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On the Self-Equivalences of H-spaces

By

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§1. Introduction

The set $\mathscr{E}_H(X)=\mathscr{E}_H(X,m)$ of all (based) homotopy classes of homotopy equivalent H-maps of an H-space $X=(X,m)$ to itself forms a subgroups of $\mathscr{E}(X)$, which consists of all homotopy classes of homotopy equivalences of X to itself.

The purpose of this note is to study the group $\mathscr{E}_H(X)$ for some H-space X. One of our main results is stated as follows:

THEOREM 3.8. *Let $X=X_1\times X_2\times\cdots\cdots\times X_n$ be the product H-complex of homotopy associative H-complexes X_i. Then the group $\mathscr{E}_H(X)$ is isomorphic to the following group of matrices:*

$$\{(a_{ij})\mid a_{ij}\in[X_j,X_i]_H,\ (a_{ij})\ \text{is invertible},$$
$$a_{ki}p_i+a_{kj}p_j=a_{kj}p_j+a_{ki}p_i\ \text{in}\ [X,X_k]\},$$

where $[X_j,X_i]_H$ is the subset of the homotopy set $[X_j,X_i]$ consisting of H-maps and $p_i\colon X\longrightarrow X_i$ is the projection.

This theorem is proved in §3. Also we study in Theorem 3.11 the condition that $\mathscr{E}_H(X_1\times X_2)$ is equal to the product group $\mathscr{E}_H(X_1)\times\mathscr{E}_H(X_2)$.

In §§4−5, we consider the H-complexes S^n, $S^1\times S^n$ ($n=3$ or 7) and $S^3\times S^7$ and obtain

THEOREMS 4.1, 4.3 *and* 5.2. *Let $n=3$ or 7. Then*

$$\mathscr{E}_H(S^n)=\mathscr{E}_H(S^3\times S^7)=1,\quad \mathscr{E}_H(S^1\times S^n)=Z_2,$$

for any multiplication on each H-complex.

Furthermore, we study the sets $\widetilde{M}(X)$ of H-equivalence classes of multiplications on these H-complexes X (Theorems 4.1, 4.4 and 5.3).

Finally in §6, we compute the group $\mathscr{E}_H(X,m)$ of a product X of two

Eilenberg-MacLane complexes for any multiplication m on X (Theorem 6.3).

As an example, we see that there are two homotopy associative multiplications m and n on $X=K(Z_3, 6)\times K(Z, 2)$ such that $\mathscr{E}_H(X, m)$ and $\mathscr{E}_H(X, n)$ are finite groups with different orders (Example 6.5). This gives a counter example to the assertion of M. Arkowitz and C. R. Curjel [1, p. 147].

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§2. The group $\mathscr{E}_H(X, m)$ and the set $\widetilde{M}(X)$

In this note, spaces, (continuous)maps and homotopies are always assumed to be based.

A topological space X is called an H-space, if there is a map $m: X\times X \longrightarrow X$, called a multiplication, such that $m \mid X\times * = 1 = m \mid * \times X$ in the homotopy set $[X, X]$. Let $M(X)$ be the subset of the homotopy set $[X\times X, X]$ consisting of homotopy classes of multiplications on an H-space X, and Let $\mathscr{E}(X)$ be the group of homotopy classes of self(homotopy)equivalences $h: X\longrightarrow X$. Then, we can define the action

$$(2.1) \qquad \chi = \chi_X : M(X)\times \mathscr{E}(X) \longrightarrow M(X)$$

by $\chi_X(m, h) = hm(h^{-1}\times h^{-1})$ for $(m, h) \in M(X)\times \mathscr{E}(X)$, where h^{-1} is a homotopy inverse of h. This action is considered by C. R. Curjel [2]. Under this action we can consider the group $\mathscr{E}(X)$ as a transformation group of $M(X)$, that is

$$(2.2) \qquad \chi(m, 1) = m, \ \chi(m, fg) = \chi(\chi(m, g), f), \quad \text{for } m \in M(X), \ f, g \in \mathscr{E}(X).$$

By the definition of χ we have immediately the following

PROPOSITION 2.3. *If $m \in M(X)$ is homotopy associative (or homotopy commutative), then so is $\chi(m, f)$ for any $f \in \mathscr{E}(X)$.*

Let $J_1: M(X)\times M(Y) \longrightarrow M(X\times Y)$ and $J_2: \mathscr{E}(X)\times \mathscr{E}(Y) \longrightarrow \mathscr{E}(X\times Y)$ be the natural inclusions. Then we have

PROPOSITION 2.4. $\chi_{X\times Y}(J_1\times J_2) = J_1(\chi_X\times \chi_Y)(1\times T\times 1)$,

where $T: M(X)\times \mathscr{E}(Y) \longrightarrow \mathscr{E}(Y)\times M(X)$ is the twisted map.

For H-spaces $X=(X, m)$ and $Y=(Y, n)$, a map $f: X\longrightarrow Y$ is called an H-map if $fm = n(f\times f)$ in $[X\times X, Y]$, and the set of homotopy classes of H-maps $f: X\longrightarrow Y$ is denote by $[(X, m), (Y, n)]_H$ or simply $[X, Y]_H$.

For an H-space $X=(X, m)$, let $\mathscr{E}_H(X, m)$ be the subgroup of $\mathscr{E}(X)$ consisting of all homotopy equivalent H-maps $f:(X, m)\longrightarrow(X, m)$. Then we have

PROPOSITION 2.5. *$\mathscr{E}_H(X, m)$ is the isotropy subgroup of m under the action $\chi: M(X)\times\mathscr{E}(X)\longrightarrow M(X)$ of* (2.1).

An H-space X is called an H-complex if X is a CW-complex. By [8, Th. 2.3] we have

LEMMA 2.6. *Let $X=(X, m_0)$ be an H-complex, then any multiplication $m\in M(X)$ can be written uniquely as*

$$m=\alpha\pi+m_0\in[X\times X, X] \quad \text{for } \alpha\in[X\wedge X, X],$$

where $\pi: X\times X\longrightarrow X\wedge X$ is the collapsing map and $+$ is the usual sum induced by m_0.

By Proposition 2.5 and the above lemma, we have

PROPOSITION 2.7. *If X is an H-complex such that $[X\wedge X, X]=0$, then we have $\mathscr{E}_H(X, m)=\mathscr{E}(X)$.*

Let m and n be multiplications on an H-space X. Then m is called H-equivalent to n if there exists a homotopy equivalent H-map $f:(X, m)\longrightarrow(X, n)$. Let $\widetilde{M}(X)$ be the quotient set of $M(X)$ by this equivalence relation. Let $M(X)/\mathscr{E}(X)$ be the set of orbits under the action χ of (2.1). Then we have

$$\widetilde{M}(X)=M(X)/\mathscr{E}(X). \tag{2.8}$$

PROPOSITION 2.9. *If $\#\{\mathscr{E}(X)/\mathscr{E}_H(X, \mathrm{m})\}$ is equal to $N(<\infty)$ for any $m\in M(X)$, then we have $\#\widetilde{M}(X)=\#M(X)/N$, where $\#A$ is the number of the elements of a set A.*

PROOF. The number of the elements contained in the orbit of m (under the action χ) is equal to $\#\{\mathscr{E}(X)/\mathscr{E}_H(X, m)\}=N$. Therefore, by (2.8) we have the desired result. q. e. d.

