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On the Reflectionless Solutions of the Modified Korteweg-de Vries Equation

By

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In [2], the present author has studied the initial value problem for the modified Korteweg-de Vries (KdV) equation

$$(1) \qquad v_t - 6v^2 v_x + v_{xxx} = 0, \quad -\infty < x, \ t < \infty$$

with the step type initial data which tend to $\pm m$ as $x \to \pm\infty$ for some positive constant m. We have constructed the smooth real valued solutions of (1) in terms of the scattering data of the Dirac operator

$$L_{iv} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D + i \begin{bmatrix} 0 & -v \\ v & 0 \end{bmatrix}, \quad D = d/dx.$$

In the present paper, we discuss the asymptotic properties of the reflectionless solution $v_n^s(x, t)$ as $t \to \pm\infty$, where the reflectionless solution $v_n^s(x, t)$ is the solution of (1) which is constructed from the reflectionless scattering data with $2n+1$ discrete eigenvalues.

The solution $v_0^s(x, t)$ takes the form of the traveling wave solution

$$(2) \qquad v_0^s(x, t) = m \tanh\{m(x + 2m^2 t + \delta)\}.$$

We call the function which takes the form $s(x+kt+\delta, c)$ $(k, c>0)$ the soliton, where $s(x, c) = c\tanh(cx)$. The main result of the present paper shows that the reflectionless solution $v_n^s(x, t)$ decomposes into $2n+1$ solitons for large t.

In section 1, we summarize the general properties of the scattering data of L_{iv}. In section 2, we construct the reflectionless solutions of (1). In section 3, we study the asymptotic properties of the reflectionless solutions.

Throughout the paper, c^* denotes the complex conjugate of c.

§1. Scattering theory of L_{iv}

We summarize the general results of the scattering theory of L_{iv} from [2]. Consider the eigenvalue problem

(3) $$L_{iv}y=\lambda y,\ \ y={}^t(y_1,\ y_2)$$

on the real axis $(-\infty,\ \infty)$.

Let $\zeta=\zeta(\lambda)$ be the two-valued algebraic function defined by

$$\zeta^2=\lambda^2-m^2$$

and R be the upper leaf of the two-sheeted Riemann surface associated with ζ. We assume $\mathrm{Im}\,\zeta>0$ for $\lambda\in R$. For $\zeta\in \mathbf{R}_m=\mathbf{R}\setminus[-m,\ m]$, put

$$\sigma=\sigma(\xi)=(\mathrm{sgn}\,\xi)\,(\xi^2-m^2)^{1/2}.$$

If we assume

$$\pm\int_x^{\pm\infty}(1+|y|)|v(y)\mp m|\,dy+\sup_{\pm y>\pm x}|v(y)\ \mp m\ |<\infty,$$

then, for $\lambda\in R$, there are unique solutions $f_\pm$ of (3) (called Jost solutions) which behave as

$$f_\pm(x,\ \lambda)=f_\pm^o(x,\ \lambda)+o(1)$$

as $x\to\pm\infty$, where

$$f_+^o(x,\ \lambda)={}^t(im^{-1}(\zeta-\lambda),1)\exp(i\zeta x)$$
$$f_-^o(x,\ \lambda)={}^t(1,\ im^{-1}(\zeta-\lambda))\exp(-i\zeta x).$$

Then $f_\pm(x,\lambda)$ are analytic in $\lambda\in R$. Moreover we have the integral expression

(4) $$f_+(x,\lambda)=E(\lambda)\exp(i\zeta x)\,\{{}^t(0,\ 1)+\int_0^\infty K(x,\ y)\exp(2i\zeta y)\,dy\},$$
$$K={}^t(K_1,\ K_2),$$

where

$$E(\lambda)=\begin{bmatrix}1 & im^{-1}(\zeta-\lambda)\\ im^{-1}(\zeta-\lambda) & 1\end{bmatrix}.$$

Note that if y is a solution of (3), then

$$y^\#={}^t(y_2^*,\ y_1^*)$$

is a solution of (3), λ being replaced by λ^*.

