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On H-Equivalences of SU(3), U(3) and Sp(2)

By

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§0. Introduction

The set $\mathscr{E}_H(X)$ of all (based) homotopy classes of H-equivalences (homotopy equivalent H-maps) of an H-space X to itself forms a subgroup of the group $\mathscr{E}(X)$ of all homotopy classes of homotopy equivalences of X to itself.

Assume that an H-space X is a CW-complex, and let $\{X_n, f_n: X \to X_n, p_n: X_n \to X_{n-1}\}$ be the Postnikov system of X. Then X_n is an H-space such that f_n and p_n are H-maps, and we obtain naturally the homomorphism

$$\tilde{\Phi}_n: \mathscr{E}_H(X) \longrightarrow \mathscr{E}_H(X_n) \quad \text{such that} \quad \tilde{\Phi}_n(h)f_n = f_n h;$$

and we can prove the following

THEOREM 1.3. (ii) *$\tilde{\Phi}_n$ is isomorphic if $n \geqq 2\dim X$.*

Furthermore, the group $\mathscr{E}(X_n)$ can be determined by the results of Y. Nomura [9] inductively on n (Lemma 1.6), and also we study a condition that some elements of $\mathscr{E}(X_n)$ belong to $\mathscr{E}_H(X_n)$ (Lemma 1.9).

By using these results, we determine the groups $\mathscr{E}(X_n)$ and $\mathscr{E}_H(X_n)$ for the case that X is the special unitary group $SU(3)$ in §§2–3, and obtain the following

THEOREM 3.1. (iv) *$\mathscr{E}_H(SU(3)) = Z_2$, generated by the conjugation $c: SU(3) \to SU(3)$.*

Also, we see that $\mathscr{E}_H(U(3)) = Z_2 \times Z_2$ (Theorem 3.5 (ii)), by using the general result that $U(n)$ is naturally H-equivalent to $S^1 \times SU(n)$.

In the same way, we consider the case that X is the symplectic group $Sp(2)$ in §§4–5, and obtain the following

THEOREM 5.7. (iv) *$\mathscr{E}_H(Sp(2)) = 1$,*

by noticing that the localization $Sp(2)_{(5)}$ at 5 is not homotopy commutative (Propositions 5.1 and 5.6).

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§1. Some relations between $\mathscr{E}_H(X)$ and $\mathscr{E}_H(X_n)$

In this note, all (topological) spaces are arcwise connected spaces with base points and have homotopy types of CW-complexes, and all (continuous) maps and homotopies preserve the base points. For any spaces X and Y, let $[X, Y]$ be the set of all homotopy classes of maps of X to Y. For a map $f: X \to Y$, we denote usually its homotopy class in $[X, Y]$ by the same letter f.

A space X is called an H-space, if there is a map $m: X \times X \to X$, called a multiplication, such that $m|X \times * = 1 = m|* \times X$ in $[X, X]$. For H-spaces $X = (X, m)$ and $Y = (Y, n)$, a map $f: X \to Y$ is called an H-map if $fm = n(f \times f)$ in $[X \times X, Y]$, and let $[X, Y]_H$ be the subset of $[X, Y]$ consisting of all homotopy classes of H-maps. Let $\mathscr{E}(X)$ be the group of all homotopy classes of self (homotopy) equivalences of X. For an H-space $X = (X, m)$, let $\mathscr{E}_H(X) = \mathscr{E}_H(X, m)$ be the subgroup of $\mathscr{E}(X)$ consisting of all homotopy classes of H-equivalences (homotopy equivalent H-maps) of (X, m) to itself.

Let $\{X_n, f_n, p_n\}$ be the Postnikov system of X, that is, it consists of spaces X_n and maps $f_n: X \to X_n$, $p_n: X_n \to X_{n-1}$ such that $\pi_i(X_n) = 0$ if $i > n$, $f_{n*}: \pi_i(X) \to \pi_i(X_n)$ is isomorphic if $i \leqq n$, $K(\pi_n(X), n) \xrightarrow{i_n} X_n \xrightarrow{p_n} X_{n-1}$ is a fiber space and $p_n f_n = f_{n-1}$ in $[X, X_{n-1}]$. Then, as is well known, a cell-structure for X_n can be given by

$$(1.1) \qquad X_n = X \cup (\bigcup_\alpha e_\alpha^{n+2}) \cup (\bigcup_\beta e_\beta^{n+3}) \cdots$$

Since $f_n^*: [X_n, X_n] \to [X, X_n]$ is bijective by (1.1), we can define a homomorphism

$$(1.2) \qquad \phi_n: \mathscr{E}(X) \longrightarrow \mathscr{E}(X_n)$$

as the restriction of the composition $[X, X] \xrightarrow{f_{n*}} [X, X_n] \xrightarrow{f_n^{*-1}} [X_n, X_n]$. Assume that X is an H-space with a multiplication m, in addition. Then it is easy to see that m induces the unique multiplication m_n on X_n up to homotopy such that p_n and f_n are H-maps, and we can define a homomorphism

$$(1.2)' \qquad \tilde{\phi}_n: \mathscr{E}_H(X) = \mathscr{E}_H(X, m) \longrightarrow \mathscr{E}_H(X_n) = \mathscr{E}_H(X_n, m_n)$$

as the restriction of ϕ_n in (1.2).

THEOREM 1.3. *Let X be an l-dimensional CW-complex and an H-space with a multiplication m. Let X_n and $\tilde{\phi}_n$ be as above. Then, we have the following* (i) *and* (ii).

(i) *If $n \geqq l$, then $\tilde{\phi}_n$ is monomorphic,*

(ii) *If $n \geqq 2l$, then $\tilde{\phi}_n$ is epimorphic.*

PROOF. By [11, Lemma 7.1], we obtain

(1.4) ϕ_n is isomorphic if $n \geqq l$.

Hence, (i) is obvious.

(ii) Since ϕ_n is isomorphic by (1.4), for any $h \in \mathscr{E}_H(X_n)$ there exists an element $\bar{h} \in \mathscr{E}(X)$ such that $f_n \bar{h} = h f_n$, and hence $f_{n*} m(\bar{h} \times \bar{h}) = f_{n*} \bar{h} m$ since f_n is an H-map. In the same way as [14, p. 405, 23 Cor.], we see that $f_{n*}: [X \times X, X] \to [X \times X, X_n]$ is injective by the assumption $n \geqq 2l$, and hence the element $\bar{h}$ belongs to $\mathscr{E}_H(X)$.

q.e.d.