§3. The group $\mathscr{E}_H(X_1\times\cdots\cdots\times X_n)$

Let $X_i=(X_i, m_i)$ $(i=1, \cdots\cdots, n)$ be homotopy associative H-complexes, and consider the product H-complex

$$X=X_1\times\cdots\cdots\times X_n,$$

whose multiplication $m=m_1\times\cdots\cdots\times m_n$ is the one induced by m_i. In this section, we consider the group $\mathscr{E}_H(X)=\mathscr{E}_H(X, m)$.

Denote by

$$M(n, \Lambda_{ij}), \quad \Lambda_{ij}=[X_j, X],$$

the set of all (n, n) matrices (a_{ij}), where $a_{ij} \in \Lambda_{ij}$. The multiplication of matrices is defined, as usually, by

$$(a_{ij})(b_{ij})=(a_{i1}b_{1j}+\cdots\cdots+a_{in}b_{nj}) \tag{3.1}$$

where $+$ is the sum in the homotopy set $[X_j, X_i]$ induced by m_i.

Now, consider the maps

$$[X, X] \underset{\psi}{\overset{\phi}{\rightleftarrows}} M(n, \Lambda_{ij}), \tag{3.2}$$

which are defined as follows:

$\phi(f)=(f_{ij}), \ f_{ij}=p_i f i_j, \quad$ for $f \in [X, X]$,

$\psi(a_{ij})=(a_{11}p_1+\cdots\cdots+a_{1n}p_n, \cdots\cdots, a_{n1}p_1+\cdots\cdots+a_{nn}p_n), \quad$ for $(a_{ij}) \in M(n, \Lambda_{ij})$

where $i_j: X_j \longrightarrow X$ is the inclusion and $p_j: X \longrightarrow X_j$ is the projection.

Then, we have the following lemmas.

LEMMA 3.3. (i) $\qquad \phi\psi = id.$

(ii) $\qquad \psi\phi = id \quad on \ [X, X]_H.$

PROOF. We can prove the lemma by the induction on n, and so we prove the lemma for $n=2$

It is clear that the cofibering

$$X_1 \vee X_2 \xrightarrow{j} X \xrightarrow{\pi} X_1 \wedge X_2 \ (X=X_1 \times X_2)$$

induces the exact sequence

$$0 \longrightarrow [X_1 \wedge X_2, X] \xrightarrow{\pi^*} [X, X] \underset{\psi}{\overset{j^*}{\rightleftarrows}} [X_1 \vee X_2, X] \longrightarrow 0,$$

where $[X_1 \vee X_2, X]=M(2, \Lambda_{ij})$, $j^*=\phi$ and $j^*\psi=$id. Futhermore, any element $f \in [X, X]$ can be written as

$$f=\psi\phi(f)+\alpha\pi \quad \text{for some } \alpha \in [X_1 \wedge X_2, X].$$

If $f \in [X, X]_H$, then $f=m(f \times f)(i_1 \times i_2)$ by definition, and so we have

$$f=m(fi_1 \times fi_2)=\psi\phi(f),$$

which shows (ii). q. e. d.

LEMMA 3.4. *ϕ of (3.2) is homomorphic on $[X, X]_H$.*

PROOF. It is clear that $p_i fg i_j=p_i f i_1 p_1 g i_j+\cdots\cdots+p_i f i_n p_n g i_j$ if $f, g \in [X, X]_H$, which shows the lemma. q. e. d.

Consider the subset $GL(n, \Lambda_{ij})$ of $M(n, \Lambda_{ij})$ consisting of all matrices (a_{ij}) which have right and left inverses, and the subset

$$(3.5) \qquad HL(n, \Lambda_{ij}) = \{(a_{ij}) \in GL(n, \Lambda_{ij}) \mid a_{ij} \in [X_j, X_i]_H,\ a_{ki}p_i + a_{kj}p_j = a_{kj}p_j + a_{ki}p_i \text{ in } [X, X_k]\}.$$

LEMMA 3.6. $\phi(\mathscr{E}_H(X)) \subset HL(n, \Lambda_{ij})$.

PROOF. It is clear $\phi(\mathscr{E}_H(X)) \subset GL(n, \Lambda_{ij})$ by Lemma 3.4.
Assume $f \in [X, X]_H$. Then it is clear $f_{ij} = p_j f i_j \in [X_j, X_i]_H$. Furthermore,

$$f_{jk}p_k + f_{jl}p_l = p_j f(i_k p_k + i_l p_l) = p_j f(i_l p_l + i_k p_k) = f_{jl}p_l + f_{jk}p_k.$$

Therefore we have the lemma. q. e. d.

LEMMA 3.7. *$\psi(HL(n, \Lambda_{ij})) \subset [X, X]_H$ and ψ is homomorphic on $HL(n, \Lambda_{ij})$.*

PROOF. For the simplicity, we prove the lemma for the case $n=2$. Let $(a_{ij}) \in HL(n, \Lambda_{ij})$, and let $q_i: X \times X \longrightarrow X$ be the projection onto the i-th factor. Then we see easily

$$(a_{11}p_1q_1 + a_{12}p_2q_1,\ a_{21}p_1q_1 + a_{22}p_2q_1) + (a_{11}p_1q_2 + a_{12}p_2q_2,\ a_{21}p_1q_2 + a_{22}p_2q_2)$$
$$= (a_{11}p_1(q_1+q_2) + a_{12}p_2(q_1+q_2),\ a_{21}p_1(q_1+q_2) + a_{22}p_2(q_1+q_2))$$

by the condition that $(a_{ij}) \in HL(2, \Lambda_{ij})$. The above equality shows immediately $m(\psi(a_{ij}) \times \psi(a_{ij})) = \psi(a_{ij})m$. The equality $\psi((a_{ij})\psi(b_{ij}) = \psi((a_{ij})(b_{ij}))$ can be proved by the similar calculations.

By the above lemmas, we have the following q. e. d.

THEOREM 3.8. *The group $\mathscr{E}_H(X)$ of the product H-complex $X = X_1 \times \cdots\cdots \times X_n$ of the homotopy associative H-complexes X_i is isomorphic to the group $HL(n, \Lambda_{ij})$ of (3.5) by the homomorphisms in (3.2).*

REMARK 3.9. Similarly we can consider the above discussions for the case that X_i is not homotopy associative, by defining $a_1 + a_2 + \cdots\cdots + a_n = (((a_1 + a_2) + \cdots\cdots) + a_n)$ in $[Y, X_i]$, and Lemmas 3.3, 3.4 and 3.6 also hold for this case.

