Mappings of Bounded Dilatation

By

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§ 1. Introduction

Let $M$ and $N$ be Riemannian manifolds of dimensions $m$ and $n$, respectively. Recently, two of the authors introduced the concept of a quasiconformal mapping $f: M \to N$ and applied it to obtain distance and (intermediate) volume decreasing properties of harmonic mappings between Riemannian manifolds of different dimensions [1]. In this paper the concept of a mapping $f: M \to N$ of bounded dilatation is introduced which is more general and natural than that of a $K$-quasiconformal mapping when $m$ and $n$ are greater than 2. An example of such a mapping which is not $K$-quasiconformal is given which is even harmonic. The main results in [1], viz., generalizations of the Schwarz-Ahlfors lemma as well as Liouville's theorem and the little Picard theorem are valid for this class of mappings. This is due to the fact that $\|f_\ast \|^2 / \|\Lambda f_\ast\|$ is bounded if $f$ is $K$-quasiconformal or if $f$ is of bounded dilatation.

Let $f: M \to N$ be a harmonic mapping of bounded dilatation of Riemannian manifolds. If the ratio function $\|f_\ast \|^2$ of distances attains its maximum at $x \in M$, then under suitable conditions on the bounds of the sectional curvatures at $x$ and $f(x)$, $f$ is distance decreasing.

If $M$ is a complete connected Riemannian manifold of constant negative curvature $-A$, in particular, if $M$ is the unit open $m$-ball with the hyperbolic metric of constant curvature $-A$, the condition on $\|f_\ast \|$ may be dropped. Indeed, let $N$ be a Riemannian manifold with sectional curvatures bounded above by a negative con-

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stant depending on $A$. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is distance decreasing.

The technique employed to prove this statement also yields the following fact.

Let $M$ be a complete connected locally flat Riemannian manifold and let $N$ be an $n$-dimensional Riemannian manifold with negative sectional curvature bounded away from zero. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is a constant mapping.

§ 2. Mappings of bounded dilatation

Let $V$ be an Euclidean vector space of dimension $m$ and let $V^\ast$ be its dual space. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $V$ with dual basis $\{\omega_1, \ldots, \omega_m\}$. A quadratic function on $V$ is an element of $(V \otimes V)^\ast$, so since $(V \otimes V)^\ast$ is canonically isomorphic to $V^\ast \otimes V^\ast$, a quadratic function on $V$ may be written as

$$f = \sum f_{ij} \omega_i \otimes \omega_j.$$  

If $f$ is positive semidefinite an orthonormal basis $\{e_i\}$ can be chosen so that $f_{ii} = 0$ for $i \neq j$ and $f_{ii} = \gamma_i^2 > 0$ for $i = 1, \ldots, k \leq m$, where $k = \text{rank } f$.

Let $W$ be an Euclidean vector space of dimension $n$ with inner product $h$, and let $F: V \to W$ be a linear mapping of rank $k \leq \min(m, n)$. We choose an orthonormal basis $\{e_i\}$ of $V$ so that

$$F^\ast h = \sum \gamma_i^2 \omega_i \otimes \omega_i.$$  

The vectors $\eta_i = (1/\gamma_i) F e_i$, $i = 1, \ldots, k$, therefore form part of an orthonormal basis of $W$. (If all of the $\gamma_i$ vanish, $F = 0$.) Let $X = \sum x^i e_i$, be a vector of unit length and assume $F \neq 0$, then $FX = \sum y^i \eta_i$, where $x^i = y^i / \gamma_i$. Consequently, if $F$ is of rank $k$, it maps a unit $(k-1)$-dimensional sphere of $V$ to a $(k-1)$-dimensional ellipsoid of $W$ with semiaxes of lengths $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k > 0$, where $\lambda_i = \gamma_i$, $i = 1, \ldots, k$ are the eigenvalues of $F^\ast F: V \to V$.

**Definition 1.** The ratio

$$l_s = \frac{\gamma_s}{\gamma_{s+1}}, \quad s = 1, \ldots, k-1$$  

will be called the $s$-th dilatation of $F$.

The mapping $F: V \to W$ induces a mapping

$$\wedge^p F: \wedge^p V \to \wedge^p W, \quad p \leq \min(m, n)$$  

given by
\[ \wedge^p F(e_{i_1} \wedge \cdots \wedge e_{i_p}) = Fe_{i_1} \wedge \cdots \wedge Fe_{i_p}, \]

where \( 1 \leq i_1 < i_2 < \cdots < i_p \leq m \). \( \wedge^p F \) may be regarded as an element of \( \wedge^p V^* \otimes \wedge^t W \). A norm \( \| \cdot \| \) can be defined on this space in terms of inner products on \( V \) and \( W \) so that

\[ \| \wedge^t F \| = \sum_{i_1 < \cdots < i_s} \lambda_{i_1} \cdots \lambda_{i_s}. \]

If \( 1 \leq p \leq q \leq s \leq k \) and \( l_i \leq K \), the following fact is easily established.