In the above theorem, we have the following example such that $\mathscr{E}_H(X_n)$ is not isomorphic to $\mathscr{E}_H(X)$ for $2l > n \geqq l$.

EXAMPLE 1.5. *Let $X = S^l$ ($l = 3, 7$) and m be any multiplication on X. Let $\{X_n\}$ be the Postnikov system of $X = S^l$. Then,*

$$\mathscr{E}_H(X_n) = \begin{cases} \mathscr{E}(X) = Z_2 & \text{if } l \leqq n < 2l \\ \mathscr{E}_H(X) = 1 & \text{if } 2l \leqq n, \end{cases}$$

PROOF. Since $X_n \wedge X_n$ is $2l-1$ connected, we have $[X_n \wedge X_n, X_n] = 0$ for $l \leqq n < 2l$ and so $\mathscr{E}_H(X_n) = \mathscr{E}(X_n) = \mathscr{E}(X) = Z_2$ for $l \leqq n < 2l$ by [12, Prop. 2.7] and (1.4). The rest of the proof is obtained by Theorem 1.3 and [12, Th. 4.1]. q.e.d.

The following lemma will be used in §§3 and 5.

LEMMA 1.6 (Y. Nomura [9, Th. 2.1, 2.9]) *For each n, the fibering $K(\pi_n(X), n) \to X_n \xrightarrow{p_n} X_{n-1}$ with the k-invariant k^{n+1} in the Postnikov system of a simply connected H-space X induces the exact sequence*

$$0 \longrightarrow H_n \xrightarrow{\kappa} \mathscr{E}(X_n) \longrightarrow G_n \longrightarrow 1 .$$

Here $$H_n = p_n^* H^n(X_{n-1}; \pi_n(X)) / (\Omega k^{n+1})_* [X_n, \Omega X_{n-1}],$$

$$G_n = \{(h, \varepsilon) \in \mathscr{E}(X_{n-1}) \times \mathscr{E}(K(\pi_n(X), n+1)),\ k^{n+1} h = \varepsilon k^{n+1} \text{ in } H^{n+1}(X_{n-1}; \pi_n(X))\},$$

κ is the homomorphism defined by the same way as (1.7) *below and $p_n^*: H^n(X_{n-1}; \pi_n(X)) \to H^n(X_n; \pi_n(X))$ is always monomorphic.*

Let (X, m_1) and (Y, m_2) be two H-spaces and $f: X \to Y$ be an H-map. Let E_f be the mapping track of f and $k: E_f \times \Omega Y \to E_f$ be the action on the induced fibering $\Omega Y \xrightarrow{i} E_f \xrightarrow{p} X$. Then it is well known that E_f is an H-space such that i, p and k are H-maps (cf. [5, Th. 2]). We define a map

(1.7) $$\kappa: [E_f, \Omega Y] \longrightarrow [E_f, E_f]$$

by $\kappa(\alpha) = k(1 \times \alpha) d$ for $\alpha \in [E_f, \Omega Y]$, where d is the diagonal map. Then we see easily

$$\kappa(\alpha)=1+i\alpha \quad \text{for} \quad \alpha\in[E_f, \Omega Y]. \tag{1.8}$$

LEMMA 1.9. *Assume that* $(\Omega f)_*: [E_f\wedge E_f, \Omega X]\to[E_f\wedge E_f, \Omega Y]$ *is trivial. Then for* $\alpha\in[E_f, \Omega Y]$, $\kappa(\alpha)$ *is an H-map if and only if* α *is an H-map.*

PROOF. Since k is an H-map, the "if" part is obvious.

Conversely, assume that $\kappa(\alpha)$ is an H-map, and let m be the multiplication on E_f as above. Then by (1.8) we see that

$$\kappa(\alpha)m=m+i\alpha m, \quad m(\kappa(\alpha)\times\kappa(\alpha))=m+im_2(\alpha\times\alpha).$$

Therefore we have $i\alpha m=im_2(\alpha\times\alpha)$ by [3, Th. 1.1]. By the assumption, $i_*: [E_f\wedge E_f, \Omega Y]\to[E_f\wedge E_f, E_f]$ is monomorphic and hence we have $\alpha m=m_2(\alpha\times\alpha)$.

q.e.d.

§2. The Postnikov system of $SU(3)$

Throughout this and the next sections, let $\{X_n, f_n: SU(3)\to X_n, p_n: X_n\to X_{n-1}\}$ be the Postnikov system of the special unitary group $SU(3)$ and we shall compute the groups $\mathscr{E}(X_n)$ and $\mathscr{E}(SU(3))$.

There is the principal bundle

$$S^3 \xrightarrow{i} SU(3) \xrightarrow{p} S^5 \tag{2.1}$$

with the characteristic element $\eta_3\in\pi_4(S^3)$. So the lower homotopy groups $\pi_i(SU(3))$ are computed from Toda's table [17], and we have

LEMMA 2.2 (M. Mimura and H. Toda [7]).

$$\begin{aligned}
&\pi_3(SU(3))=Z && \textit{with generator} \quad i_*\iota_3,\\
&\pi_5(SU(3))=Z && \textit{with generator} \quad [2\iota_5],\\
&\pi_6(SU(3))=Z_6 && \textit{with generator} \quad i_*\omega,\\
&\pi_8(SU(3))=Z_{12} && \textit{with generator} \quad [\alpha_1]+[2\iota_5]\nu_5,\\
&\pi_i(SU(3))=0 && \textit{otherwise for} \quad i\leqq 8=\dim SU(3),
\end{aligned}$$

where $[\alpha]$ *denotes an element such that* $p_*[\alpha]=\alpha\in\pi_i(S^5)$, *and the elements* $[\alpha_1]$ *and* $[2\iota_5]\nu_5$ *in* $\pi_8(SU(3))$ *are of order* 3 *and* 4, *respectively.*

From this lemma, the partial Postnikov system $\{X_n\}_{n\leqq 8}$ of $SU(3)$ is given by the diagram

(2.3)
$$\begin{array}{ccccccccccc} X_8 & \xrightarrow{p_8} & X_7 & \xrightarrow{p_7} & X_6 & \xrightarrow{p_6} & X_5 & \xrightarrow{p_5} & X_4 & \xrightarrow{p_4} & X_3=K(Z,3) \\ & & \downarrow k^9 & & & & \downarrow k^7 & & \downarrow k^6 & & \\ & & K(Z_{12},9) & & & & K(Z_6,7) & & K(Z,6), & & \end{array}$$

and $X_2=X_1=*$.