By applying this theorem to $X = S^3 \times S^3$ or $S^7 \times S^7$ we have

EXAMPLE 3.10. *For the H-complex S^l ($l=3$ or 7) with the usual multiplication,*

$$\mathscr{E}_H(S^l \times S^l) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, Z),\ a \equiv d \equiv \frac{1+\varepsilon}{2},\ b \equiv c \equiv \frac{1-\varepsilon}{2} (k_l) \}$$

where $\varepsilon = ad - bc$ $(=\pm 1)$ and $k_3 = 24$, $k_7 = 240$.

RROOF. Let $n: S^3 \longrightarrow S^3$ be the map of degree n, then by M. Arkowitz and C. R. Curjel[1, Prop. D] we have

$$n \in [S^3, S^3]_H \text{ if and only if } n(n-1) \equiv 0 \ (24).$$

Also, we have easily

$$n+m=m+n \text{ in } [S^3 \times S^3, S^3] \text{ if and only if } nm \equiv 0 \ (12).$$

By these facts, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in HL(2, Z)$ if and only if

$$ad-bc=\varepsilon, \quad ad \equiv cd \equiv 0 \ (12) \quad \text{and } n(n-1) \equiv 0 \ (24) \quad \text{for } n=a, b, c, d.$$

By the easy calculation we see that this condition is equivalent to

$$ad-bc=\varepsilon, \quad a \equiv d \equiv \frac{1+\varepsilon}{2} \ (24) \quad \text{and } b \equiv c \equiv \frac{1-\varepsilon}{2} \ (24).$$

Therefore we have the result for S^3.

By the same way, we see that $HL(2, \Lambda_{ij})$ is given by the left hand side of the example for $n=2$ and $X_1=X_2=S^7$. Since S^7 is not homotopy associative, we have to prove Theorem 3.8 for $n=2$ and $X_1=X_2=S^7$ more carefully. The obstructions for $f+(g+(h+k))=f+((g+h)+k)=(f+(g+h))+k=((f+g)+h)+k=(f+g)+(h+k)$ in $[S^7 \times S^7 \times S^7 \times S^7, S^7]$ $(f, g, h, k \in \pi_7(S^7))$ are contained in $\pi_{21}(S^7)$ or $\pi_{28}(S^7)$. Since $\pi_{21}(S^7)=Z_{24}+Z_4$ and $\pi_{28}(S^7)=Z_6+Z_2$ by [13, p, 187 and 5, p. 325], the above obstructions vanish if the degrees of two of f, g, h, k vanish mod 24. Therefore we see by the same proof that Lemma 3.7 holds for our case. q. e. d.

Now, we consider the condition that $\mathscr{E}_H(X_1 \times X_2)= \mathscr{E}_H(X_1) \times \mathscr{E}_H(X_2)$.

An H-map $f: X_1 \longrightarrow X_2$ is called central if $fp_1+p_2=p_2+fp_1$ in $[X_1 \times X_2, X_2]$. We denote by $[X_1, X_2]_{CH}$ the subset of $[X_1, X_2]_H$ consisting of central H-maps of X_1 to X_2, which is an abelian subgroup of $[X_1, X_2]$ if X_2 is homotopy associative (cf. [1]).

THEOREM 3.11. *Let $X_i=(X_i, m_i)$ be an H-complex for $i=1, 2$. If $[X_1, X_2]_H=[X_2, X_1]_{CH}=0$ or $[X_1, X_2]_{CH}=[X_2, X_1]_H=0$, then we have*

$$\mathscr{E}_H(X_1 \times X_2)= \mathscr{E}_H(X_1) \times \mathscr{E}_H(X_2).$$

Conversely, if X_1 and X_2 are homotopy associative and the above equality holds, then we have $[X_1, X_2]_{CH}=[X_2, X_1]_{CH}=0$.

PROOF. Assume $[X_1, X_2]_H=0$, and take $h \in \mathscr{E}_H(X_1 \times X_2)$. Then $\phi(h)=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in HL(2, \Lambda_{ij})$ by Remark 3.9 and Lemma 3.6. Therefore $\phi(h)$ has the left inverse, and so there is $a' \in [X_1, X_1]$ such that $a'a=1$ in $[X_1, X_1]$. Since

$a\in[X_1, X_1]_H$, we have also $a'\in[X_1, X_1]_H$. Hence the condition $ap_1+bp_2=bp_2+ap_1$ implies $p_1+a'bp_2=a'bp_2+p_1$, which shows $a'b\in[X_2, X_1]_{CH}$. Therefore $b=0$ if $[X_2, X_1]_{CH}=0$, and we have the first half of the theorem.

Conversely, if $f\in[X_2, X_1]_{CH}$ is not zero, then it is easy to see that

$$\psi\begin{pmatrix}1 & f\\ 0 & 1\end{pmatrix}\in\mathscr{E}_H(X_1\times X_2), \text{ and } \notin\mathscr{E}_H(X_1)\times\mathscr{E}_H(X_2). \qquad \text{q. e. d.}$$

EXAMPLE 3.12. *$\mathscr{E}_H(X\times Y)=\mathscr{E}_H(X)\times\mathscr{E}_H(Y)$ holds for the following cases:*

(i) *X is a simply connected H-complex and Y is the product of q-copies of the group S^1.*

(ii) *X is the product of p-copies of the group S^3 and Y is the product of q-copies of the H-space S^7.*

PROOF. (i) is obtained immediately by Theorem 3.11. The case (ii) is proved immediately by Theorem 3.11 and the following lemma.

LEMMA 3.13. (F. Sigrist[12, Th.]) *Let ξ be the nonzero element of $\pi_7(S^3)$, then $\xi:(S^7, m)\longrightarrow(S^3, n)$ is not an H-map for any multiplications m and n.*

§4. The groups $\mathscr{E}_H(S^n)$ and $\mathscr{E}_H(S^1\times S^n)$ for $n=3,7$

It is clear that $\mathscr{E}_H(S^1)=Z_2$ and $\#\widetilde{M}(S^1)=1$.

For the H-space $S^n(n=3$ or $7)$ we have the following theorem which is well-known.

THEOREM 4.1. *For any multiplication m on S^3 or S^7, we have*

$$\mathscr{E}_H(S^3, m)=1, \quad \mathscr{E}_H(S^7, m)=1,$$

and hence

$$\#\widetilde{M}(S^3)=6, \quad \#\widetilde{M}(S^7)=60.$$

PROOF. Let $\iota_n\in\pi_n(S^n)$ ($n=3$ *or* 7) be the identity class, then $\mathscr{E}(S^n)=Z_2$ is generated by $-\iota_n$. For any multiplication m we have $\chi(m, -\iota_n)=mT$ by the definition (2.1), and $mT\neq m$ since S^n has not a homotopy commutative multiplication [4, p. 176]. Therefore, we have $\mathscr{E}_H(S^n, m)=1$ by Proposition 2.5. Also, since $\# M(S^n)=\#\pi_{2n}(S^n)$ by Lemma 2.6, we have the desired results by Proposition 2.9 and [13, p. 186]. q. e. d.