**Lemma 2.1.**

\[ \left( \frac{\| \wedge^t F \|^2}{\binom{k}{p}} \right)^{1/p} \leq K \left( \frac{\| \wedge^t F \|^2}{\binom{s}{q}} \right)^{1/q}. \]

In the sequel, it is assumed that \( M \) and \( N \) are Riemannian manifolds of dimensions \( m \) and \( n \), respectively. Let \( f: M \to N \) be a \( C^r \) mapping and \( (f_x)_*: T_x(M) \to T_{f(x)}(N) \) be the induced mapping of tangent spaces at \( x \).

**Definition 2.** If either \( (f_x)_* = 0 \) at each point \( x \in M \) or if any one of the dilatations \( l_i(x), i = 1, \ldots, k-1 \) is bounded on \( M \), then \( f \) is said to be of \emph{bounded dilatation}.

For a nonconstant mapping of bounded dilatation, \( l_i(x) \) is always bounded. In this case, \( K \) will denote the l. u. b. of \( l_i(x) \) and \( f \) will be said to be of \emph{bounded dilatation of order} \( K \).

**Remark.** Since \( l_i(x) \leq l_j(x) \) for \( i \leq j \leq k \), a \( K \)-quasiconformal mapping in the sense of [1] and [3] is a mapping of bounded dilatation. If \( m = n = 2 \) the two notions are identical. However, for \( m \) and \( n \) greater than 2 a mapping of bounded dilatation is not necessarily quasiconformal as the following example shows.

Let \( U \) be the open submanifold of \( E^3 \) given by \( \{(x, y, z) \in E^3; \ x^2 + y^2 > 1/(a+1)^2, \ a \neq -1 \} \), and let \( f: U \to E^3 \) be defined by

\[ f = \left( \frac{1}{3} (x^2-y^2), \ 3xy, \ \frac{1}{a+1}z \right). \]
Then, the eigenvalues of $\phi f \phi$ are $\lambda_1 = 9(x^2 + y^2)$, $\lambda_2 = x^2 + y^2$ and $\lambda_3 = 1/(a+1)^2$. Consequently, $l_1(x, y, z) = 3$ and $l_2(x, y, z) = 3(a+1)/(x^2 + y^2)^4$. Observe that $f$ is also harmonic.

In the sequel, a mapping of bounded dilatation will be assumed to have the same rank $k$ at each point of $M$.

**Lemma 2.2.** A $C^\infty$ mapping $f : M \to N$ is of bounded dilatation of order $K$ if and only if

$$\|f_*\|^2 \leq kK\|\wedge^2 f_*\|.$$ 

**Proof.** The necessity follows from Lemma 2.1. For the sufficiency suppose that $l_1 = (\lambda_1/\lambda_2)^k$ is unbounded. Then,

$$\|f_*\|^2 \leq \sum \frac{\lambda_i}{\lambda^2} = \frac{\sum \lambda_i}{\lambda^2} \leq \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_2} + \frac{\lambda_3}{\lambda_2} = l_1(\frac{k}{2})^k = \frac{l_1}{\lambda_2}(\frac{k}{2})^k,$$

so $\|f_*\|^2 / \|\wedge^2 f_*\|$ is unbounded.

### § 3. Harmonic mappings of bounded dilatation

The principal results in [1] may now be extended to mappings of bounded dilatation. Only statements of theorems are given. Details will be presented elsewhere.

**Theorem 3.1.** Let $M$ and $N$ be Riemannian manifolds of dimensions $m$ and $n$, respectively and let $f : M \to N$ be a harmonic mapping with rank $f \geq 2$. If $\|f_*\|^2$ attains a maximum at a point $x \in M$, and if (a) the sectional curvatures of $M$ at $x$ are bounded below by a non-positive constant $-A$, or $M$ is an Einstein manifold with the scalar curvature $R$ at $x$ satisfying $R \leq -m(m-1)A$, and (b) the sectional curvatures of $N$ at $f(x)$ are bounded above by a nonpositive constant $-B$, then

$$B\|f_*\|^2 \leq \frac{m-1}{2} - k^2 l_1(x) A.$$
Corollary 3.1. Let $f: M \to N$ be a mapping of bounded dilatation. If $M$ is locally flat and the sectional curvatures of $N$ are bounded above by a negative constant $-B$ then either $\|f_*\|$ does not attain its maximum or $f$ is a constant.

The following result generalizes Theorem 5.3 in [1].