PROPOSITION 2.4. *In the above diagram, the group* $H^{n+1}(X_{n-1}; \pi_n(SU(3)))$, $n \leqq 8$, *is given by*

n	1, 2, 3, 4	5	6	7	8
$H^{n+1}(X_{n-1}; \pi_n(SU(3)))$	0	Z_2	Z_6	0	Z_{12}

and the k-invariant k^{n+1} *generates this group.*

PROOF. We consider the Serre's exact sequence of the integral cohomology groups derived from the fibering $X_5 \xrightarrow{p_5} K(Z, 3) \xrightarrow{k^6} K(Z, 6)$:

$$H^6(Z, 6) \xrightarrow{k^{6*}} H^6(Z, 3) \xrightarrow{p_5^*} H^6(X_5).$$

Since $SU(3)=S^3 \cup e^5 \cup e^8$, we have $X_5=(S^3 \cup e^5 \cup e^8) \cup e^7 \cup \cdots$ by (1.1) and so $H^6(X_5)=0$. Also, we have $H^6(Z, 6)=Z$ and $H^6(Z, 3)=Z_2$ by [2, Th. 5]. Therefore, k^6 is the generator of $H^6(Z, 3)$ by the above exact sequence. Next, since $X_6=(S^3 \cup e^5 \cup e^8) \cup e^8 \cup \cdots$ by (1.1), we have $H^6(X_6; Z_6)=H^7(X_6; Z_6)=0$. Therefore, by the Serre's exact sequence derived from the fibering $X_6 \xrightarrow{p_6} X_5 \xrightarrow{k^7} K(Z_6, 7)$, we see that $H^7(X_5; Z_6)=Z_6$ which is generated by k^7. Finally, we consider the following exact sequence:

$$\pi_8(S^3 \cup e^5) \xrightarrow{j_*} \pi_8(SU(3)) \longrightarrow \pi_8(SU(3), S^3 \cup e^5),$$

where $j: S^3 \cup e^5 \to SU(3)$ is the inclusion. Here, $\pi_8(SU(3))=Z_{12}$ by Lemma 2.2 and $\pi_8(SU(3), S^3 \cup e^5)$ is isomorphic to $\pi_8(S^8)=Z$ by [1, Th. II]. Therefore j_* is epimorphic. By (1.1) and by this fact, X_7 has a cell-structure of $(S^3 \cup e^5 \cup e^8) \cup e^9 \cup \cdots$ in which the attaching element of e^9 is $j_*(\zeta)$ for some element $\zeta \in \pi_8(S^3 \cup e^5)$. Hence we have $H^8(X_7; Z_{12})=Z_{12}$. Also, $H^8(X_8; Z_{12})=Z_{12}$ and $H^9(X_8; Z_{12})=0$ by (1.1), and we see that $H^9(X_7; Z_{12})=Z_{12}$ which is generated by k^9 by the Serre's exact sequence derived from the fibering $X_8 \xrightarrow{p_7} X_7 \xrightarrow{k^9} K(Z_{12}, 9)$. q.e.d.

As is well known, the integral cohomology ring of $SU(3)$ is the exterior algebra

(2.5) $$H^*(SU(3))=\wedge(x_3, x_5), \quad \deg x_i=i,$$

and each generator x_i is primitive. Then we have

LEMMA 2.6. *Let* $c: SU(3) \to SU(3)$ *and* $v: SU(3) \to SU(3)$ *be the maps given by* $c(\alpha)=\bar{\alpha}$ *(conjugate matrix of* α*) and* $v(\alpha)=\alpha^{-1}$*, respectively. Then,*

$$c^*(x_3)=x_3,\ c^*(x_5)=-x_5 \quad \textit{and} \quad c^*(x_3\cdot x_5)=-x_3\cdot x_5\,,$$

$$v^*(x_3)=-x_3,\ v^*(x_5)=-x_5 \quad \textit{and} \quad v^*(x_3\cdot x_5)=x_3\cdot x_5\,.$$

PROOF. Since the diagram

$$\begin{array}{ccccc} S^3 & \xrightarrow{i} & SU(3) & \xrightarrow{p} & S^5 \\ \downarrow{\scriptstyle \iota_3} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle -\iota_5} \\ S^3 & \xrightarrow{i} & SU(3) & \xrightarrow{p} & S^5 \end{array} \tag{2.7}$$

commutes (up to homotopy) by the definition of c, we have $c^*(x_3)=x_3$, $c^*(x_5)=-x_5$ and also $c^*(x_3\cdot x_5)=c^*(x_3)c^*(x_5)=-x_3\cdot x_5$. q.e.d.

Since $f_8^*: H^8(X_8;\ Z_{12})\to H^8(SU(3);\ Z_{12})$ and $\phi_8: \mathscr{E}(SU(3))\to\mathscr{E}(X_8)$ of (1.2) are isomorphic, we can define the elements $\xi_n\in\mathscr{E}(X_n)$ $(n\geqq 8)$ by

$$\xi_8=\kappa((f_8^*)^{-1}(x_3\cdot x_5)),\quad \xi_n=\phi_n\phi_8^{-1}(\xi_8)\,, \tag{2.8}$$

where $x_i\in H^i(SU(3);\ Z_{12})$ is the mod 12 reduction of x_i in (2.5). We also define

$$\xi=\phi_8^{-1}(\xi_8)\in\mathscr{E}(SU(3))\,. \tag{2.8$'$}$$

PROPOSITION 2.9. *Put* $c_n=\phi_n(c)\in\mathscr{E}(X_n)$ *and* $v_n=\phi_n(v)\in\mathscr{E}(X_n)$, *where* c *and* v *are the maps in the above lemma. Then*

(i) $\mathscr{E}(X_n)=Z_2$ *with generator* v_n *for* $n=3, 4$,

(ii) $\mathscr{E}(X_n)=Z_2\times Z_2$ *with generators* c_n *and* v_n *for* $5\leqq n\leqq 7$,

(iii) $\mathscr{E}(X_n)=D_{12}\times Z_2$ *for* $n\geqq 8$, *where* $D_{12}=D_{12}(\xi_n, v_n)$, *and the second factor* Z_2 *is generated by* c_n. *Here* $D_i(a, b)$ *is the dihedral group of order* $2i$ *generated by* a *and* b *with the relations* $a^i=1=b^2$, $ab=ba^{-1}$.