Put

$$f_\pm(x,\ \xi)=f_\pm(x,\ \xi+i0),\ \ \xi\in\mathbf{R}_m,$$

and f_+ and $f_+^\#$ are linearly independent solution of (3). Therefore we can express as

(5) $$f_-(x,\ \xi)=a(\xi)f_+^{\#}(x,\ \xi)+b(\xi)f_+(x,\ \xi).$$

From (5), we have

$$a(\xi)=m^2\det(f_-,\ f_+)/2\sigma(\xi-\sigma),$$

so $a(\xi)$ can be extended to the analytic function

$$a(\lambda)=m^2\det(f_-(x,\ \lambda),\ f_+(x,\ \lambda))/2\zeta(\lambda-\zeta),\ \lambda\in R.$$

The coefficient $a(\lambda)$ does not vanish for $\lambda\in\mathbf{R}_m$ and has only a finite number of zeros $\pm\varkappa_j(j=0, 1, \cdots, n)$ which are simple, where $0=\varkappa_0<\varkappa_1<\cdots<\varkappa_n<m$. There are non zero real numbers d_j such that

$$f_-(x,\pm\varkappa_j)=d_jf_+(x,\ \pm\varkappa_j).$$

The constants $\pm\varkappa_j$ are eigenvalues of the eigenvalue problem (3) and corresponding eigenfunctions are given by $f_+(x,\ \pm\varkappa_j)$. Put

$$c_0=id_0/2a'(0)$$

$$c_j=imd_j/\eta_ja'(\varkappa_j),\quad j=1, 2, \cdots,\ n,$$

where $\eta_j=(m^2-\varkappa_j^2)^{1/2}$. The coefficients c_j are positive. Put

$$r(\xi)=b(\xi)/a(\xi)$$

(called the reflection coefficient). We call the collection

$$\{r(\xi),\ c_j,\ \varkappa_j,\ j=0, 1, \cdots,\ n\}$$

the scattering data of L_{iv}.

The scattering data are related to the kernel $K(x,\ y)$ (defined by (4)) by the integral equation (called the fundamental equation)

(6) $$K^\tau(x,\ y)+F(x+y)^t(0,\ 1)+\int_0^\infty F(x+y+z)K(x,\ z)dz=0\ \ (y>0),$$

where

$$F(x)=2\sum_{j=0}^n c_j\begin{bmatrix}-\eta_j/m & 1\\ 1 & -\eta_j/m\end{bmatrix}\exp(-2\eta_jx)$$
$$+\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r(\xi)\begin{bmatrix}\sigma & -im\\ -im & \sigma\end{bmatrix}\exp(2i\sigma x)\ d\sigma,$$

and $K^\tau={}^t(K_2,\ K_1)$. Then one can reconstruct the potential v from the scattering data by solving the fundamental equation as the integral equation for the kernel K and putting

(7) $$v(x)=-K_1(x,\ 0)+m.$$

If the reflection coefficient is identically zero, the potential v is more explicitly written by the scattering data as follows. The assumption $r(\xi)\equiv 0$ implies

$$(8)\qquad F(x)=2\sum_{j=0}^{n} c_j \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} \exp(-2\eta_j x).$$

Let $K(x, y)$ be the solution of the fundamental equation (5). Putting (8) into (6), we see that $K(x, y)$ has the form

$$K(x, y)=2\sum_{j=0}^{n} c_j \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} g_j(x)\exp(-2\eta_j(x+y)),$$

where $g_j(x)={}^t(g_{1j}(x), g_{2j}(x))$. Substitute this into the fundamental equation (6), and we have the system of $2(n+1)$ linear algebraic equations

$$(9)\qquad g_i(x)+\sum_{j=0}^{n} c_j \begin{bmatrix} -\eta_j/m & 1 \\ 1 & -\eta_j/m \end{bmatrix} (\eta_i+\eta_j)^{-1}\exp(-2\eta_j x) g_j(x)=-{}^t(1, 0), \quad (i=0, 1, \cdots, n),$$

whose coefficient matrix is easily seen to be nondegenerate. Let $g_{ij}(x)$ ($i=1, 2$ and $j=0, 1, \cdots, n$) be the unique solutions of (9). By (7) we have the reflectionless potential

$$(10)\qquad v_n^s(x)=2\sum_{j=0}^{n} c_j (m^{-1}\eta_j g_{1j}(x)-g_{2j}(x))\exp(-2\eta_j x)+m.$$

