr{T}_\alpha \mid \alpha \in \Delta\}$ and $\mathscr{U} = \times\{\mathscr{U}_\alpha \mid \alpha \in \Delta\}$. Then $\mathscr{T}$, $\mathscr{U} \in \mathrm{T}(X)$.

III. 2. 1 THEOREM. *Let* $(X, \mathscr{T}) = \times\{(X_\alpha, \mathscr{T}_\alpha) | \alpha \in \Delta\}$ *and* $(X, \mathscr{U}) = \times\{(X_\alpha, \mathscr{U}_\alpha) | \alpha \in \Delta\}$. *Then* $\mathscr{T}$ *imp* $\mathscr{U}$ *iff* (i) *there exists* $\delta \in \Delta$ *such that* $\mathscr{T}_\delta$ *imp* $\mathscr{U}_\delta$ *and* (ii) $\mathscr{T}_\alpha = \mathscr{U}_\alpha$ *is the indiscrete topology on* X_α *for all* $\alpha \in \Delta$ *such that* $\alpha \neq \delta$.

PROOF. Necessity: Suppose $\mathscr{T}$ *imp* $\mathscr{U}$. By Lemma III. 1. 2 $\mathscr{T}_\alpha \subseteq \mathscr{U}_\alpha$ for $\alpha \in \Delta$. If $\mathscr{T}_\alpha = \mathscr{U}_\alpha$, for all $\alpha \in \Delta$, then $\mathscr{T} = \mathscr{U}$, a contradiction. Hence, there exists $\delta \in \Delta$ such that $\mathscr{T}_\delta \neq \mathscr{U}_\delta$. Let $U_\delta \in \mathscr{U}_\delta - \mathscr{T}_\delta$. Then $P_\delta^{-1}[U_\delta] \in \mathscr{U} - \mathscr{T}$. To prove that $\mathscr{T}_\alpha = \mathscr{U}_\alpha$ is the indiscrete topology on X_α for all $\alpha \neq \delta$ it suffices to prove that $\mathscr{U}_\alpha$ is the indiscrete topology on X_α for all $\alpha \neq \delta$. Suppose we deny it. There exists $\beta \neq \delta$ such that $\mathscr{U}_\beta$ is not indiscrete. Choose U_β a proper open set in $\mathscr{U}_\beta$. If $P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta] \in \mathscr{T}$, then $P_\delta[P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta]] = U_\delta \in \mathscr{T}_\delta$, a contradiction. Hence $P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta] \notin \mathscr{T}$ and we have $\mathscr{T} \subseteq \mathscr{T}(P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta]) \subseteq \mathscr{U}$, the first inclusion being proper. We assert that the second inclusion is also proper or that $P_\delta^{-1}[U_\delta] \notin \mathscr{T}(P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta])$. Suppose $P_\delta^{-1}[U_\delta] \in \mathscr{T}(P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta])$. Then we can write $P_\delta^{-1}[U_\delta] = O_1 \cup (O_2 \cap P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta])$ where $O_1, O_2 \in \mathscr{T}$. Pick $x_\beta \notin U_\beta$ and $x_\delta \in U_\delta - \mathrm{Int}_{\mathscr{T}_\delta} U_\delta$. Also, choose $x_\alpha \in X_\alpha$ for all $\alpha \neq \beta, \delta$. Define $x: \Delta \to \cup\{X_\alpha | \alpha \in \Delta\}$ as follows:

$$x(\alpha) = \begin{cases} x_\delta & \text{if } \alpha = \delta \\ x_\beta & \text{if } \alpha = \beta \\ x_\alpha & \text{if } \alpha \neq \delta, \beta. \end{cases}$$