(iv) $\mathscr{E}(SU(3))=D_{12}\times Z_2$, *where* $D_{12}=D_{12}(\xi, v)$, *and the second factor* Z_2 *is generated by* c.

PROOF. We shall compute $\mathscr{E}(X_n)$ by using Lemma 1.6 repeatedly. (i) follows from the facts $X_n=K(Z, 3)$, $\operatorname{Aut} Z=Z_2$ and Lemma 2.6.

(ii) We have easily $H^n(X_{n-1};\ \pi_n(SU(3)))=0$ for $5\leqq n\leqq 7$, and so $\mathscr{E}(X_n)$ is isomorphic to the subgroup G_n of $\mathscr{E}(X_{n-1})\times\operatorname{Aut}\pi_n(SU(3))$ in Lemma 1.6. k^6 is the generator of $H^6(X_4;\ \pi_5(SU(3)))=Z_2$ by Proposition 2.4 and $\operatorname{Aut}\pi_5(SU(3))=Z_2$ by Lemma 2.2. Therefore from the definition of G_n, we have $G_5=Z_2\times Z_2$ and so $\mathscr{E}(X_5)=Z_2\times Z_2$ which is generated by c_5 and v_5 from Lemma 2.6. Since k^7 is the generator of $H^7(X_5;\ \pi_6(SU(3)))=Z_6$ by Proposition 2.4 and $\operatorname{Aut}\pi_6(SU(3))=Z_2$ by Lemma 2.2, we have

$$h^*k^7=k^7 \quad \text{or} \quad -k^7 \qquad (h\in\mathscr{E}(X_5))\,,$$

and $k^7 \neq -k^7$. Therefore from the definition of G_n and from the fact $X_7 = X_6$, we have $\mathscr{E}(X_7) = \mathscr{E}(X_6) \simeq \mathscr{E}(X_5)$.

(iii) and (iv) By Proposition 2.4, k^9 is the generator of $H^9(X_7; \pi_6(SU(3))) = Z_{12}$. Since the elements of order 12 of $H^9(X_7, \pi_6(SU(3)))$ are $\pm k^9$ and $\pm 5k^9$, we have the following equality

$$h^* k^9 = \varepsilon k^9 \qquad (\varepsilon \in \mathrm{Aut}\, Z_{12} = \{\pm 1, \pm 5\})$$

for each $h \in \mathscr{E}(X_7)$. Therefore, from the definition of G_n in Lemma 1.6, we have $G_8 \simeq \mathscr{E}(X_7) \simeq Z_2 \times Z_2$. Next, we shall compute the subgroup H_8 of $\mathscr{E}(X_8)$ in Lemma 1.6. We have easily $H^8(X_7; \pi_8(SU(3))) = Z_{12}$ by (1.1). Let $L = S^3 \cup e^5$ be the 5-skeleton of $SU(3)$ and let $S^7 \xrightarrow{\beta} L \xrightarrow{j} SU(3)$ be the cofibering. Then we have the following commutative diagram of the exact sequences:

$$\begin{array}{ccccccc}
 & & [SU(3), \Omega X_8] & \xrightarrow{j^*} & [L, \Omega X_8] & \xrightarrow{\beta^*} & \pi_7(\Omega X_8) \\
 & & \downarrow{\scriptstyle (\Omega p_8)_*} & & \downarrow{\scriptstyle (\Omega p_8)_*} & & \\
\pi_8(\Omega X_7) & \longrightarrow & [SU(3), \Omega X_7] & \xrightarrow{j^*} & [L, \Omega X_7] & \xrightarrow{\beta^*} & \pi_7(\Omega X_7) \\
 & & \downarrow{\scriptstyle (\Omega k^9)_*} & & \downarrow{\scriptstyle (\Omega k^9)_*} & & \\
 & & H^8(SU(3); Z_{12}) & & H^8(L; Z_{12}), & &
\end{array}$$

Since $\pi_8(\Omega X_7) = \pi_7(\Omega X_7) = 0$ in this diagram, the lower j^* is isomorphic. Since $H^8(L; Z_{12}) = 0$, the right $(\Omega p_8)_*$ is epimorphic. Since $S\beta\colon S^8 \to SL$ passes through S^4 by [4, (3.1)] and $\pi_4(X_8) = 0$, we see that $(S\beta)^* = 0\colon [SL, X_8] \to \pi_8(X_8)$, which is equivalent to the triviality of the upper β^*. Hence the upper j^* is epimorphic. From these facts, the left $(\Omega p_8)_*$ is epimorphic. Therefore, $(\Omega k^9)_*[SU(3), \Omega X_7] = 0$. This shows $(\Omega k^9)_*[X_8, \Omega X_7] = 0$ because $f_8^*\colon H^8(X_8; Z_{12}) \to H^8(SU(3); Z_{12})$ is monomorphic. So, we have $H_8 = p_8^* H^8(X_7; \pi_8(SU(3)))/(\Omega k^9)_*[X_8, \Omega X_7] = Z_{12}$. Thus the short exact sequence in Lemma 1.6 is given as follows:

$$0 \longrightarrow Z_{12} \longrightarrow \mathscr{E}(X_8) \longrightarrow \mathscr{E}(X_7) = Z_2 \times Z_2 \longrightarrow 1\,.$$