From now on, we determine the group $\mathscr{E}_H(S^1\times S^n, \mathrm{m})$ ($n=3$ or 7) for any multiplication m.

Let $\eta_n\in\pi_{n+1}(S^n)$ be the generator, $\pi_1: S^1\times S^n\longrightarrow S^{n+1}$ be the collapsing map, and $p_1: S^1\times S^n\longrightarrow S^1$ and $p_2: S^1\times S^n\longrightarrow S^n$ be the projections. Put

$$\sigma_n=(-\iota_1)\times\iota_n, \quad \tau_n=\iota_1\times(-\iota_n) \text{ and } \nu_n=(p_1, \eta_n\pi_1+p_2),$$

where + denotes the sum induced by the usual multiplication on S^n. Then, we have by [11, Example 4]

(4.2) $\qquad \mathscr{E}(S^1\times S^n)=Z_2+Z_2+Z_2$, generated by σ_n, τ_n, ν_n ($n=3$ or 7).

Now, our main results of this section are summarized as follows.

THEORFM 4.3. *Let $n=3$ or 7. For any multiplicaton m on $S^1\times S^n$, we have*

$$\mathscr{E}_H(S^1\times S^n, m)=Z_2, \text{ generated by } \sigma_n.$$

THEOREM 4.4. $\#\widetilde{M}(S^1\times S^3)=2^7\cdot 3$, $\#\widetilde{M}(S^1\times S^7)=2^{15}\cdot 3\cdot 5$.

we prove the above theorems by showing the following lemmas.

LEMMA 4.5. *Let χ be the action of (2.1), then*

$$\chi(m, \sigma_n)=m \quad \text{for any } m\in M(S^1\times S^n).$$

PROOF. Let m_0 and m_0' be the usual multiplications on S^1 and S^n respectively. Then, by Lemma 2.6, any multiplication $m\in M(S^1\times S^n)$ can be written as

(4.6) $\qquad m=(m_0, \alpha\pi+m_0')$ for some $\alpha\in[(S^1\times S^n)\wedge(S^1\times S^n), S^n]$.

This and the definitions of χ and σ_n show that

$$\chi(m, \sigma_n)=((-\iota_1)\times\iota_n)(m_0, \alpha\pi+m_0')((-\iota_1)\times\iota_n\times(-\iota_1)\times\iota_n)$$
$$=(m_0, \alpha\pi((-\iota_1)\times\iota_n\times(-\iota_1)\times\iota_n)+m_0'=(m_0, \alpha(\sigma_n\wedge\sigma_n)\pi'+m_0').$$

Therefore, it is sufficient to show that

(4.7) $\qquad \alpha(\sigma_n\wedge\sigma_n)=\alpha \quad$ for $\alpha\in[(S^1\times S^n)\wedge(S^1\times S^n), S^n]$.

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [S^1\wedge S^n\wedge X, S^n] & \xrightarrow{\pi_2^*} & [X\wedge X, S^n] & \xrightarrow{j^*} & [(S^1\vee S^n)\wedge X, S^n] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \{((-\iota_1)\wedge\iota_n)\wedge\sigma_n\}^*} & & \downarrow{\scriptstyle \{\sigma_n\wedge\sigma_n\}^*} & & \downarrow{\scriptstyle \{((-\iota_1)\vee\iota_n)\wedge\sigma_n\}^*} & & \\
0 & \longrightarrow & [S^1\wedge S^n\wedge X, S^n] & \xrightarrow{\pi_2^*} & [X\wedge X, S^n] & \xrightarrow{j^*} & [(S^1\vee S^n)\wedge X, S^n] & \longrightarrow & 0
\end{array}$$

where $X=S^1\times S^n$, and $(S^1\vee S^n)\wedge X\xrightarrow{j}X\wedge X\xrightarrow{\pi_2}(S^1\wedge S^n)\wedge X$ is the cofibering. In this diagram, the horizontal sequences are exact [8, Cor. 2.2]. Also, we can define a splitting map $\theta: [(S^1\vee S^n)\wedge X, S^n]\longrightarrow[(S^1\times S^n)\wedge X, S^n]$ by the similar way as in (3.2), and this satisfies

(4.8) $\qquad \theta\{(-\iota_1)\vee\iota_n)\}\wedge\sigma_n\}^* =\{\sigma_n\wedge\sigma_n\}^*\theta.$

By the fact that $\pi_7(S^3)=Z_2$, $\pi_{15}(S^7)=Z_2+Z_2+Z_2$ and $\pi_{n+1}(S^n)=Z_2$ [13, p. 186] and by using the homotopy equivalence $S(S^1\times S^n)\simeq SS^1\vee SS^n\vee S(S^1\wedge S^n)$ ([3, 11.10]), we see that the left and right vertical homomorphisms in the above diagram are the identity maps. Therefore, by (4.8) the middle vertical homomorphism is also so . This shows (4.7). q. e. d.

LEMMA 4.9. *For any* $m\in M(S^1\times S^n)$, *we have*

(i) $\chi(m, \tau)\neq m,$

(ii) $\chi(m, \nu)\neq m,$

(iii) $\chi(m, \nu\tau)\neq m,$

PROOF. (i) Let $i_2: S^n\longrightarrow S^1\times S^n$ be the inclusion. Then by the difinitions of χ and τ and (4.6) we have

$$\begin{aligned}\chi(m, \tau)(i_2\times i_2)&=(\iota_1\times(-\iota_n))(m_0, \alpha\pi+m_0')(\iota_1\times(-\iota_n)\times\iota_1\times(-\iota_n))(i_2\times i_2)\\&=(\ast, m_0'T+(-\iota_n)\alpha\pi(i_2\times i_2))=(\ast, (m_0'+\alpha\pi(i_2\times i_2))T)\\&\neq(\ast, m_0'+\alpha\pi(i_2\times i_2))=m(i_2\times i_2),\end{aligned}$$

because the multiplication $m_0'+\alpha\pi(i_2\times i_2)$ on S^n is not homotopy commutative.