Corollary 3.2. Let $f: M \to N$ be a harmonic mapping with rank $f \geq 2$. Suppose that the function $\|f_*\|^2$ attains its maximum at $x \in M$. If (a) the sectional curvatures of $M$ at $x$ are bounded below by a non-positive constant $-A$ or if $M$ is an Einstein manifold with scalar curvature $R$ at $x$ satisfying $R \geq -m(m-1)A$, and (b) the sectional curvatures of $N$ at $f(x)$ are bounded above by a negative constant $-B$, then

\[(3.1) \quad \|\wedge f_*\|^{2p} \leq k\left(\frac{k}{p}\right)^{m-1} \frac{A}{B} l_1^p(x), \quad 1 \leq p \leq k.\]

In particular, we get

Corollary 3.3. Under the assumptions of Corollary 3.2, if $B \geq \frac{(m-1)k}{(m+1)A}l_1^p(x)A/2$ and $M$ is connected the mapping $f$ is distance decreasing. If $m = n$ and $B \geq n(n-1)l_1^p(x)A/2$, then $f$ is volume decreasing.

Corollary 3.4. Let $M$ and $N$ be Riemannian manifolds of nonpositive constant curvature and $f: M \to N$ a harmonic mapping with rank $f \geq 2$. Then, if $M$ is locally flat so is $N$ (cf. Theorem 3.6).

If $M$ is the unit open m-ball with the hyperbolic metric of constant curvature $-A$, the requirement that $\|f_*\|$ attain its maximum on $M$ may be omitted and we obtain

Theorem 3.2. Let $B^n$ be the $m$-dimensional unit open ball with the metric $ds^2 = 4\sum dx_i^2 / A(1-r^2)^2$, $A > 0$, and let $N$ be an $n$-dimensional Riemannian manifold with sectional curvatures bounded above by a negative constant $-B$. Then, if $f: B^n \to N$ is a harmonic mapping of bounded dilatation of order $K$, the inequality (3.1) is satisfied, if $l_1(x)$ is replaced by $K$.

Let $E^m$ denote Euclidean $m$-space with the standard flat metric. Then the same method of proof as that of Theorem 3.2 yields
THEOREM 3.3. Let \( N \) be an \( n \)-dimensional Riemannian manifold with negative sectional curvature bounded away from zero, and let \( f : E^m \to N \) be a harmonic mapping of bounded dilatation. Then, \( f \) is a constant mapping.

If \( \pi : S \to M \) is a Riemannian covering we have easily

**Lemma 3.1.** Let \( f : M \to N \) be a \( C^\infty \) mapping and \( \tilde{f} = f \circ \pi \). Then,

\[
\| \bigwedge^k \tilde{f}_* \| = \| \bigwedge^k f_* \|_{(x,\xi)}, \quad x \in S.
\]

If \( M \) is a complete connected Riemannian manifold of constant curvature \( c \), then its simply connected covering is

\[ S^m \text{ if } c > 0, \quad E^m \text{ if } c = 0 \text{ and } B^m \text{ if } c < 0, \]

where \( S^m \) is the \( m \)-sphere of constant curvature \( c(>0) \) and \( B^m \) is the unit open \( m \)-ball with the metric \( ds^2 = -4\sum dx_i^2/c(1-r_i^2) \) of constant curvature \( c(<0) \). Hence, by Proposition 4.1 of [1], Theorems 3.2 and 3.1 above we get

THEOREM 3.4. Let \( M \) be a complete connected Riemannian manifold of positive constant curvature and let \( N \) be a manifold with non-positive sectional curvature. Then, if \( f : M \to N \) is a harmonic mapping, it is a constant mapping.

This fact is well-known (see [1]).

THEOREM 3.5. Let \( M \) be a complete connected Riemannian manifold of constant negative curvature \(-A\) and let \( N \) be a Riemannian manifold with sectional curvatures bounded above by a negative constant \(-B\). Then, if \( f : M \to N \) is a harmonic mapping of bounded dilatation of order \( K \), the inequality (3.1) is satisfied, if \( \ell(x) \) is replaced by \( K \).

Thus, if \( B \geq (m-1)\kappa^2 K/A/2 \), the mapping \( f \) is distance decreasing. In the equidimensional case, if \( B \geq n(n-1)K^4 A/2 \), \( f \) is volume decreasing.

THEOREM 3.6. Let \( M \) be a complete connected locally flat Riemannian manifold and let \( N \) be a Riemannian manifold with neg-
ative sectional curvature bounded away from zero. Then, if \( f: M \to N \) is a harmonic mapping of bounded dilatation, it is a constant mapping.

Theorem 3.6 generalizes Liouville's theorem and the little Picard theorem. For, in the first case, a bounded domain in \( C \) is contained in a disc which has constant negative curvature with respect to the Poincaré metric, and in the latter case, \( C \cdot \{2 \text{ points}\} \) carries an hermitian metric of constant negative curvature.

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Bibliography