Therefore $\mathscr{E}(X_8)$ is generated by the three elements $\xi_8 = \kappa(x)$ $(x = f_8^{*-1}(x_3 \cdot x_5))$, c_8 and v_8 of order 12, 2 and 2, respectively. By Lemma 2.2 and (2.7), we have $c_* = -1\colon \pi_8(SU(3)) \to \pi_8(SU(3))$. Thus $c_7^* k^9 = -k^9$ by [6, Th. 2.2] and hence $k(c_8 \times (-\iota)) = c_8 k$. Also we have $c_8^* x = -x$ by Lemma 2.6. These facts show that

$$\begin{aligned}
\xi_8 c_8 &= k(1 \times x)(c_8 \times c_8) d \\
&= k(c_8 \times (-\iota))(1 \times x) d \\
&= c_8 k(1 \times x) d = c_8 \xi_8.
\end{aligned}$$

Obviously $v_* = -1\colon \pi_8(SU(3)) \to \pi_8(SU(3))$ and $v_7^* k^9 = -k^9$. Also, we have $v_8^* x = x$ by Lemma 2.6. Therefore by the similar way as the above, we have $\xi_8 v_8 = v_8(\xi_8)^{-1}$.

These and the facts $c_8 v_8 = \phi_8(cv) = \phi_8(vc) = v_8 c_8$ show the desired results for $\mathscr{E}(X_8)$. Since $\dim SU(3) = 8$, (iv) and (iii) for $n \geqq 9$ are obtained by (1.4). q.e.d.

REMARK 2.10. The above (iv) gives a different proof of our previous result [10, Example 4.5]. For the element $\lambda(\alpha)$ defined in [10, Example 4.5] by using the cofibering $L \xrightarrow{j} SU(3) \to S^8$ and for the element ξ defined in the above (2.8)' by using the fibering $K(Z_{12}, 8) \xrightarrow{i_8} X_8 \xrightarrow{p_7} X_7$, the following equality holds:

$$\lambda(\alpha) = \xi^k \qquad \text{for some} \quad 0 \leqq k \leqq 11,$$

where k can be determined uniquely for $\alpha \in \pi_8(SU(3))$ as satisfying $\phi_8(\alpha\pi) f_8 = i_8 k(x_3 \cdot x_7)$ in $[X_8, X_8]$.

§ 3. *H*-equivalences of $SU(3)$ and $U(3)$

In this section, we shall determine the subgroup $\mathscr{E}_H(X_n)$ of $\mathscr{E}(X_n)$ by using the results for $\mathscr{E}(X_n)$ of the previous section, and we obtain the following theorem which is the one of our main results.

THEOREM 3.1. *Let $\{X_n\}$ be the Postnikov system of the special unitary group $SU(3)$. Then, we have*

(i) $\mathscr{E}_H(X_n) = Z_2$, *generated by* v_n *for* $n = 3, 4$,

(ii) $\mathscr{E}_H(X_5) = Z_2 \times Z_2$, *generated by* c_5 *and* v_5,

(iii) $\mathscr{E}_H(X_n) = Z_2$, *generated by* c_n *for* $n \geqq 6$,

(iv) $\mathscr{E}_H(SU(3)) = Z_2$, *generated by* c,

where c_n and v_n are the elements of $\mathscr{E}(X_n)$ defined in Proposition 2.9 and c and v are the elements of $\mathscr{E}(SU(3))$ defined in Lemma 2.6.

PROOF. Since $[X_n \wedge X_n, X_n] = 0$ for $n \leqq 5$, we see $\mathscr{E}_H(X_n) = \mathscr{E}(X_n)$ by [12, Prop. 2.7]. Therefore (i) and (ii) are obtained by Proposition 2.9 (i) and (ii).

Since $c \in \mathscr{E}_H(SU(3))$, we have

$$(3.2) \qquad c_n \in \mathscr{E}_H(X_n) \qquad \text{for all} \quad n.$$

Let $i\colon S^3 \to SU(3)$ be the inclusion in (2.1). By the definition of v_6, we have $v_6 f_6 i = -f_6 i$. Assume that $v_6 \in \mathscr{E}_H(X_6)$. Then, this equality shows that $-f_6 i$ is an H-map, and hence $(f_6 i)_* \phi = 0$ in $[S^3 \times S^3, X_6]$, where $\phi \in [S^3 \times S^3, S^3]$ is the commutator map. Let $\pi\colon (S^3 \times S^3, S^3 \vee S^3) \to (S^6, *)$ be the collapsing map, and consider the commutative diagram

$$\begin{array}{ccc} \pi_6(S^3) & \xrightarrow{i_*} \pi_6(SU(3)) \xrightarrow[\cong]{f_{6*}} & \pi_6(X_6) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ [S^3 \times S^3, S^3] & \xrightarrow{(f_6 i)_*} & [S^3 \times S^3, X_6]\,. \end{array}$$

In this diagram, $\pi_6(S^3)=Z_{12}$, $\pi_6(SU(3))=Z_6$, i_* is an epimorphism by Lemma 2.2 and π^*'s are monomorphisms. Also it is clear that $\phi \in \pi^*(\pi_6(S^3))$. Therefore $(f_6 i)_* \phi = 0$ implies $6\phi=0$, which is contradictory to the result of I. M. James [3, p. 176]. Thus $v_6 \notin \mathscr{E}_H(X_6)$. By Proposition 2.9 (ii), c_6 and v_6 generate the group $\mathscr{E}(X_6)=Z_2 \times Z_2$ and we obtain (iii) for $n=6, 7$.