Then $x \in P_\delta^{-1}[U_\delta]$. Since $x_\beta \notin U_\beta$, $x \notin O_2 \cap P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta]$. But then $x \in O_1 \subseteq P_\delta^{-1}[U_\delta]$. This implies that $x_\delta \in P_\delta[O_1] \subseteq U_\delta$ or that $x_\delta \in \mathrm{Int}_{\mathscr{T}_\delta} U_\delta$, a contradiction. Hence, $P_\delta^{-1}[U_\delta] \notin \mathscr{T}(P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta])$ and $\mathscr{T} \subseteq \mathscr{T}(P_\beta^{-1}[U_\beta] \cap P_\delta^{-1}[U_\delta]) \subseteq \mathscr{U}$, the inclusions being proper. This contradicts the fact that $\mathscr{T}$ *imp* $\mathscr{U}$. Thus $\mathscr{U}_\alpha$ is the indiscrete topology on X_α for all $\alpha \neq \delta$ and (ii) is taken care of. To complete the proof of necessity we now show that $\mathscr{T}_\delta$ *imp* $\mathscr{U}_\delta$. Let $\mathscr{V}_\delta \in \mathrm{T}(X_\delta)$ such that $\mathscr{T}_\delta \subseteq \mathscr{V}_\delta \subseteq \mathscr{U}_\delta$, $\mathscr{T}_\delta \neq \mathscr{V}_\delta$. Choose $V_\delta \in \mathscr{V}_\delta - \mathscr{T}_\delta$. Let $\mathscr{V} = \times\{\mathscr{V}_\alpha | \alpha \in \Delta\}$ where $\mathscr{V}_\alpha = \mathscr{T}_\alpha$ if $\alpha \neq \delta$ and $\mathscr{V}_\alpha = \mathscr{V}_\delta$ if $\alpha = \delta$. Then $P_\delta^{-1}[V_\delta] \in \mathscr{V} - \mathscr{T}$ and therefore, $\mathscr{T} \subseteq \mathscr{V} \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{V}$. Thus $\mathscr{V} = \mathscr{U}$ which in turn implies that $\mathscr{V}_\delta = \mathscr{U}_\delta$. Hence $\mathscr{T}_\delta$ *imp* $\mathscr{U}_\delta$ and (i) is proved.

Sufficiency: Suppose there exists $\delta \in \Delta$ such that $\mathscr{T}_\delta$ *imp* $\mathscr{U}_\delta$ and $\mathscr{T}_\alpha = \mathscr{U}_\alpha$ is the indiscrete topology on X_α for all $\alpha \neq \delta$. Clearly $\mathscr{T} \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{U}$. Let $\mathscr{V} \in \mathrm{T}(X)$ such that $\mathscr{T} \subseteq \mathscr{V} \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{V}$. By Lemma III. 1. 2 $P_\alpha \mathscr{T} = \mathscr{T}_\alpha \subseteq P_\alpha \mathscr{V} \subseteq \mathscr{U}_\alpha = P_\alpha \mathscr{U}$ for all $\alpha \in \Delta$. Then, by hypothesis $\mathscr{T}_\alpha = P_\alpha \mathscr{V} = \mathscr{U}_\alpha$ is the indiscrete topology on X_α for all $\alpha \neq \delta$. Let $V \in \mathscr{V} - \mathscr{T}$. Then $P_\delta^{-1}[P_\delta[V]] = V$ since $\mathscr{U}_\alpha$ is the indiscrete topology on X_α for all $\alpha \neq \delta$. Therefore, $P_\delta[V]$

$\in P_\delta\mathscr{V}-\mathscr{T}_\delta$. Hence we have $\mathscr{T}_\delta \subseteq P_\delta\mathscr{V} \subseteq \mathscr{U}_\delta$, $\mathscr{T}_\delta \neq P_\delta\mathscr{V}$ which implies $P_\delta\mathscr{V} = \mathscr{U}_\delta$. Now, we show $\mathscr{U}=\mathscr{V}$ as follows: Let $U\in\mathscr{U}$. Then $P_\delta^{-1}[P_\delta[U]]=U$. But $P_\delta[U]\in\mathscr{U}_\delta=P_\delta\mathscr{V}$ implies that $U=P_\delta^{-1}[P_\delta[U]]\in\mathscr{V}$. Thus $\mathscr{V}=\mathscr{U}$. Therefore, $\mathscr{T}$ *imp* $\mathscr{U}$ and the proof is complete.

