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Mappings of Bounded Dilatation

By

S. I. Goldberg[1], T. Ishihara and N. C. Petridis

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§ 1. Introduction

Let M and N be Riemannian manifolds of dimensions m and n, respectively. Recently, two of the authors introduced the concept of a quasiconformal mapping $f: M \to N$ and applied it to obtain distance and (intermediate) volume decreasing properties of harmonic mappings between Riemannian manifolds of different dimensions [1]. In this paper the concept of a mapping $f: M \to N$ of bounded dilatation is introduced which is more general and natural than that of a K-quasiconformal mapping when m and n are greater than 2. An example of such a mapping which is not K-quasiconformal is given which is even harmonic. The main results in [1], viz., generalizations of the Schwarz-Ahlfors lemma as well as Liouville's theorem and the little Picard theorem are valid for this class of mappings. This is due to the fact that $\| f_* \|^2 / \| \wedge^2 f_* \|$ is bounded if f is K-quasiconformal or if f is of bounded dilatation.

Let $f: M \to N$ be a harmonic mapping of bounded dilatation of Riemannian manifolds. If the ratio function $\|f_*\|^2$ of distances attains its maximum at $x \in M$, then under suitable conditions on the bounds of the sectional curvatures at x and $f(x)$, f is distance decreasing.

If M is a complete connected Riemannian manifold of constant negative curvature $-A$, in particular, if M is the unit open m-ball with the hyperbolic metric of constant curvature $-A$, the condition on $\| f_* \|$ may be dropped. Indeed, let N be a Riemannian manifold with sectional curvatures bounded above by a negative con-

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stant depending on A. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is distance decreasing.

The technique employed to prove this statement also yields the following fact.

Let M be a complete connected locally flat Riemannian manifold and let N be an n-dimensional Riemannian manifold with negative sectional curvature bounded away from zero. Then, if f: $M \to N$ is a harmonic mapping of bounded dilatation, it is a constant mapping.

§ 2. Mappings of bounded dilatation

Let V be an Euclidean vector space of dimension m and let V^* be its dual space. Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of V with dual basis $\{\omega_1, \cdots, \omega_m\}$. A quadratic function on V is an element of $(V \otimes V)^*$, so since $(V \otimes V)^*$ is canonically isomorphic to $V^* \otimes V^*$, a quadratic functionon V may be written as $f = \sum f_{ij}\, \omega_i \otimes \omega_j$. If f is positive semidefinite an orthonormal basis $\{e_i\}$ can be chosen so that $f_{ij} = 0$ for $i \neq j$ and $f_{ii} = \gamma_i^2 > 0$ for $i = 1, \cdots, k \leq m$, where $k = rank\, f$.

Let W be an Euclidean vector space of dimension n with inner product h, and let $F: V \to W$ be a linear mapping of $rank\ k \leq min(m, n)$. We choose an orthonormal basis $\{e_i\}$ of V so that

$$F^* h = \sum \gamma_i^2\, \omega_i \otimes \omega_i .$$

The vectors $\eta_i = (1/\gamma_i) F e_i$, $i = 1, \cdots, k$, therefore form part of an orthonormal basis of W. (If all of the γ_i vanish, $F = 0$.) Let $X = \sum_1^m x^i e_i$ be a vector of unit length and assume $F \neq 0$, then $FX = \sum y^i \eta_i$, where $x^i = y^i / \gamma_i$. Consequently, if F is of rank k, it maps a unit $(k-1)$-dimensional sphere of V to a $(k-1)$-dimensional ellipsoid of W with semiaxes of lengths $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k > 0$, where $\gamma_i^2 = \lambda_i$, $i = 1, \cdots, k$ are the eigenvalues of ${}^t FF: V \to V$.

DEFINITION 1. The ratio

$$l_s = \frac{\gamma_1}{\gamma_{s+1}}, \quad s = 1, \cdots, k-1$$

will be called the s-th *dilatation* of F.

The mapping $F: V \to W$ induces a mapping $\wedge^p F: \wedge^p V \to \wedge^p W$, $p \leq min(m, n)$ given by

$$\wedge^p F(e_{i_1} \wedge \cdots \wedge e_{i_p}) = Fe_{i_1} \wedge \cdots \wedge Fe_{i_p},$$

where $1 \le i_1 < i_2 < \cdots < i_p \le m$. $\wedge^p F$ may be regarded as an element of $\wedge^p V^* \otimes \wedge^p W$. A norm $\| \quad \|$ can be defined on this space in terms of inner products on V and W so that

$$\| \wedge^p F \|^2 = \sum_{i_1 < \cdots < i_p} \lambda_{i_1} \cdots \lambda_{i_p}.$$

If $1 \le p \le q \le s < k$ and $l_s \le K$, the following fact is easily established.