Since $X_8 \wedge X_8$ is 5-connected and $\pi_i(\Omega X_7)=0$ for $i \geqq 6$, it holds $[X_8 \wedge X_8, \Omega X_7]=0$ and hence the assumption of Lemma 1.9 is satisfied for $f=k^9$, $X=X_7$, $Y=K(Z_{12}, 9)$. Then, for $\alpha \in H^8(X_8; Z_{12})$,

$$\kappa(\alpha) \in \mathscr{E}_H(X_8) \quad \text{if and only if} \quad \alpha \in [X_8, K(Z_{12}, 8)]_H\,.$$

The later condition means that α is primitive in $H^*(X_8; Z_{12})$ (cf. [16, Th. 10.1]). But $H^8(SU(3); Z_{12})$ has no non-trivial primitive element and hence $H^8(X_8; Z_{12})$ is also so. Therefore

$$\kappa(\alpha) \notin \mathscr{E}_H(X_8) \qquad \text{for any} \quad \alpha(\neq 0) \in H^8(X_8; Z_{12})\,. \tag{3.3}$$

We see that $\kappa(\alpha) f_8 i = f_8 i$ by the definition of κ and so $\kappa(\alpha) v_8 f_8 i = \kappa(\alpha) f_8 v i = -f_8 i$. Thus, in the same way as the above proof of $v_6 \notin \mathscr{E}_H(X_6)$, we have

$$\kappa(\alpha) v_8 \notin \mathscr{E}_H(X_8) \qquad \text{for any} \quad \alpha \in H^8(X_8; Z_{12})\,. \tag{3.4}$$

From (3.2–4), we have

$$\kappa(\alpha) c_8 \notin \mathscr{E}_H(X_8) \qquad \text{for} \quad \alpha(\neq 0) \in H^8(X_8; Z_{12})\,,$$

$$\kappa(\alpha) v_8 c_8 \notin \mathscr{E}_H(X_8) \qquad \text{for} \quad \alpha \in H^8(X_8; Z_{12})\,.$$

Thus, the proof of (iii) for $n=8$ is completed.

For $n \geqq 9$, $\tilde{\phi}_8 \colon \mathscr{E}_H(X_n) \to \mathscr{E}_H(X_8)$ is monomorphic by Theorem 1.3, and $\mathscr{E}_H(X_n)$ contains the non-trivial element c_n by (3.2). Hence, $\tilde{\phi}_8$ is isomorphic, and the proofs of (iii) for $n \geqq 9$ and (iv) are also completed. q. e. d.

For the unitary group $U(n)$, we have the following

THEOREM 3.5. (i) *The natural map* $\rho \colon S^1 \times SU(n) \to U(n)$, $\rho(\alpha, \beta)=\alpha\beta$ $(\alpha \in S^1 = SU(1),\ \beta \in SU(n))$, *is an H-equivalence.*

(ii) $\mathscr{E}_H(U(n))=Z_2 \times \mathscr{E}_H(SU(n))$, *and* $\mathscr{E}_H(U(3))=Z_2 \times Z_2$.

PROOF. (i) The homotopy $\rho_t \colon S^1 \times SU(n) \to U(n)$, given by

$$\rho_t(\alpha, \beta)=\alpha^{1-t}\beta\alpha^t \qquad (\alpha \in S^1, \beta \in SU(n)),$$

satisfies $\rho_0=\rho$ and $\rho_1(\alpha, \beta)=\beta\alpha$. This implies that ρ is an H-map.

(ii) follows immediately from (i), [12, Example 3.12 (i)] and Theorem 3.1.

q. e. d.

§4. The Postnikov system of $Sp(2)$

In the rest of this paper, let $\{X_n, f_n: Sp(2)\to X_n, p_n: X_n\to X_{n-1}\}$ be the Postnikov system of the symplectic group $Sp(2)$, and compute the groups $\mathscr{E}(X_n)$ and $\mathscr{E}(Sp(2))$.

There is the principal bundle

$$S^3 \xrightarrow{i} Sp(2) \xrightarrow{p} S^7 \tag{4.1}$$

with the characteristic element $\omega\in\pi_6(S^3)$, and we have

LEMMA 4.2 (M. Mimura and H. Toda [7]).

$\pi_3(Sp(2))=Z$ *with generator* $i_*\iota_3$,

$\pi_4(Sp(2))=Z_2$, $\pi_5(Sp(2))=Z_2$,

$\pi_7(Sp(2))=Z$ *with generator* $[12\iota_7]$,

$\pi_{10}(Sp(2))=Z_{120}$ *with generator* $i_*\alpha_2+i_8\alpha_{1,5}+[\nu_7]$,

$\pi_i(Sp(2))=0$ *otherwise for* $i\leqq 10=\dim Sp(2)$,

where $[\alpha]$ *denotes an element such that* $p_*[\alpha]=\alpha\in\pi_i(S^7)$, *and the elements* $i_*\alpha_2$, $i_*\alpha_{1,5}$ *and* $[\nu_7]$ *in* $\pi_{10}(Sp(2))$ *are of order* 3, 5 *and* 8, *respectively.*

From this lemma, the partial Postnikov system $\{X_n\}_{n\leqq 10}$ of $Sp(2)$ is given by the diagram

$$\begin{array}{ccccccccccccccc} X_{10} & \xrightarrow{p_{10}} & X_9 & \xrightarrow[=]{p_9} & X_8 & \xrightarrow[=]{p_8} & X_7 & \xrightarrow{p_7} & X_6 & \xrightarrow[=]{p_6} & X_5 & \xrightarrow{p_5} & X_4 & \xrightarrow{p_4} & X_3=K(Z,3) \\ & & \downarrow{\scriptstyle k^{11}} & & & & & & \downarrow{\scriptstyle k^8} & & & & \downarrow{\scriptstyle k^6} & & \downarrow{\scriptstyle k^5} \\ & & K(Z_{120},11) & & & & & & K(Z,8) & & & & K(Z_2,6) & & K(Z_2,5), \end{array} \tag{4.3}$$

and $X_2=X_1=*$.

PROPOSITION 4.4. *In the above diagram, the group* $H^{n+1}(X_{n-1}; \pi_n(Sp(2)))$, $n\leqq 10$, *is given by*

n	1, 2, 3	4	5	6	7	8, 9	10
$H^{n+1}(X_{n-1}; \pi_n(Sp(2)))$	0	Z_2	Z_2	0	Z_{12}	0	Z_{120}

and the k-invariant k^{n+1} generates this group.

PROOF. In the same way as Proposition 2.4, this proposition is proved easily. q. e. d.

