J. Math. Tokushima Univ.
Vol. 12 (1978), 19 - 42

Galerkin Approximations of Periodic Solution and its Period to van der Pol Equation

By

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(Received April 30, 1978)

§ 1. Introduction

In the previous paper [10], one of the authors has developed numerical procedure for the computation of periodic solutions and their periods to nonlinear autonomous differential systems, and computed a Chebyshev-series-approximation of periodic solution and its period $\omega(\lambda)$ to van der Pol equation

$$(1) \qquad \frac{d^2x}{d\tau^2} - \lambda(1-x^2)\frac{dx}{d\tau} + x = 0$$

with $\lambda = 0.01$. But, for not small λ, the order of finite Chebyshev series become too high. Hence, in the present paper, taking account of the symmetric character of the orbits of equation (1), we shall use the Galerkin method for computing the periodic solutions and their periods to equation (1) with $\lambda = 1 \sim 3$.

Numerical results with error estimation are shown in Tables $1 \sim 3$. Table 4 shows that the present results are better than Urabe et al.'s ones [2, 3], Krogdahl's ones [4] and Strasberg's ones [11].

§ 2. Basic Theorems

Now, by the transformation $\tau = \frac{\omega t}{2\pi}$, equation (1) is rewritten in the following form

$$(2) \qquad \frac{d^2x}{dt^2} - \lambda(1-x^2)\frac{\omega}{2\pi}\frac{dx}{dt} + \left(\frac{\omega}{2\pi}\right)^2 x = 0$$

and the problem is reduced to the one of finding a 2π-periodic solution of the boundary value problem:

$$(3)\qquad \begin{cases} \dfrac{dx}{dt} = \lambda^k y, \\ \dfrac{dy}{dt} = -\dfrac{1}{\lambda^k}\left(\dfrac{\omega}{2\pi}\right)^2 x + \dfrac{\omega}{2\pi}\lambda(1-x^2)y, \\ \dfrac{d\omega}{dt} = 0, \end{cases}$$

$$(4)\qquad \begin{cases} x(0) - x(2\pi) = 0, \\ y(0) - y(2\pi) = 0. \end{cases}$$

The boundary value problem (3)−(4) is clearly incomplete. Hence, we consider an additional condition of the form

$$(5)\qquad l(\boldsymbol{u}) \equiv \frac{1}{\pi}\int_0^{2\pi} x(t)\cos \hat{n}t\,dt = \beta,$$

where $\boldsymbol{u}(t) = \mathrm{col}[x(t),\ y(t),\ \omega(t)]$.
Then, the boundary value problem (3)−(5) can also be written briefly as

$$(6)\qquad \frac{d\boldsymbol{u}}{dt} = \boldsymbol{X}(\boldsymbol{u}),$$

$$(7)\qquad \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0},$$

where

$$(8)\qquad \boldsymbol{X}(\boldsymbol{u}) = \begin{pmatrix} \lambda^k y \\ -\dfrac{1}{\lambda^k}\left(\dfrac{\omega}{2\pi}\right)^2 x + \dfrac{\omega}{2\pi}\lambda(1-x^2)y \\ 0 \end{pmatrix},$$

$$(9)\qquad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} x(0) - x(2\pi) \\ y(0) - y(2\pi) \\ l(\boldsymbol{u}) - \beta \end{pmatrix}$$

$$= L_1\boldsymbol{u}(0) - L_1\boldsymbol{u}(2\pi) + L_2\int_0^{2\pi}\frac{1}{\pi}\boldsymbol{u}(t)\cos\hat{n}t\,dt - \boldsymbol{\beta} = \boldsymbol{0},$$

and

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\quad \boldsymbol{\beta} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}.$$

Let D be a domain in the $\boldsymbol{u}$-space. Consider a product space $\Omega=I\times D$, where $I=[0, 2\pi]$, and put

$$S=\{\boldsymbol{u}(t) \mid (t, \boldsymbol{u}(t))\in\Omega \text{ for all } t\in I, \boldsymbol{u}(t)\in M\equiv C^1[I]\},$$

$$S'=\{\boldsymbol{u}(t) \mid (t, \boldsymbol{u}(t))\in\Omega \text{ for all } t\in I, \boldsymbol{u}(t)\in C[I]\}.$$

We shall denote the Euclidean norm by $\|\cdot\|$, and for any $\boldsymbol{u}(t)\in C[I]$ we define its norm $\|\boldsymbol{u}\|_c$ by $\|\boldsymbol{u}\|_c=\sup_{t\in I}\|\boldsymbol{u}(t)\|$.