Put

$$h_{\pm j}(x)=c_j(1\mp m^{-1}\eta_j)\exp(-2\eta_j x)(g_{1j}(x)\pm g_{2j}(x)),$$

where $j=1, 2, \cdots, n$ for + and $j=0, 1, \cdots, n$ for -, and we have the another expression of (10)

$$(11)\qquad v_n^s(x)=\sum_{j=0}^{n} h_{-j}(x)-\sum_{j=1}^{n} h_{+j}(x)+m.$$

The formula (11) is more convinient for studying the asymptotic properties of the reflectionless solution. The functions $h_{\pm j}$ satisfy the systems of linear algebraic equations

$$(12\pm)\qquad a_{\pm i}^{-1}\exp(2\eta_i x)h_{\pm i}(x)+\sum_j(\eta_i+\eta_j)^{-1}h_{\pm j}(x)=-1,$$

where $a_{\pm i}=c_i(1\mp m^{-1}\eta_i)$. The coefficient matrices of (12±) are easily seen to be nondegenerate.

§2. Reflectionless solution

Let $v(t)=v(x, t)$ be a smooth solution of the modified KdV equation (1) which tend to $\pm m$ as $x\to\pm\infty$. We assume that v_x belongs to S, the space of C^∞-functions which are rapidly decreasing together with all its derivatives. In [2], it is shown that the eigenvalues of $L_{iv}(t)$ do not depend on t

and time dependencies of the coefficients $r(\xi)$ and c_j are given as

$$r(\xi, t)=r(\xi)\exp\{i(8\sigma^3+12m^2\sigma)t\}$$

and

$$c_j(t)=c_j\exp\{(8\eta_j^3-12m^2\eta_j)t\}. \tag{13}$$

Converse statements of this fact are valid (see the present author [2, Theorem 5.2]). Especially, we have

THEOREM 1. *Let $v_n^s(x, t)$ be the reflectionless potential which corresponds to the reflectionless scattering data*

$$\{0, c_j(t), \varkappa_j, j=0, 1, \cdots, n\}$$

for each t, where $c_j(t)$ is defined by (13). *Then $v_n^s(x, t)$ is a solution of the modified KdV equation* (1).

We call the solution $v_n^s(x, t)$ the reflectionless solution with $2n+1$ discrete eigenvalues.

Put

$$z_j=x+\rho_j^2 t,$$

where $\rho_j=(6m^2-4\eta_j^2)^{1/2}$. Note that ρ_j are positive and $\rho_i<\rho_j (i<j)$.

Put

$$b_{\pm ij}(x, t)=a_{\pm i}^{-1}\exp(2\eta_i z_i)\delta_{ij}+(\eta_i+\eta_j)^{-1}$$

and $h_{\pm j}(x, t)$ be the unique solutions of the systems of linear algebraic equations

$$\textstyle\sum_j b_{\pm ij}h_{\pm j} =-1 \tag{14}$$

whose coefficient matrices $A_\pm(x, t)=(b_{\pm ij}(x, t))$ are easily seen to be non-degenerate. Then by (11) and (12), we have the formula for the reflectionless solution with $2n+1$ discrete eigenvalues

$$v_n^s(x, t)=\textstyle\sum_{j=0}^n h_{-j}(x, t)-\sum_{j=1}^n h_{+j}(x, t)+m. \tag{15}$$

§3. Asymptotic properties of $v_n^s(x, t)$

The reflectionless solution $v_0^s(x, t)$ takes the form of the traveling wave solution

$$v_0^s(x, t)=s(x+2m^2t+\delta_0, m),$$

where

$$s(x, k)=k\tanh(kx)=k(\exp(kx)-\exp(-kx))/(\exp(kx)+\exp(-kx))$$

and

$$\delta_0 = (2m)^{-1}\log(m/c_0).$$

We call the function which takes the form $s(x+k_1t+\delta, k_2)(k_j>0)$ the soliton.