As is well known, the integral cohomology ring of $Sp(2)$ is the exterior algebra

(4.5) $$H^*(Sp(2)) = \wedge(x_3, x_7), \quad \deg x_i = i,$$

and each generator x_i is primitive. Then, we have

LEMMA 4.6. *Let v: $Sp(2) \to Sp(2)$ be the map given by $v(\alpha) = \alpha^{-1}$. Then,*

$$v^*(x_i) = -x_i \quad \textit{for} \quad i = 3, 7 \quad \textit{and} \quad v^*(x_3 \cdot x_7) = x_3 \cdot x_7 .$$

Since f_{10}^*: $H^{10}(X_{10}; Z_{120}) \to H^{10}(Sp(2); Z_{120})$ and ϕ_{10}: $\mathscr{E}(Sp(2)) \to \mathscr{E}(X_{10})$ are isomorphic, we can define the elements $\xi_n \in \mathscr{E}(X_n)$ $(n \geqq 10)$ by

(4.7) $$\xi_{10} = \kappa((f_{10}^*)^{-1}(x_3 \cdot x_7)), \quad \xi_n = \phi_n \phi_{10}^{-1}(\xi_{10}),$$

where $x_i \in H^i(Sp(2); Z_{120})$ is the mod 120 reduction of x_i in (4.5). We also define

(4.7)′ $$\xi = \phi_{10}^{-1}(\xi_{10}) \in \mathscr{E}(Sp(2)) .$$

PROPOSITION 4.8. *Put $v_n = \phi_n(v) \in \mathscr{E}(X_n)$, where v is the map in the above lemma. Then*

(i) *$\mathscr{E}(X_n) = Z_2$ with generator v_n for $3 \leqq n \leqq 9$,*

(ii) *$\mathscr{E}(X_n)$ is the dihedral group $D_{120}(\xi_n, v_n)$ for $n \geqq 10$, with generators v_n and ξ_n in* (4.7),

(iii) *$\mathscr{E}(Sp(2))$ is the dihedral group $D_{120}(\xi, v)$ with generators v and ξ in* (4.7)′.

PROOF. We shall compute $\mathscr{E}(X_n)$ by using Lemma 1.6 repeatedly. Since $Sp(2) = S^3 \cup e^7 \cup e^{10}$, we have easily $H^n(X_{n-1}; \pi_n(Sp(2))) = 0$ for $n \leqq 9$ by (1.1) and Lemma 4.2, and so $\mathscr{E}(X_n)$ is isomorphic to the subgroup G_n of $\mathscr{E}(X_{n-1}) \times \mathrm{Aut}\, \pi_n(Sp(2))$ in Lemma 1.6. Using Lemma 4.2 and Proposition 4.4, we see easily that G_n is isomorphic to $\mathscr{E}(X_{n-1})$ for $4 \leqq n \leqq 10$. Hence $\mathscr{E}(X_9) \simeq \mathscr{E}(X_8) \simeq \cdots \simeq \mathscr{E}(X_3) = \mathscr{E}(K(Z, 3)) = Z_2$ and (i) is proved.

By easy calculations, we see $H_{10} = H^{10}(X_{10}; Z_{120}) = Z_{120}$ generated by $x = (f_{10}^*)^{-1}(x_3 \cdot x_7)$. So, we have the short exact sequence

$$1 \longrightarrow Z_{120} \xrightarrow{\kappa} \mathscr{E}(X_{10}) \longrightarrow \mathscr{E}(X_9) = Z_2 \longrightarrow 1 ,$$

and $\mathscr{E}(X_{10})$ is generated by the two elements $\xi_{10} = \kappa(x)$ $(x = f_{10}^*(x_3 \cdot x_7))$ and v_{10} of order 120 and 2, respectively. By the similar way as in the proof of Proposition

2.9 (iii), we have $\xi_{10}v_{10}=v_{10}(\xi_{10})^{-1}$ by Lemma 4.6. Thus we obtain the result for $\mathscr{E}(X_{10})$.

Since $\dim Sp(2)=10$, (iii) and (ii) for $n\geqq 11$ are obtained by (1.4). q.e.d.

REMARK 4.9. The above (iii) gives a different proof of our previous result [10, Example 4.5]. Furthermore, a similar equality as Remark 2.10 holds for the element $\lambda(\alpha)$ defined in [10, Example 4.5] by using the cofibering $S^3\cup e^7\xrightarrow{j}Sp(2)\xrightarrow{\pi}S^{10}$ and for the element ξ defined in (4.7)$'$ by using the fibering $K(Z_{120}, 10)\xrightarrow{i_{10}}X_{10}\xrightarrow{p_{10}}X_9$.

§5. *Sp*(2) localized at 5 and *H*-equivalences of *Sp*(2)

For a simply connected *CW*-complex X, we denote its localization at 5 by $X_{(5)}$ (cf. [8]). In this section, we shall study the localization $Sp(2)_{(5)}$ of $Sp(2)$ at 5, and compute the group $\mathscr{E}_H(Sp(2))$.

PROPOSITION 5.1. *Any loop multiplication on* $Sp(2)_{(5)}$ *is not homotopy commutative. In particular, the usual multiplication* m *on* $Sp(2)$ *induces the multiplication* $m_{(5)}$ *on* $Sp(2)_{(5)}$, *which is not homotopy commutative.*

PROOF. Put $Sp(2)_{(5)}=\Omega Y$. Let $E\colon S\Omega Y\to Y$ be the evaluation map. Then, by [13, p. 501], we have

$$(5.2)\qquad \begin{aligned} &H^*(Y;Z_5)=Z_5[y_4, y_8], \quad \deg y_i=i,\\ &E^*(y_i)=\sigma(x_{i-1}), \end{aligned}$$

where $\sigma\colon \tilde{H}^i(\Omega Y;Z_5)\to\tilde{H}^{i+1}(S\Omega Y;Z_5)$ is the suspension isomorphism and x_i is the mod 5 reduction of x_i in (4.5).