(ii) Also, let $i_1: S^1\longrightarrow S^1\times S^n$ be the inclusion, and $p_1': S^n\times S^1\longrightarrow S^1$, p_2': $S^n\times S^1\longrightarrow S^n$ be the projections. Then

$$\begin{aligned}\chi(m,\nu)(i_2\times i_1)&=(p_1, \eta_n\pi_1+p_2)(m_0, \alpha\pi+m_0')\{(p_1, \eta_n\pi_1+p_2)\times(p_1, \eta_n\pi_1+p_2)\}(i_2\times i_1)\\&=(p_1, \eta_n\pi_1+p_2)(p_1', \alpha\pi(i_2\times i_1)+p_2')\\&=(p_1', \eta_n\pi_1(p_1', \alpha\pi(i_2\times i_1)+p_2')+\alpha\pi(i_2\times i_1)+p_2')\\&\neq(p_1', \alpha\pi(i_2\times i_1)+p_2')=m(i_2\times i_1),\end{aligned}$$

because $\eta_n\pi_1(p_1', \alpha\pi(i_2\times i_1)+p_2')=\eta_n\pi_1\neq 0$

(iii) By the definitions of χ and ν we have

$$\begin{aligned}\chi(m, \nu)(i_2\times i_1)&=(p_1, \eta_n\pi_1+p_2)(m_0, \alpha\pi+m_0')(i_2\times i_1)\\&=(\ast, \alpha\pi(i_2\times i_2)+m_0')=m(i_2\times i_2).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\chi(m, \nu\pi)(i_2\times i_2)&=\chi(\chi(m, \tau), \nu)(i_2\times i_2) && \text{by } (2.2)\\&=\chi(m, \tau)(i_2\times i_2) && \text{by } \chi(n, \nu)(i_2\times i_2)=n(i_2\times i_2)\\&\neq m(i_2\times i_2) && \text{by (ii).}\end{aligned}$$

q. e. d.

PROOF OF THEOREM 4.3. The theorem is obvious by Proposition 2.5, (4.2) and Lemmas 4.5—4.9. q. e. d.

PROOF OF THEOREM 4.4. By using Lemma 2.6 and [13, p. 186], we have

(4.10) $$\sharp M(S^1\times S^3)=2^9\cdot 3,\quad \sharp M(S^1\times S^7)=2^{17}\cdot 3\cdot 5.$$

By (4.2) and Theorem 4.3, we have

$$\mathscr{E}(S^1\times S^n)/\ \mathscr{E}_H(S^1\times S^n, m)=Z_2+Z_2 \quad \text{for any multiplication } m.$$

Hence the desired results follow from Proposition 2.7 and (4.10). q. e. d.

§5. The group $\mathscr{E}_H(S^3\times S^7, m)$ and the set $\widetilde{M}(S^3\times S^7)$.

This section is devoted to compute the group $\mathscr{E}_H(S^3\times S^7, m)$ for any multiplication m and the set $\widetilde{M}(S^3\times S^7)$. The computations are based on previous result [10] on the group $\mathscr{E}(S^3\times S^7)$.

Let p_1: $S^3\times S^7\longrightarrow S^3$ and p_2: $S^3\times S^7\longrightarrow S^7$ be the projections, and let π_1: $S^3\times S^7\longrightarrow S^{10}=S^3\wedge S^7$ be the collapsing map. The usual multiplication on S^3 (resp. S^7) is denoted by m_0(resp. n_0). The homotopy group $\pi_7(S^3)$ is isomorphic to Z_2[13, p. 186] and we denote its generator by ξ.

Define the elements σ, τ, ν and $\lambda(\alpha,\beta)\in\mathscr{E}(S^3\times S^7)$ for $\alpha\in\pi_{10}(S^3)$, $\beta\in\pi_{10}(S^7)$ as follows:

$$\sigma=(-\iota_3)\times\iota_7,\quad \tau=\iota_3\times(-\iota_7),$$
$$\nu=(p_1+\xi p_2, p_2),\quad \lambda(\alpha,\beta)=(p_1+\alpha\pi_1, p_2+\beta\pi_1).$$

Then, λ is a homomrphism contained in the following split exact sequence [10; Cor. 5.8]:

$$0\longrightarrow\pi_{10}(S^3)+\pi_{10}(S^7)\xrightarrow{\lambda}\mathscr{E}(S^3\times S^7)\longrightarrow Z_2+Z_2+Z_2\longrightarrow 0,$$

and the subgroup $Z_2+Z_2+Z_2$ of $\mathscr{E}(S^3\times S^7)$ generated by σ, τ and ν is mapped isomorphically onto the right group. In particular, any element of $\mathscr{E}(S^3\times S^7)$ can be written uniquely as

(5.1) $$\lambda(\alpha,\beta)\nu^{\varepsilon_1}\tau^{\varepsilon_2}\sigma^{\varepsilon_3}\quad \text{for } \varepsilon_i=0 \text{ or } 1,\ \alpha\in\pi_{10}(S^3),\ \beta\in\pi_{10}(S^7).$$

Now, our main results of this section are stated as follows:

THEOREM 5.2. *The group* $\mathscr{E}_H(S^3\times S^7, m)$ *is the unit group for any multiplication* m *on* $S^3\times S^7$.

THEOREM 5.3. $\sharp\widetilde{M}(S^3\times S^7)=2^{32}\cdot 3^{13}\cdot 5^4\cdot 7.$

Before we prove the above theorems, we show several lemmas.

Let $\pi: S^3\times S^7\times S^3\times S^7\longrightarrow(S^3\times S^7)\wedge(S^3\times S^7)$ and $\pi_2: S^7\times S^7\longrightarrow S^7\wedge S^7=S^{14}$ be the collapsing maps. Then we have

LEMMA 5.4. *$\{\pi(\nu\times\nu)(i_2\times i_2)\}^*=\{\pi(i_2\times i_2)\}^*$ as a homomorphism of $[(S^3\times S^7)\wedge(S^3\times S^7), S^3\times S^7]$ to $[S^7\times S^7, S^3\times S^7]$.*

PROOF. Consider the following commutative diagram for $n=3,7$ and $X=S^7\times S^7$

$$\begin{array}{ccccccccc}
0\longrightarrow & [S^3\wedge S^7\wedge X, S^n] & \xrightarrow{(\pi_1\wedge 1)^*} & [(S^3\times S^7)\wedge X, S^n] & \xrightarrow{j^*} & [(S^3\vee S^7)\wedge X, S^n] & \longrightarrow 0 \\
& \downarrow{\scriptstyle \{(\xi\wedge\iota_7)\wedge 1\}^*} & & \downarrow{\scriptstyle \{\xi\times\iota_7\wedge 1\}^*} & & \downarrow{\scriptstyle \{\xi\vee\iota_7)\wedge 1\}^*} & \\
0\longrightarrow & [S^7\wedge S^7\wedge X, S^n] & \xrightarrow{(\pi_2\wedge 1)^*} & [(S^7\times S^7)\wedge X, S^n] & \xrightarrow{j^*} & [(S^7\vee S^7)\wedge X, S^n] & \longrightarrow 0
\end{array}$$