We now states the main result of the present paper.

THEOREM 2. *As* $t\to\pm\infty$, *the reflectionless solution* $v_n^s(x, t)$ *decomposes into* $2n+1$ *solitons;*

$$v_n^s(x, t) - s(x+2m^2t+\delta_0^{\pm}(+), m) - \sum_{j=1}^n \{s(x+\rho_j^2 t+\delta_j^{\pm}(+), \eta_j) - s(x+\rho_j^2t+\delta_j^{\pm}(-), \eta_j)\} \to 0$$

uniformly in x, where

$$\delta_j^+(+) = (2\eta_j)^{-1}\log\{2\eta_j m \Pi_{i=0}^{j-1}(\eta_i+\eta_j)^2/c_j(m+\eta_j)\Pi_{i=0}^{j-1}(\eta_i-\eta_j)^2\}(0\leqslant j)$$

$$\delta_j^+(-) = (2\eta_j)^{-1}\log\{2\eta_j m \Pi_{i=1}^{j-1}(\eta_i+\eta_j)^2/c_j(m-\eta_j)\Pi_{i=1}^{j-1}(\eta_i-\eta_j)^2\}(1\leqslant j)$$

$$\delta_j^-(+) = (2\eta_j)^{-1}\log\{2\eta_j m \Pi_{i=j+1}^{n}(\eta_j+\eta_i)^2/c_j(m+\eta_j)\Pi_{i=j+1}^{n}(\eta_j-\eta_i)^2\}(0\leqslant j)$$

$$\delta_j^-(-) = (2\eta_j)^{-1}\log\{2\eta_j m \Pi_{i=j+1}^{n}(\eta_j+\eta_i)^2/c_j(m-\eta_j)\Pi_{i=j+1}^{n}(\eta_j-\eta_i)^2\}(1\leqslant j).$$

PROOF. We express $h_{\pm j}(x, t)$ by the Cramer's formula as

$$h_{\pm j}(x, t) = D_{\pm k}(x, t)/\det A_{\pm}(x, t) \quad (k=j \text{ for } + \text{ and } k=j+1 \text{ for } -),$$

where $D_{\pm i}(x, t)$ are the determinants obtained by replacing the i-th column of det $A_{\pm}$ by ${}^t(-1, \cdots, -1)$. Note that $\det A_{\pm}$ are polynomials in $\exp(2\eta_i z_i)$ with positive coefficients and non-zero constant terms(Lemma 2 of [1]). We have

$$(16)\qquad |h_{\pm j}(x, t)| < C(1+\exp(2\eta_j z_j))^{-1}, \quad t>0.$$

Hence, $h_{\pm j}(x, t)$ converge to zero as $t\to\infty$ in the half space $x>(-\rho_j^2 t+\varepsilon)t$, $t>0$, where convergence is uniform. By (16), $v_n^s(x, t)$ converges to m as $t\to\infty$ uniformly in the half space $x<(-\rho_0^2+\varepsilon)t$, $t>0$, ε being sufficiently small positive number.

Now we consider the behavior of $h_{\pm j}(x, t)$ in the infinite sectors $(-\rho_k^2+\varepsilon)t<x<(-\rho_{k-1}^2-\varepsilon)t$, $t>0$, $(k>j)$. Let $\alpha_1, \cdots, \alpha_r$ be positive numbers different from each other. Put

$$D(\alpha_1, \cdots, \alpha_r) = \det((\alpha_p+\alpha_q)^{-1})_{p, q=1,\cdots,r}$$

and let $D_j^{\circ}(\alpha_1\cdots, \alpha_r)$ be the determinant obtained by replacing the i-th column of $D(\alpha_1, \cdots, \alpha_r)$ by ${}^t(1, \cdots, 1)$.