Assume that the loop multiplication on $\Omega Y=Sp(2)_{(5)}$ is homotopy commutative. Then, by J. D. Stasheff [15, Th. 1.10], there is an extension $f\colon S\Omega Y\times S\Omega Y\to Y$ of $E\vee E\colon S\Omega Y\vee S\Omega Y\to Y$, and hence we have the commutative diagram

$$(5.3)\qquad \begin{array}{ccc} H^*(Y;Z_5) & \xrightarrow{E^*} & H^*(S\Omega Y;Z_5)\\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \nabla^*}\\ H^*(S\Omega Y\times S\Omega Y;Z_5) & \xrightarrow{j^*} & H^*(S\Omega Y\vee S\Omega Y;Z_5) \end{array}$$

where ∇ is the folding map and j is the inclusion. Since $H^*(S\Omega Y;Z_5)$ is generated by 1, $\sigma x_{i-1}=E^*y_i$ $(i=4, 8)$ and $\sigma(x_3\cdot x_7)$, the commutativity of (5.3) shows that

$$(5.4)\qquad f^*(y_4)=\sigma x_3\otimes 1+1\otimes\sigma x_3,$$

and $f^*(y_8)=\sigma x_7\otimes 1+1\otimes\sigma x_7 \mod \sigma x_3\otimes\sigma x_3$. But $\sigma x_3\otimes\sigma x_3=(1/2)f^*(y_4^2)$ by (5.4), and so we can replace y_8 such that it satisfies

(5.5) $$f^*(y_8)=\sigma x_7\otimes 1+1\otimes\sigma x_7.$$

Consider the reduced power operation $\mathscr{P}^1$ on $H^*(Y; Z_5)$. By the dimensional reason, we have

$$\mathscr{P}^1 y_4\equiv 0 \mod (y_4), \quad \mathscr{P}^1 y_8\equiv\alpha y_8^2 \mod (y_4)$$

for some $\alpha\in Z_5$, where (y_4) is the ideal generated by y_4 and is invariant under $\mathscr{P}^1$. Since $(\mathscr{P}^1)^4 y_8=-\mathscr{P}^4 y_8=-y_8^5$, the coefficient α is non-trivial.

By (5.5) and (5.4), we see that $f^*(y_8^2)=2\sigma x_7\otimes\sigma x_7$ and $f^*u=0$ for other monomial u of degree 16. Hence $f^*(\mathscr{P}^1 y_8)=2\alpha(\sigma x_7\otimes\sigma x_7)\neq 0$. On the other hand, $\mathscr{P}^1\sigma x_{i-1}=\sigma\mathscr{P}^1 x_{i-1}=0$, and so $\mathscr{P}^1 f^*(y_8)=0$ by (5.5). These facts are contradictory to the naturality of $\mathscr{P}^1$. Thus, the loop multiplication on ΩY is not homotopy commutative. q.e.d.

Proposition 5.6. *The multiplication on $(X_n)_{(5)}$ induced from m on $Sp(2)$ is not homotopy commutative for $n\geqq 14$.*

Proof. Let $\{W_n, q_n, g_n\}$ be the Postnikov system of $BSp(2)$. Then, $\{\Omega W_{n+1}, \Omega q_{n+1}, \Omega g_{n+1}\}$ is the Postnikov system of $Sp(2)$, and it is easy to see that $\{(\Omega W_{n+1})_{(5)}, (\Omega q_{n+1})_{(5)}, (\Omega g_{n+1})_{(5)}\}$ is the one of $Sp(2)_{(5)}$. Furthermore $(\Omega W)_{(5)}=\Omega(W_{(5)})$ by [8, Prop. 3.3]. On the other hand, $(W_{15})_{(5)}=(W_{16})_{(5)}=(W_{17})_{(5)}$ by [7], and we see that $(g_{15})_{(5)}^*: H^i((W_{15})_{(5)}; Z_5)\to H^i(BSp(2)_{(5)}; Z_5)$ is isomorphic for $i\leqq 17$. Therefore, we can proved the proposition in the same way as the above proof, by replacing Y by $(W_n)_{(5)}$. q.e.d.

The rest of this paper is devoted to prove the following main theorem.

Theorem 5.7. *Let $\{X_n\}$ be the Postnikov system of the symplectic group $Sp(2)$. Then, we have*

(i) $\mathscr{E}_H(X_n)=Z_2$ *with generator* v_n *for* $3\leqq n\leqq 9$,

(ii) $\mathscr{E}_H(X_n)=1$ *or* Z_2 *with generator* v_n *or* Z_2 *with generator* $\xi_n^{60}v_n$ *for* $10\leqq n\leqq 13$,

(iii) $\mathscr{E}_H(X_n)=1$ *for* $n\geqq 14$,

(iv) $\mathscr{E}_H(Sp(2))=1$,

where v_n and ξ_n are defined in Proposition 4.8.

Proof. (i) Since we have easily $[X_n\wedge X_n, X_n]=0$, it holds $\mathscr{E}_H(X_n)=\mathscr{E}(X_n)$ by [12, Prop. 2.7]. Therefore (i) is obtained by Proposition 4.8 (i).

(ii) We see easily that $\Omega p_{10*}: [Sp(2)\wedge Sp(2), \Omega X_{10}]\to[Sp(2)\wedge Sp(2), \Omega X_9]$ is epimorphic since $Sp(2)=S^3\cup e^7\cup e^{10}$. This implies that $\Omega p_{10*}: [X_{10}\wedge X_{10},$

$\Omega X_{10}]\to[X_{10}\wedge X_{10}, \Omega X_9]$ is also so. Hence the assumption of Lemma 1.9 is satisfied for $f=k^{11}$, $X=X_{10}$, $Y=K(Z_{120}, 11)$. By the similar discussion to the proof of (3.3), we can prove that

$$\xi_{10}^k \notin \mathscr{E}_H(X_{10}) \qquad \text{for any} \quad 0<k<120. \tag{5.8}$$

Assume that $\xi_n^k v_n \in \mathscr{E}_H(X_n)$ for some $0\leqq k<120$ and $n\geqq 10$. Then, $(\xi_n^k v_n)v_n = v_n(\xi_n^k v_n)$ and so $k=0 \bmod 60$ by Proposition 4.8 (ii). This and (5.8) show (ii).

(iii) Assume that $\xi_n^k v_n \in \mathscr{E}_H(X_n)$. Then $(v_n)_{(5)}$ is an H-map since $(\xi_n^k v_n)_{(5)} = v_{(5)}$ for $k=0 \bmod 60$. Thus the multiplication on $(X_n)_{(5)}$ is homotopy commutative. This is contradictory to the above proposition if $n\geqq 14$. Hence we have

$$\xi_n^k v_n \notin \mathscr{E}_H(X_n) \qquad \text{for any} \quad 0\leqq k<120.$$

By this result, (5.8) and Proposition 4.8 (ii), we have (iii).

(iv) The result is obvious by Theorem 1.3 and (iii). q. e. d.

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