III. 3 Topological Sums

In this section we will prove a theorem for topological sums (Theorem III. 3. 6) which is analogous to Theorem III. 2. 1. In this direction we list a couple of lemmas.

III. 3. 1 Lemma. *Let* $(X, \mathscr{T})$ *be a topological space and* $X=\cup\{O_\alpha \mid \alpha\in\Delta\}$ *where* $O_\alpha\in\mathscr{T}$ *for all* $\alpha\in\Delta$. *Then* $O\in\mathscr{T}$ *iff* $O\cap O_\alpha\in\mathscr{T}\cap O_\alpha$ *for all* $\alpha\in\Delta$.

III. 3. 2 Lemma. *Let* $\mathscr{T}, \mathscr{U}\in T(X)$ *and* $X=\cup\{O_\alpha \mid \alpha\in\Delta\}$ *where* $O_\alpha\in\mathscr{T}$, $\mathscr{U}$ *for all* $\alpha\in\Delta$. *Then* $\mathscr{T}\subseteq\mathscr{U}$ *if and only if* $\mathscr{T}\cap O_\alpha\subseteq\mathscr{U}\cap O_\alpha$ *for all* $\alpha\in\Delta$.

Proof: If $\mathscr{T}\subseteq\mathscr{U}$, then clearly $\mathscr{T}\cap O_\alpha\subseteq\mathscr{U}\cap O_\alpha$ for all $\alpha\in\Delta$. Conversely, suppose $\mathscr{T}\cap O_\alpha\subseteq\mathscr{U}\cap O_\alpha$ for all $\alpha\in\Delta$. If $O\in\mathscr{T}$, then by Lemma III. 3. 1 $O\cap O_\alpha\in\mathscr{T}\cap O_\alpha$ for all $\alpha\in\Delta$ which implies that $O\cap O_\alpha\in\mathscr{U}\cap O_\alpha$ for all $\alpha\in\Delta$. Therefore, $O\in\mathscr{U}$ by Lemma III. 3. 1.

As a direct consequence of III. 3. 2 we get the following lemma.

III. 3. 3 Lemma. *Let* $\mathscr{T}, \mathscr{U}\in T(X)$ *and* $X=\cup\{O_\alpha \mid \alpha\in\Delta\}$ *where* $O_\alpha\in\mathscr{T}$, $\mathscr{U}$ *for all* $\alpha\in\Delta$. *Then* $\mathscr{T}=\mathscr{U}$ *if and only if* $\mathscr{T}\cap O_\alpha=\mathscr{U}\cap O_\alpha$ *for all* $\alpha\in\Delta$.

III. 3. 4 Theorem. *Let* $\mathscr{T}, \mathscr{U}\in T(X)$ *and* $X=\cup\{O_\alpha \mid \alpha\in\Delta\}$ *where* $O_\alpha \in\mathscr{T}$, $\mathscr{U}$ *for all* $\alpha\in\Delta$ *and* O'_α*s are pairwise disjoint. Then* $\mathscr{T}$ *imp* $\mathscr{U}$ *iff there exists* $\delta\in\Delta$ *such that* $\mathscr{T}\cap O_\delta$ *imp* $\mathscr{U}\cap O_\delta$ *and* $\mathscr{T}\cap O_\alpha=\mathscr{U}\cap O_\alpha$ *for all* $\alpha\neq\delta$.