LEMMA 2. 1.

$$\left[\frac{\| \wedge^p F \|^2}{\binom{k}{p}} \right]^{1/p} \le K^2 \left[\frac{\| \wedge^q F \|^2}{\binom{s}{q}} \right]^{1/q} .$$

In the sequel, it is assumed that M and N are Riemannian manifolds of dimensions m and n, respectively. Let $f: M \to N$ be a C^∞ mapping and $(f_*)_x : T_x(M) \to T_{f(x)}(N)$ be the induced mapping of tangent spaces at x.

DEFINITION 2. If either $(f_*)_x = 0$ at each point $x \in M$ or if any one of the dilatations $l_i(x)$, $i = 1, \cdots, k-1$ is bounded on M, then f is said to be of *bounded dilatation.*

For a nonconstant mapping of bounded dilatation, $l_1(x)$ is always bounded. In this case, K will denote the l. u. b. of $l_1(x)$ and f will be said to be of *bounded dilatation of order K.*

REMARK. Since $l_i(x) \le l_j(x)$ for $i \le j \le k$, a K-quasiconformal mapping in the sense of [1] and [3] is a mapping of bounded dilatation. If $m = n = 2$ the two notions are identical. However, for m and n greater than 2 a mapping of bounded dilatation is not necessarily quasiconformal as the following example shows.

Let U be the open submanifold of E^3 given by $\{(x, y, z) \in E^3;\ x^2 + y^2 > 1/(a+1)^2,\ a \ne -1\}$, and let $f: U \to E^3$ be defined by

$$f = (\tfrac{1}{2}(x^2 - y^2),\ 3xy,\ \frac{1}{a+1} z).$$

Then, the eigenvalues of ${}^t f_* f_*$ are $\lambda_1 = 9(x^2+y^2)$, $\lambda_2 = x^2+y^2$ and $\lambda_3 = 1/(a+1)^2$. Consequently, $l_1(x, y, z) = 3$ and $l_2(x, y, z) = 3(a+1)/(x^2+y)^{\frac{1}{2}}$. Observe that f is also harmonic.

In the sequel, a mapping of bounded dilatation will be assumed to have the same rank k at each point of M.

LEMMA 2. 2. *A C^∞ mapping $f: M \to N$ is of bounded dilatation of order K if and only if*

$$\| f_* \|^2 \leq kK \, \| \wedge^2 f_* \| .$$

PROOF. The necessity follows from Lemma 2. 1. For the sufficiency suppose that $l_1 = (\lambda_1/\lambda_2)^{\frac{1}{2}}$ is unbounded. Then,

$$\begin{aligned}\frac{\| f \|^2}{\| \wedge^2 f_* \|} &= \frac{\sum \lambda_i}{(\sum_{i<j} \lambda_i \lambda_j)^{\frac{1}{2}}} \\ &= \left(\frac{\lambda_1}{\lambda} + 1 + \frac{\lambda_3}{\lambda_2} + \cdots + \frac{\lambda_k}{\lambda_2}\right) / \left(\frac{\lambda_1}{\lambda_2} + terms \leq \frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}} \\ &\geq \frac{\lambda_1}{\lambda_2} / \left(\frac{k}{2}\right)^{\frac{1}{2}} \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}} = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}} / \left(\frac{k}{2}\right)^{\frac{1}{2}} = l_1 / \left(\frac{k}{2}\right)^{\frac{1}{2}},\end{aligned}$$

so $\| f_* \|^2 / \| \wedge^2 f_* \|$ is unbounded.

§ 3. Harmonic mappings of bounded dilatation

The principal results in [1] may now be extended to mappings of bounded dilatation. Only statements of theorems are given. Details will be presented elsewhere.

THEOREM 3. 1. *Let M and N be Riemannian manifolds of dimensions m and n, respectively and let $f: M \to N$ be a harmonic mapping withrank* f ≥ 2. *If $\| f_* \|^2$ attains a maximum at a point $x \in M$, and if (a) the sectional curvatures of M at x are bounded below by a nonpositive constant $-A$ or M is an Einstein manifold with the scalar curvature R at x satisfying $R \geq -m(m-1)A$, and (b) the sectional curvatures of N at f(x) are bounded above by a nonpositive constant $-B$, then*

$$B \| f_* \|^2 \leq \frac{m-1}{2} k^2 l_1^4(x) A.$$

COROLLARY 3. 1. *Let f: $M\rightarrow N$ be a mapping of bounded dilatation. If M is locally flat and the sectional curvatures of N are bounded above by a negative constant $-B$ then either $\|f_*\|$ does not attain its maximum or f is a constant.*

The following result generalizes Theorem 5. 3 in [1].