We express det $A_+(x, t)$ and $D_{+j}(x, t)$ as

$$\det A_+(x, t) = \sum_{i=k}^n a_{+i}^{-1}\exp(2\eta_i z_i)\, D(\eta_1, \cdots, \eta_{k-1})(1+P_k(x, t))$$

and

$$D_{+j}(x,\ t) = -\sum_{i=k}^{n} a_{+i}^{-1} \exp(2\eta_i z_i) D_j^o(\eta_1, \cdots, \eta_{k-1})(1+Q_k^j(x,\ t)).$$

Then we have the estimate

$$|P_k(x,\ t)| < C(\sum_{i=1}^{K-1} \exp(2\eta_i z_i) + \sum_{i=k}^{n} \exp(-2\eta_i z_i))$$

in the sector. The same estimate holds for $Q_k^j(x,\ t)$. Hence we have

$$h_{+j}(x,\ t) \to -D_j^o(\eta_1, \cdots, \eta_{k-1})/D(\eta_1, \cdots, \eta_{k-1}) \quad (t\to\infty,\ j<k),$$

where convergence is uniform in the infinite sector $(-\rho_k^2+\varepsilon)t<x<(-\rho_{k-1}^2-\varepsilon)t,\ t>0$. By the same way, we have

$$h_{-j}(x,\ t) \to -D_{j+1}^o(\eta_0, \eta_1, \cdots, \eta_{k-1})/D(\eta_0, \eta_1, \cdots, \eta_{k-1})\ (t\to\infty,\ j<k),$$

where convergence is uniform in the infinite sector. Hence we have

$$v_n^s(x,\ t) \to -\sum_{j=1}^{k} D_j^o(\eta_0, \cdots, \eta_{k-1})/D(\eta_0, \cdots, \eta_{k-1}) + \sum_{j=1}^{k-1} D_j^o(\eta_1, \cdots, \eta_{k-1})/D(\eta_1, \cdots \eta_{k-1}) + m \quad (t\to\infty),$$

where convergence is uniform in the infinite sector $(-\rho_k^2+\varepsilon)\ t<x<(-\rho_{k-1}^2-\varepsilon)t,\ t>0$.

By brief calculation, we have

$$D_j^o(\alpha_1, \cdots, \alpha_r)/D(\alpha_1, \cdots, \alpha_r) = 2\alpha_j \Pi_{i\neq j}(\alpha_j+\alpha_i)/(\alpha_j-\alpha_i).$$

Hence, we have

$$\sum_{j=1}^{k} D_j^o(\eta_0, \cdots, \eta_{k-1})/D(\eta_0, \cdots, \eta_{k-1}) - \sum_{j=1}^{k-1} D_j^o(\eta_1, \cdots, \eta_{k-1})/D(\eta_1, \cdots, \eta_{k-1}) = 2m\sum_{j=0}^{k-1} 2\eta_j D_j^o(\eta_0, \cdots, \eta_{k-1})/(\eta_j+m) D(\eta_0, \cdots, \eta_{k-1}). \tag{17}$$

$D_j^o(\eta_0, \cdots, \eta_{k-1})/D(\eta_0, \cdots, \eta_{k-1})\ (j=1, \cdots, k)$ are the solutions of the system of k linear algebraic equations

$$\sum_{j=0}^{k-1} (\eta_i+\eta_j)^{-1} X_j = 1 \quad (i=0, \cdots, k-1).$$

Therefore, the right hand side of (17) coincides with $2m$. This implies that $v_n^s(x,\ t)$ converges to $-m$ uniformly in the infinite sector $(-\rho_k^2+\varepsilon)t<x<(-\rho_{k-1}^2-\varepsilon)t,\ t>0$, as $t\to\infty$. Similar consideration is valid in the half space $x<(-\rho_n^2-\varepsilon)t,\ t>0$ and we have

$$v_n^s(x,\ t) \to -m \quad (t\to\infty),$$

where convergence is uniform.