Proof: Sufficiency: Soppose there exists $\delta\in\Delta$ such that $\mathscr{T}\cap O_\delta$ *imp* $\mathscr{U}\cap O_\delta$ and $\mathscr{T}\cap O_\alpha=\mathscr{U}\cap O_\alpha$ for all $\alpha\neq\delta$. Then, by Lemma III.3.2 and III.3.3 $\mathscr{T}\subseteq\mathscr{U}$, $\mathscr{T}\neq\mathscr{U}$. We assert that $\mathscr{T}$ *imp* $\mathscr{U}$. So suppose $\mathscr{T}\subseteq\mathscr{V}\subseteq\mathscr{U}$, $\mathscr{T}\neq\mathscr{V}$. By Lemma III.3.2 and by the hypotheses (1), $\mathscr{T}\cap O_\alpha=\mathscr{V}\cap O_\alpha=\mathscr{U}\cap O_\alpha$ for all $\alpha\neq\delta$. Now, choose $V\in\mathscr{V}-\mathscr{T}$. If $V\cap O_\delta\in\mathscr{T}\cap O_\delta$, then by (1) $V\cap O_\alpha\in\mathscr{T}\cap O_\alpha$ for all $\alpha\in\Delta$ which in turn implies that $V\in\mathscr{T}$, a contradiction. Hence $V\cap O_\delta \notin\mathscr{T}\cap O_\delta$. This together with III. 3. 2 implies that $\mathscr{T}\cap O_\delta\subseteq\mathscr{V}\cap O_\delta \subseteq\mathscr{U}\cap O_\delta$, $\mathscr{T}\cap O_\delta\neq\mathscr{V}\cap O_\delta$. Since $\mathscr{T}\cap O_\delta$ *imp* $\mathscr{U}\cap O_\delta$, we have (2) $\mathscr{V}\cap O_\delta =\mathscr{U}\cap O_\delta$. From (1) and (2) we can conclude that $\mathscr{V}\cap O_\alpha=\mathscr{U}\cap O_\alpha$ for all $\alpha\in\Delta$ or that $\mathscr{V}=\mathscr{U}$. Thus, $\mathscr{T}$ *imp* $\mathscr{U}$.

Necessity: Suppose $\mathscr{T}$ *imp* $\mathscr{U}$. By Lemma III. 3. 3 there exists $\delta\in\Delta$

such that $\mathscr{T} \cap O_\delta \subseteq \mathscr{U} \cap O_\delta$, $\mathscr{T} \cap O_\delta \neq \mathscr{U} \cap O_\delta$. But then by Corollary III. 1. 6 $\mathscr{T} \cap O_\delta$ *imp* $\mathscr{U} \cap O_\delta$. Because of the same corollary either $\mathscr{T} \cap O_\alpha = \mathscr{U} \cap O_\alpha$ or $\mathscr{T} \cap O_\alpha$ *imp* $\mathscr{U} \cap O_\alpha$ for all $\alpha \neq \delta$. We claim that $\mathscr{T} \cap O_\alpha = \mathscr{U} \cap O_\alpha$ for all $\alpha \neq \delta$. Suppose not, that is, suppose there exists $\beta \neq \delta$ such that $\mathscr{T} \cap O_\beta$ *imp* $\mathscr{U} \cap O_\beta$. Now, choose $U_1 \cap O_\delta \in \mathscr{U} \cap O_\delta - \mathscr{T} \cap O_\delta$ and $U_2 \cap O_\beta \in \mathscr{U} \cap O_\beta - \mathscr{T} \cap O_\beta$. Then U_1, $U_2 \in \mathscr{U} - \mathscr{T}$ (Lemma III. 3. 1). We consider two cases:

Case 1: $U_1 \cap O_\beta \in \mathscr{T} \cap O_\beta$

We have $\mathscr{T} \subseteq \mathscr{T}(U_1) \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{T}(U_1)$. We claim that $U_2 \notin \mathscr{T}(U_1)$. Suppose we deny it. Then, $U_2 = O_1 \cup (O_2 \cap U_1)$ where $O_1, O_2 \in \mathscr{T}$. Then, $U_2 \cap O_\beta = (O_1 \cup (O_2 \cap U_1)) \cap O_\beta = (O_1 \cap O_\beta) \cup (O_2 \cap U_1 \cap O_\beta)$. This implies that $U_2 \cap O_\beta \in \mathscr{T} \cap O_\beta$, a contradiction. Hence, $U_2 \notin \mathscr{T}(U_1)$. Therefore, $\mathscr{T} \subseteq \mathscr{T}(U_1) \subseteq \mathscr{U}$, inclusions being proper, which contradicts the hypothesis $\mathscr{T}$ *imp* $\mathscr{U}$.