Sinse $S^2\xi=0$ by [13, (5.9)], the left vertical homomorphism is trivial and the right vertical homomorphism is equal to $\{(*\vee\iota_7)\wedge 1\}^*$. Therefore, by the similar way as in Lemma 4.5, we have $\{(\xi\times\iota_7)\wedge 1\}^*=\{(*\times\iota_7)\wedge 1\}^*$. Also we have $\{1\wedge(\xi\times\iota_7)\}^*=\{1\wedge(*\times\iota_7)\}^*: [(S^3\times S^7)\wedge(S^3\times S^7), S^n]\longrightarrow[(S^3\times S^7)\wedge(S^7\times S^7), S^n]$ by the similar way. According to these equalities, we have $\{(\xi\times\iota_7)\wedge(\xi\times\iota_7)\}^*=\{(*\times\iota_7)\wedge(*\times\iota_7)\}^*$, and so we have $\{\pi(\nu\times\nu)(i_2\times i_2)\}^*=\{((\xi,\iota_7)\wedge(\xi,\iota_7))\pi_2\}^*=\{((*,\iota_7)\wedge(*,\iota_7))\pi_2\}^*=\{\pi(i_2\times i_2)\}^*$ q. e. d.

LEMMA 5.5. *Let $\zeta\in\pi_{14}(S^3)$ and $f\in[S^7\times S^7, S^7]$, then $\pi_1(\xi\pi_2, f)$ is trivial in $[S^7\times S^7, S^{10}]$.*

PROOF. This lemma follows from the following commutative diagram

$$\begin{array}{ccccc}
S^7\times S^7 & \xrightarrow{(\pi_2,f)} & S^{14}\times S^7 & \xrightarrow{\pi_3} & S^{21} \\
\downarrow{\scriptstyle (\zeta\pi_2,f)} & & \downarrow{\scriptstyle \zeta\times 1} & & \downarrow{\scriptstyle S^7\zeta} \\
S^3\times S^7 & = & S^3\times S^7 & \xrightarrow{\pi_1} & S^{10}
\end{array}$$

Where $\pi_3: S^{14}\times S^7\longrightarrow S^{21}$ is a collapsing map. q. e. d.

LEMMA 5.6. *For any multiplication m on $S^3\times S^7$, we have the following relations.*

(i) $\chi(m,f)\neq m$ *if* $f=\lambda(\alpha,\beta)\tau$, $\lambda(\alpha,\beta)$, $\lambda(\alpha,\beta)\nu\tau$,

(ii) $\chi(m,f)\neq m$ *if* $f=\lambda(\alpha,\beta)\nu^{\varepsilon_1}\tau^{\varepsilon_2}\sigma$ *for* $\varepsilon_i=0,1$,

(iii) $\chi(m,\chi(\alpha,\beta))\neq m$ *if* $(\alpha,\beta)\neq 0$ *in* $\pi_{10}(S^3)+\pi_{10}(S^7)$.

PROOF. By Lemma 2.6, m can be written as

$m=(a\pi+m_0, b\pi+n_0)$ for some $(a, b)\in[(S^3\times S^7)\wedge(S^3\times S^7), S^3\times S^7]$.

(i) By the definitions of χ and τ we have $p_2\chi(m, \tau)(i_2\times i_2)=n_0T+(-b\pi(i_2\times i_2))$ and by the similar way as in [10, Lemma 6.5] we have $n_0T+(-b\pi(i_2\times i_2))=(-b\pi(i_2\times i_2))+n_0T=(b\pi(i_2\times i_2)+n_0)T$. Therefore, we have

(5.7) $$p_2\chi(m, \tau)(i_2\times i_2) = p_2m(i_2\times i_2)T\neq p_2m(i_2\times i_2) \text{ in } [S^7\times S^7, S^7],$$

because the multiplication $p_2m(i_2\times i_2)$ on S^7 is not homotopy commutative. By the easy calculation we have

$$\chi(m, \lambda(\alpha, \beta))(i_2\times i_2)$$
$$=(a\pi(i_2\times i_2)+\alpha\pi_1(a\pi(i_2\times i_2), b\pi(i_2\times i_2)+n_0), b\pi(i_2\times i_2)+n_0+\beta\pi_1(a\pi(i_2\times i_2), b\pi(i_2\times i_2)+n_0))$$

and so by Lemma 5.5 we have

(5.8) $$\chi(m, \lambda(\alpha, \beta))(i_2\times i_2)=m(i_2\times i_2).$$

By (2.2), (5.8) and (5.7) we have

(5.9) $$\chi(m, \lambda(\alpha, \beta)\tau)(i_2\times i_2)=\chi(m, \tau)(i_2\times i_2)\neq m(i_2\times i_2).$$

By Lemma 5.4 we have

(5.10) $$\chi(m,\nu)(i_2\times i_2)=(a\pi(i_2\times i_2)+\xi p_1'+\xi p_2'+\xi(b\pi(i_2\times i_2)+n_0), b\pi(i_2\times i_2)+n_0),$$

where $p_i'\colon S^7\times S^7\longrightarrow S^7$ is the projection to the i-th factor. On the other hand, by Lemma 3.17 we can write

$$\xi p_i'+\xi p_2'+\xi(b\pi(i_2\times i_2)+n_0 = c\pi_2 \quad \text{for some nonzero element } c \text{ of } \pi_{14}(S^7).$$

This and (5.10) show that

(5.11) $$\chi(m, \nu)(i_2\times i_2)\neq m(i_2\times i_2).$$

By (2.2), (5.8) and (5.11) we have

(5.12) $$\chi(m, \lambda(\alpha, \beta)\nu)(i_2\times i_2) = \chi(m, \nu)(i_2\times i_2)\neq m(i_2\times i_2).$$

Also, by (2.2), (5.8) and (5.10) we have

$$\chi(m, \lambda(\alpha, \beta)\nu\tau)(i_2\times i_2)=(\iota_3\times(-\iota_7))\chi(m, \nu)(i_2\times i_2)((-\iota_7)\times(-\iota_7))$$
$$=(a\pi(i_2\times i_2)+c\pi_2, n_0T+((-b)\pi(i_2\times i_2))).$$

Since $c\neq 0$, we have

(5.13) $$\chi(m, \lambda(\alpha, \beta)\nu\tau)(i_2\times i_2)\neq m(i_2\times i_2).$$

Now, (5.9), (5.12) and (5.13) show (i).

(ii) By [10, Lemma 6.5] We have $m_0T+((-a)\pi(i_1\times i_1))=((-a)\pi(i_1\times i_1)+m_0T$. Therefore, by the definitions of χ and σ we have

(5.14) $$\chi(m, \sigma)(i_1\times i_1)=(m_0T+(-a)\pi(i_1\times i_1), *)= m(i_1\times i_1)T\neq m(i_1\times i_1),$$

because S^3 has not a homotopy commutative multiplication.
By the easy calculation we have

(5.15) $$\chi(m,f)(i_1\times i_1)=m(i_1\times i_1) \quad \text{for } f=\tau, \nu \text{ and } \lambda(\alpha,\beta).$$

By (2.2), (5.14) and (5.15), we have

$$\chi(m,f)(i_1\times i_1)=\chi(m, \sigma)(i_1\times i_1)\neq m(i_1\times i_1)$$
$$\text{for } f=\lambda(\alpha, \beta)\sigma,\ \lambda(\alpha, \beta)\tau\sigma,\ \lambda(\alpha, \beta)\nu\sigma \text{ and } \lambda(\alpha, \beta)\nu\tau\sigma.$$

This shows (ii).