Consider a product space $N\equiv C[I]\times \boldsymbol{R}^{n+1}$; and for any $\boldsymbol{n}=[\boldsymbol{u}(t), \boldsymbol{v}]\in N$ we define its norm $|||\boldsymbol{n}|||$ by

$$|||\boldsymbol{n}|||=\|\boldsymbol{u}\|_c+\|\boldsymbol{v}\|.$$

Then the product space N is evidently a Banach space with respect to the norm $|||\cdot|||$.

Now we consider an additive operator T mapping M into N of the following form:

$$T\boldsymbol{h}=[\frac{d\boldsymbol{h}}{dt}-A(t)\boldsymbol{h},\ L\boldsymbol{h}],$$

where $A(t)$ is an $(n+1)\times(n+1)$ matrix continuous on I and L is a linear operator mapping $C[I]$ into $\boldsymbol{R}^{n+1}$. By $\Phi(t)$, let us denote an arbitrary fundamental matrix of the linear homogeneous system

$$\frac{d\boldsymbol{z}}{dt}=A(t)\boldsymbol{z},$$

and by $L[\Phi(t)]$ we denote the matrix whose column vectors are $L[\boldsymbol{\varphi}_i(t)]$ $(i=1, 2, \cdots, n+1)$, where $\boldsymbol{\varphi}_i(t)$ $(i=1, 2, \cdots, n+1)$ are column vectors of the matrix $\Phi(t)$.

Then we have the following theorem.

THEOREM 1 (Urabe [5]).

If the matrix $G\equiv L[\Phi(t)]$ is non-singular, namely,

$$\det G=\det L[\Phi(t)]\neq 0,$$

then the operator T has a linear inverse operator T^{-1}, and for $\|T^{-1}\|_c$ we have

$$\|T^{-1}\|_c\leqslant\max(\|H_1\|_c, \|H_2\|_c).$$

Here H_1 is the linear operator mapping $C[I]$ into $M=C^1[I]\subset C[I]$ such that

$$H_1\boldsymbol{\varphi}=\Phi(t)\int_0^t\Phi^{-1}(s)\boldsymbol{\varphi}(s)ds-\Phi(t)G^{-1}L[\Phi(t)\int_0^t\Phi^{-1}(s)\boldsymbol{\varphi}(s)ds]$$

and H_2 is the linear operator mapping $\boldsymbol{R}^{n+1}$ into M such that

$$H_2 \boldsymbol{v} = \Phi(t) G^{-1} \boldsymbol{v}.$$

When an approximate solution $\bar{\boldsymbol{u}}(t)$ of the boundary value problem (6)$-$(7) has been obtained by Galerkin method, it is necessary to find an error bound for $\bar{\boldsymbol{u}}(t)$.
For this purpose we take $A(t)$ and L respectively such that $A(t) = \boldsymbol{X}_u(\bar{\boldsymbol{u}}(t))$, $L = \boldsymbol{f}'(\bar{\boldsymbol{u}}(t))$, where $\boldsymbol{X}_u(\bar{\boldsymbol{u}})$ and $\boldsymbol{f}'(\bar{\boldsymbol{u}})$ denote the Jacobian matrix of $\boldsymbol{X}(\boldsymbol{u})$ and the Fréchet derivative of $\boldsymbol{f}(\boldsymbol{u})$ at $\bar{\boldsymbol{u}}$ respectively.
Then we have the following theorem.

THEOREM 2 ([10]).
Assume that the boundary value problem (6)$-$(7) possesses an approximate solution $\boldsymbol{u} = \bar{\boldsymbol{u}}(t)$ in S such that the matrix

(10) $$G \equiv \boldsymbol{f}'(\bar{\boldsymbol{u}})[\Phi(t)]$$

is non-singular, where $\Phi(t)$ is the fundamental matrix of the following linear system satisfying the initial condition $\Phi(0) = E$ (unit matrix):

(11) $$\frac{d\boldsymbol{z}}{dt} = \boldsymbol{X}_u(\bar{\boldsymbol{u}}(t))\boldsymbol{z}.$$