Next we consider the behavior of $h_{\pm j}(x,\ t)$ in the infinite sector

$(-\rho_k^2-\varepsilon)t<x<(-\rho_k^2+\varepsilon)t$, $t>0$, for $j<k$ $(k=1, \cdots, n)$. In this case, we express $\det A_+(x, t)$ and $D_{+j}(x, t)$ as

$$\det A_+(x, t)=\sum_{i=k+1}^{n} a_{+i}^{-1}\exp(2\eta_i z_i)B_k(x, t)(1+R_k(x, t))$$

and

$$D_{+j}(x, t)=\sum_{i=k+1}^{n} a_{+i}^{-1}\exp(2\eta_i z_i)\,C_{kj}(x, t)(1+S_{kj}(x, t)),$$

where

$$B_k(x, t)=D(\eta_1, \cdots, \eta_{k-1})a_{+k}^{-1}\exp(2\eta_k z_k)+D(\eta_1, \cdots, \eta_k)$$

and $C_{kj}(x, t)$ is the determinant obtained by replacing the j-th column of $B_k(x, t)$ by ${}^t(-1, -1, \cdots, -1)$. Then we have the estimate

$$|R_k(x, t)|<C(\sum_{i=1}^{k-1}\exp(2\eta_i z_i)+\sum_{i=k+1}^{n}(-2\eta_i z_i))$$

in the sector. The same estimate holds for $S_{kj}(x, t)$. Similar consideration holds for $h_{-j}(x, t)$. Therefore, the reflectionless solution $v_n^s(x, t)$ behaves as

$$(18)\qquad -\frac{a_{-k}^{-1}\sum_{j=1}^{k}D_j^o(\eta_0, \cdots, \eta_{k-1})\exp(2\eta_k z_k)+\sum_{j=1}^{k+1}D_j^o(\eta_0, \cdots, \eta_k)}{a_{-k}^{-1}D(\eta_0, \cdots, \eta_{k-1})\exp(2\eta_k z_k)+D(\eta_0, \cdots, \eta_k)}$$

$$+\frac{a_{+k}^{-1}\sum_{j=1}^{k-1}D_j^o(\eta_1, \cdots, \eta_{k-1})\exp(2\eta_k z_k)+\sum_{j=1}^{k}D_j^o(\eta_1, \cdots, \eta_k)}{a_{+k}^{-1}D(\eta_1, \cdots, \eta_{k-1})\exp(2\eta_k z_k)+D(\eta_1, \cdots, \eta_k)}+m.$$

By direct calculation, we can show that the function (18) coincides with

$$(19)\qquad s(x+\rho_k^2 t+\delta_k^+(+), \eta_k)-s(x+\rho_k^2 t+\delta_k^+(-), \eta_k).$$

It is easy to see that the function (19) belongs to S for each t.

In the infinite sector $(-\rho_0^2-\varepsilon)t<x<(-\rho_0^2+\varepsilon)t$, $t>0$, as $t\to\infty$, $h_{-0}(x, t)$ behaves as $s(x+2m^2t+\delta_0^+(+), m)$ and $h_{\pm j}(x, t)$ $(j=1, \cdots, n)$ converge to zero uniformly.

The proof for the behavior of $v_n^s(x, t)$ as $t\to-\infty$ can be obtained by the parallel discussion to above. Q. E. D.

Moreover, we have the formulea for the phase shift of each soliton;

$$\delta_j^+(\pm)-\delta_j^-(\pm)=\eta_j^{-1}\sum_{i=0}^{j-1}\log\frac{\eta_i+\eta_j}{\eta_i-\eta_j}+\eta_j^{-1}\sum_{i=j+1}^{n}\log\frac{\eta_j-\eta_i}{\eta_j+\eta_i}\quad(j=0, \cdots, n)$$

$$\delta_j^\pm(+)-\delta_j^\pm(-)=\pm(2\eta_j)^{-1}\log\frac{m+\eta_j}{m-\eta_j}\quad(j=1, \cdots, n).$$

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