(iii) Let $\overline{p_1}: S^7\times S^3\longrightarrow S^3$ and $\overline{p_2}: S^7\times S^3\longrightarrow S^7$ be the projections. Then, by the definitions of χ and $\lambda(\alpha, \beta)$, we have

$$\begin{aligned}
&\chi(m, \lambda(\alpha, \beta))(i_2\times i_1)\\
&=(p_1+\alpha\pi_1, p_2+\beta\pi_1)(a\pi+m_0, b\pi+n_0)\{(p_1+(-\alpha)\pi_1,\ p_2+(-\beta)\pi_2)\times(p_1+(-\alpha)\\
&\qquad \pi_1, p_2+(-\beta)\pi_2)\}(i_2\times i_1)\\
&=(p_1+\alpha\pi_1, p_2+\beta\pi_1)(a\pi+m_0, b\pi+n_0)\{(*, \overline{p_2})\times(\overline{p_1}, *)\}\\
&=(a\pi(i_2\times i_1)+\overline{p_1}+\alpha\pi_1(a\pi(i_2\times i_1)+\overline{p_1}, b\pi(i_2\times i_1)+\overline{p_2}), b\pi(i_2\times i_1)+\overline{p_2}\\
&\qquad +\beta\pi_1(a\pi(i_2\times i_1)+p_1, b\pi(i_2\times i_1)+\overline{p_2})).
\end{aligned}$$

On the other hand, it is easy to see that

$$\pi_1(a\pi(i_2\times i_1)+\overline{p_1}, b\pi(i_2\times i_1)+\overline{p_2})=(-\iota_{10})\pi_1' \quad \text{in } [S^7\times S^3, S^{10}],$$

where $\pi_1': S^7\times S^3\longrightarrow S^7\wedge S^3=S^{10}$ is a collapsing map.
Therefore, we have

$$\chi(m, \lambda(\alpha, \beta))(i_2\times i_1)=(a\pi(i_2\times i_1)+\overline{p_1}+(-\alpha)\pi_1', b\pi(i_2\times i_1)+\overline{p_2}+(-\beta)\pi_1').$$

Since $m(i_2\times i_1)=(a\pi(i_2\times i_1)+\overline{p_1}, b\pi(i_2\times i_1)+\overline{p_2})$, we have (iii). q. e. d.

PROOF OF THEOREM 5.2. By Proposition 2.5, (5.1) and Lemma 5.6, we have the desired results. q. e. d.

PROOF OF THEOREM 5.3. By using Lemma 2.6 and [13, pp. 186–187], we have

$$\# M((S^3\times S^7)=2^{38}\cdot 3^{15}\cdot 5^5\cdot 7. \tag{5.16}$$

By (5.1) and Theorem 5.2, we have

$$\#\{\mathscr{E}(S^3\times S^7)/\mathscr{E}_H(S^3\times S^7, m)\}=\#\mathscr{E}(S^3\times S^7)=2^6\cdot 3^2\cdot 5$$

for any multiplication m.

Hence, the desired results follow from Proposition 2.7. q. e. d.

§6. The group $\mathscr{E}_H(K(A,p)\times K(B,q), m)$

In this section, we shall deal with the product space $K(A,p)\times K(B,q)$ of two Eilenberg-MacLane complexes. We shall also given an example of X such that $\mathscr{E}_H(X, m)\not\cong \mathscr{E}_H(X, n)$ for some m, $n\in M(X)$.

Let A, B be abelian groups and let

$$K_1=K(A,p),\quad K_2=K(B,q),\quad p>q\geqq 1.$$

The usual loop multiplications on K_1 and K_2 are denoted by m_1 and m_2.

By [6, Th. 2.10], [7, Cor. 5.9] and [9, Example 4.1], there is a split exact sequence:

$$0\longrightarrow H^p(B,q;A)\longrightarrow \mathscr{E}(K_1\times K_2)\longrightarrow \mathscr{E}(K_1)\times\mathscr{E}(K_2)\longrightarrow 1.$$

More precisely

$$\mathscr{E}(K_1\times K_2)=\{(\alpha p_1+\beta p_2, \gamma p_2)\mid \alpha\in\mathscr{E}(K_1), \gamma\in\mathscr{E}(K_2), \beta\in H^p(B,q;A)\}, \tag{6.1}$$

where p_i: $K_1\times K_2\longrightarrow K_i$ is the projection, and $+$ is induced by the usual multiplication m_1 on K_1. Futhermore, $\mathscr{E}(K_1)=\text{Aut } A$, $\mathscr{E}(K_2)=\text{Aut } B$ and any element $(\alpha p_1+\beta p_2, \gamma p_2)$ can be written as a matrix $\begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix}$.

By the easy calculation we can see that the group $[(K_1\times K_2)\wedge(K_1\times K_2), K_1\times K_2]$ is isomorphic to $H^p(K_2\wedge K_2; A)$ and so by Lemma 2.6 any multiplication m on $K_1\times K_2$ can be written as

$$m=(m_1+\theta\pi, m_2)\quad \text{for some } \theta\in H^p(K_2\wedge K_2; A), \tag{6.2}$$

where $\pi: K_1\times K_2\times K_1\times K_2\xrightarrow{p_2\times p_2} K_2\times K_2\xrightarrow{\pi_2} K_2\wedge K_2$ is the composition of the projection and the collapsing map.

Let p' and p'' be the projections: $K_2\times K_2\longrightarrow K_2$ to the first and the second factors, respectively.

THEOREM 6.3. *With the above notations, the group* $\mathscr{E}_H(K(A,p)\times K(B,q),$

$(m_1+\theta\pi, m_2))$ *is the subgroup*

$$\{(\alpha p_1+\beta p_2, \gamma p_2) \mid p'^*\beta+p''^*\beta-m_2^*\beta=\pi_2^*(\alpha\theta-\theta(\gamma\wedge\gamma)) \quad in\ [K_2\times K_2, K_1]\}$$

of $\mathscr{E}(K(A, p)\times K(B, q))$.