COROLLARY 3. 2. *Let f: $M\rightarrow N$ be a harmonic mapping with rank $f\geq 2$. Suppose that the function $\|f_*\|^2$ attains its maximum at $x\in M$. If (a) the sectional curvatures of M at x are bounded below by a nonpositive constant $-A$ or if M is an Einstein manifold with scalar curvature R at x satisfying $R\geq -m(m-1)A$, and (b) the sectional curvatures of N at $f(x)$ are bounded above by a negative constant $-B$, then*

$$\|\wedge^p f_*\|^{2/p}\leq k\binom{k}{p}^{1/p}\frac{m-1}{2}\frac{A}{B}l_1^4(x),\quad 1\leq p\leq k. \tag{3.1}$$

In particular, we get

COROLLARY 3. 3. *Under the assumptions of Corollary 3. 2, if $B\geq (m-1)k^2l_1^4(x)A/2$ and M is connected the mapping f is distance decreasing. If $m=n$ and $B\geq n(n-1)l_1^4(x)A/2$, then f is volume decreasing.*

COROLLARY 3. 4. *Let M and N be Riemannian manifolds of nonpositive constant curvature and f: $M\rightarrow N$ a harmonic mapping with rank $f\geq 2$. Then, if M is locally flat so is N* (cf. Theorem 3. 6).

If M is the unit open m-ball with the hyperbolic metric of constant curvature $-A$, the requirement that $\|f_*\|$ attain its maximum on M may be omitted and we obtain

THEOREM 3. 2. *Let B^m be the m-dimensional unit open ball with the metric $ds^2=4\sum dx_i^2/A(1-r^2)^2$, $A>0$, and let N be an n-dimensional Riemannian manifold with sectional curvatures bounded above by a negative constant $-B$. Then, if f: $B^m\rightarrow N$ is a harmonic mapping of bounded dilatation of order K, the inequality* (3. 1) *is satisfied, if $l_1(x)$ is replaced by K.*

Let E^m denote Euclidean m-space with the standard flat metric. Then the same method of proof as that of Theorem 3. 2 yields

THEOREM 3.3. *Let N be an n-dimensional Riemannian manifold with negative sectional curvature bounded away from zero, and let $f: E^m \to N$ be a harmonic mapping of bounded dilatation. Then, f is a constant mapping.*

If $\pi: S \to M$ is a Riemannian covering we have easily

LEMMA 3.1. *Let $f: M \to N$ be a C^∞ mapping and $\tilde{f} = f \circ \pi$. Then,*

$$\| \wedge^p \tilde{f}_* \|_x = \| \wedge^p f_* \|_{\pi(x)}, \quad x \in S.$$

If M is a complete connected Riemannian manifold of constant curvature c, then its simply connected covering is

$$S^m \text{ if } c>0, \quad E^m \text{ if } c=0 \text{ and } B^m \text{ if } c<0,$$

where S^m is the m-sphere of constant curvature $c(>0)$ and B^m is the unit open m-ball with the metric $ds^2 = -4\sum dx_i^2 / c(1-r^2)^2$ of constant curvature $c(<0)$. Hence, by Proposition 4.1 of [1], Theorems 3.2 and 3.1, and Lemma 3.1 above we get

THEOREM 3.4. *Let M be a complete connected Riemannian manifold of positive constant curvature and let N be a manifold with non-positive sectional curvature. Then, if $f: M \to N$ is a harmonic mapping, it is a constant mapping.*

This fact is well-known (see [1]).

THEOREM 3.5. *Let M be a complete connected Riemannian manifold of constant negative curvature $-A$ and let N be a Riemannian manifold with sectional curvatures bounded above by a negative constant $-B$. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation of order K, the inequality (3.1) is satisfied, if $l_1(x)$ is replaced by K.*

Thus, if $B \geq (m-1)k^2K^4A/2$, the mapping f is distance decreasing. In the equidimensional case, if $B \geq n(n-1)K^4A/2$, f is volume decreasing.

THEOREM 3.6. *Let M be a complete connected locally flat Riemannian manifold and let N be a Riemannian manifold with neg-*

ative sectional curvature bounded away from zero. Then, if $f: M \to N$ is a harmonic mapping of bounded dilatation, it is a constant mapping.

Theorem 3. 6 generalizes Liouville's theorem and the little Picard theorem. For, in the first case, a bounded domain in $\boldsymbol{C}$ is contained in a disc which has constant negative curvature with respect to the Poincaré metric, and in the latter case, $\boldsymbol{C}$ - {2 points} carries an hermitian metric of constant negative curvature.

University of Illinois, Urbana
Tokushima University, Josanjima
Eastern Illinois University, Charleston

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