Case 2: $U_1 \cap O_\beta \notin \mathscr{T} \cap O_\beta$

$U_1 \cap O_\delta \notin \mathscr{T}$, for otherwise $U_1 \cap O_\delta \in \mathscr{T} \cap O_\delta$, a contradiction. Thus, we have $\mathscr{T} \subseteq \mathscr{T}(U_1 \cap O_\delta) \subseteq \mathscr{U}$, $\mathscr{T} \neq \mathscr{T}(U_1 \cap O_\delta)$. We show that $U_1 \cap O_\beta \notin \mathscr{T}(U_1 \cap O_\delta)$ as follows: If $U_1 \cap O_\beta \in \mathscr{T}(U_1 \cap O_\delta)$, then $U_1 \cap O_\beta = O_1 \cup (O_2 \cap U_1 \cap O_\delta)$ where O_1, $O_2 \in \mathscr{T}$. Since $O_\beta \cap O_\delta = \phi$, we have $U_1 \cap O_\beta = O_1 \cap O_\beta$. This in turn implies that $U_1 \cap O_\beta \in \mathscr{T} \cap O_\beta$, a contradiction. Hence $U_1 \cap O_\beta \notin \mathscr{T}(U_1 \cap O_\delta)$. Thus, $\mathscr{T} \subseteq \mathscr{T}(U_1 \cap O_\delta) \subseteq \mathscr{U}$, inclusions being proper. But this again contradicts the assumption $\mathscr{T}$ *imp* $\mathscr{U}$.

Thus, in any case we have a contradiction; therefore, $\mathscr{T} \cap O_\alpha = \mathscr{U} \cap O_\alpha$ for all $\alpha \neq \delta$ and the proof is complete.

In the above theorem we cannot replace "open sets" by "closed sets" as may be seen from the ensuing example.

III. 3. 5 EXAMPLE. *Let $\mathscr{T}$ be the cofinite topology on an infinite set X. Then $X = \cup\{\{x\} \mid x \in X\}$ where $\{x\}$'s are closed, pairwise disjoint sets. If $y \in X$, then $\mathscr{T}(\{y\}) - \mathscr{T} = \{\{y\}\}$ which implies $\mathscr{T}$ imp $\mathscr{T}(\{y\})$; but $\mathscr{T} \cap \{x\} = \mathscr{T}(\{y\}) \cap \{x\}$ for all $x \in X$.*

The next theorem results directly from Theorem III. 3. 4.

III. 3. 6 THEOREM. *Let $(X, \mathscr{T}) = \Sigma\{(X_\alpha, \mathscr{T}_\alpha) \mid \alpha \in \Delta\}$ and $(X, \mathscr{U}) = \Sigma\{(X_\alpha, \mathscr{U}_\alpha)\ \alpha \in \Delta\}$. Then $\mathscr{T}$ imp $\mathscr{U}$ iff there exists $\delta \in \Delta$ such that $\mathscr{T}_\delta$ imp $\mathscr{U}_\delta$ and $\mathscr{T}_\alpha = \mathscr{U}_\alpha$ for all $\alpha \neq \delta$.*

The Ohio State University

References

[1] Levine N., Dense Topologies, Amer. Math. Monthly **75** (1968), 847–852.

[2] Levine N., Generalized Closed Sets, Rend. Circ. Mat. Palermo, ser. 2, Vol. **19** (1970), 89–96.

[3] Levine N. and Nachman L., Adjacent Uniformities, J. Math. Tokushima Univ. Vol. **5** (1971), 29–36.

[4] Levine N., Simple Extensions of Topologies, Amer. Math. Monthly **71** (1964), 22–25.