PROOF. Since $(\alpha p_1+\beta p_2, \gamma p_2)^{-1}=(\alpha^{-1}p_1-\alpha^{-1}\beta\gamma^{-1}p_2, \gamma^{-1}p_2)$ and $\pi\{(\alpha p_1+\beta p_2, \gamma p_2)\times(\alpha\overline{p_1}+\beta\overline{p_2}, \gamma\overline{p_2})\}=(\gamma\wedge\gamma)\pi$, we have

$$\chi(m, (\alpha p_1+\beta p_2, \gamma p_2)^{-1})$$
$$=(\alpha^{-1}p_1-\alpha^{-1}\beta\gamma^{-1}p_2, \gamma^{-1}p_2)(m_1+\theta\pi, m_2)\{(\alpha p_1+\beta p_2, \gamma p_2)\times(\alpha\overline{p_1}+\beta\overline{p_2}, \gamma\overline{p_2})\}$$
$$=(\alpha^{-1}p_1-\alpha^{-1}\beta\gamma^{-1}p_2, \gamma^{-1}p_2)(\alpha p_1+\beta p_2+\alpha\overline{p_1}+\beta\overline{p_2}+\theta(\gamma\wedge\gamma)\pi, \gamma p_2+\gamma\overline{p_2})$$
$$=(p_1+\alpha^{-1}\beta p_2+\overline{p_1}+\alpha^{-1}\beta\overline{p_2}+\alpha^{-1}\theta(\gamma\wedge\gamma)\pi-\alpha^{-1}\beta(p_2+\overline{p_2}), p_2+\overline{p_2})$$
$$=(m_1+\alpha^{-1}\beta p_2+\alpha^{-1}\beta\overline{p_2}+\alpha^{-1}\theta(\gamma\wedge\gamma)\pi-\alpha^{-1}\beta m_2, m_2).$$

Therefore, by Proposition 2.5 we have the desired results. q. e. d.

By this theorem, we have immediately the following.

COROLLARY 6.4. *For the product multiplication* $m_1\times m_2=(m_1, m_2)$ *on* $K(A, p)\times K(B, q)$, $\mathscr{E}_H(K(A, p)\times K(B, q), (m_1, m_2))$ *is the subgroup*

$$\{(\alpha p_1+\beta p_2, \gamma p_2) \mid \beta\in H^p(B, q; A) \text{ is primitive}\}$$

of $\mathscr{E}(K(A, p)\times K(B, q))$.

Now, we apply Theorem 6.1. for $K_1=K(Z_3, 6)$ and $K_2=K(Z, 2)$ and we have

EXAMPLE 6.5. *Let* $X=K(Z_3, 6)\times K(Z, 2)$. *Then*

$$\mathscr{E}(X)=Z_2\times S_3, \quad \# M(X)=9,$$

where S_3 *is the symmetric group on three elements; and*

$$\mathscr{E}_H(X, m) = \begin{cases} \mathscr{E}(X) & \text{if } m \text{ is the product multiplication,} \\ Z_6 & \text{otherwise;} \end{cases}$$

$$\# \widetilde{M}(X)=5.$$

Furthermore, the multiplication $m=(m_1+\theta\pi, m_2)$ *on* X *is homotopy associative if and only if* $\theta=0$ *or* $\pm(a\otimes a^2+a^2\otimes a)$, *where* a *is a generator of* $H^2(Z, 2; Z_3)$.

PROOF. By (6.1), $\mathscr{E}(X)$ is given by the following group of matrices:

(6,6) $$\mathscr{E}(X)=\{\begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix} \mid \alpha\in Z_2, \beta\in Z_3, \ \gamma\in Z_2\}$$

$(Z_2=\{1, \ -1\}, Z_3=\{0,1,2\})$, which is isomorphic to $Z_2\times S_3$. It is easy to see that

(6.7) $$H^6(K(Z,2)\wedge K(Z,2); Z_3)=Z_3+Z_3,$$

and so $\# M(X)=9$ by (6.2).

On the other hand, any element of $H^6(Z,2;Z_3)$ is primitive. Thus we have the result for the product multiplication m by Corollary 6.4. It is easy to see that

(6.8) $$(-1)_*\theta=-\theta, \ \ ((-1)\wedge(-1))^*\theta=-\theta,$$

for any element θ of (6.7). Therefore, by Theorem 6.3, (6.6) and (6.8) we obtain

$$\mathscr{E}_H(X,m)=\{\begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix} \mid \alpha=\gamma=\pm 1, \ \beta\in Z_3\}=Z_6,$$

for $m=(m_1+\theta\pi, m_2)$, $\theta\neq 0$.

By the above argument, we see that the number of the elements contained in the orbit of $(m_1+\theta\pi, m_2)$(under the action χ)is equal to 1 if $\theta=0$, or 2 if $\theta\neq 0$. Therefore, we have $\#\widetilde{M}(K(Z_3,6)\times K(Z,2))=5$.

For any multiplication $m=(m_1+\theta\pi, m_2)$ on X of (6.2), we see easily that $m(m\times 1)$ and $m(1\times m)$ are equal to

$$(p_1p'_1+p_1p'_2+\theta\pi_2(p_2p'_1\times p_2p'_2)+p_1p'_3+\theta\pi_2((p_2p'_1+p_2p'_2)\times p_2p'_3), p_2p'_1+p_2p'_2+p_2p'_3),$$

$$(p_1p'_1+p_1p'_2+p_1p'_3+\theta\pi_2(p_2p'_2\times p_2p'_3)+\theta\pi_2(p_2p'_1\times(p_2p'_2+p_2p'_3)), p_2p'_1+p_2p'_2+p_2p'_3),$$

respectively, where $p'_i: X\times X\times X\longrightarrow X$ is the projection onto the i-th factor. Therefore, m is homotopy associative if and only if

(6.9) $$\theta\pi_2(p''_1\times p''_2)+\theta\pi_2((p''_1+p''_2)\times p''_3)=\theta\pi_2(p''_2\times p''_3)+\theta\pi_2(p''_1\times(p''_2+p''_3))$$

in $H^6(K_2\times K_2\times K_2; Z_3)$,

where $p''_i: K_2\times K_2\times K_2\longrightarrow K_2$ is the projection onto the i-th factor and $K_2=K(Z,2)$.

The element θ of (6.7) can be written as

$$\theta=ka\otimes a^2+la^2\otimes a \quad \text{for some } k, l\in Z_3,$$

and we see easily that

$$\theta\pi_2(p''_1\times p''_2)=\theta\otimes 1, \ \ \theta\pi_2(p''_2\times p''_3)=1\otimes\theta,$$

$$\theta\pi_2((p_1'' + p_2'') \times p_3'') = (m_2^* \otimes 1)\theta = km_2^* a \otimes a^2 + l(m_2^* a)^2 \otimes a,$$

$$\theta\pi_2(p_1'' \times (p_2'' + p_3'')) = ka \otimes (m_2^* a)^2 + la^2 \otimes m_2^* a,$$

and $m_2^* a = a \otimes 1 + 1 \otimes a$.

Therefore, (6.9) is equivalent to $2(k-l)(a \otimes a \otimes a) = 0$, and we have the desired result. q. e. d.

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