Let μ and r be the positive numbers such that

(12) $$\mu = \max(\|H_1\|_c, \|H_2\|_c) \geqq \|T^{-1}\|_c,$$

(13) $$r \geqq \left\|\frac{d\bar{\boldsymbol{u}}}{dt} - \boldsymbol{X}(\bar{\boldsymbol{u}})\right\|_c + \|\boldsymbol{f}(\bar{\boldsymbol{u}})\|.$$

If there exist a positive number δ and a non-negative number $\kappa < 1$ such that

(14) (i) $$D'_\delta = \{\boldsymbol{u} \mid \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_c \leqq \delta,\ \boldsymbol{u} \in C[I]\} \subset S',$$

(15) (ii) $$\|\boldsymbol{X}_u(\boldsymbol{u}) - \boldsymbol{X}_u(\bar{\boldsymbol{u}})\|_c + \|\boldsymbol{f}'(\boldsymbol{u}) - \boldsymbol{f}'(\bar{\boldsymbol{u}})\| \leqq \frac{\kappa}{\mu} \quad \text{on } D'_\delta,$$

(16) (iii) $$\frac{\mu r}{1-\kappa} \leqq \delta,$$

then the boundary value problem (6)$-$(7) has one and only one solution $\boldsymbol{u} = \hat{\boldsymbol{u}}(t)$ in

(17) $$D_\delta = \{\boldsymbol{u} \mid \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_c \leqq \delta,\ \boldsymbol{u} \in M\},$$

and for this exact solution $\hat{\boldsymbol{u}}(t)$ we have

(18) $$\|\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_c \leqq \frac{\mu r}{1-\kappa}.$$

§ 3. Numerical Computation

In order to get a 2π-periodic approximate solution $\bar{\boldsymbol{u}}(t)$ in Theorem 2, let us consider a trigonometric polynomial of the form

$$\boldsymbol{u}_m(t)=\boldsymbol{a}_0+\sum_{n=1}^{m}(\boldsymbol{a}_{2n-1}\sin nt+\boldsymbol{a}_{2n}\cos nt) \tag{19}$$

with unknown coefficients $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\cdots$, $\boldsymbol{a}_{2m-1}$, $\boldsymbol{a}_{2m}$.
By Galerkin′s method, we determine unknown coefficients $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{2m-1}$, $\boldsymbol{a}_{2m}$ so that

$$\frac{d\boldsymbol{u}_m}{dt}=P_m\boldsymbol{X}[\boldsymbol{u}_m(t)], \tag{20}$$

and

$$\boldsymbol{f}(\boldsymbol{u}_m(t))=\boldsymbol{0} \tag{21}$$

may be valid, where P_m denotes a truncation of the Fourier series of the $2\pi-$periodic operand function discarding all harmonic terms of the order higher than m.
A trigonometric polynomial $\boldsymbol{u}_m(t)$ of the form (19) satisfying (20) and (21) is called a Galerkin approximation of order m.
The equalities (20) and (21) are clearly equivalent to the system of $3(2m+1)+3$ equations

$$\begin{cases} \boldsymbol{F}_0(\boldsymbol{\alpha}) \equiv \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\,ds=\boldsymbol{0}, \\ \boldsymbol{F}_{2n-1}(\boldsymbol{\alpha}) \equiv \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\sin ns\,ds+n\boldsymbol{a}_{2n}=\boldsymbol{0}, \\ \boldsymbol{F}_{2n}(\boldsymbol{\alpha}) \equiv \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{u}_m(s)]\cos ns\,ds-n\boldsymbol{a}_{2n-1}=\boldsymbol{0}, \\ \boldsymbol{F}_f(\boldsymbol{\alpha}) \equiv \boldsymbol{f}[\boldsymbol{u}_m(t)]=\boldsymbol{0} \end{cases} \tag{22}$$

$$(n=1,\ 2,\cdots,\ m),$$

where $\boldsymbol{\alpha}=\mathrm{col}[\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{2m-1}, \boldsymbol{a}_{2m}]$ is a $3(2m+1)-$dimensional vector. But, taking account of the vector forms (8) and (9), the system (22) essentially consists of $3(2m+1)$ equations. Hence the determining equation (22) of Galerkin approximation (19) can be solved.
Noticing the symmetric character of the orbits of system (3), we see that

$$x(t+\pi)\equiv -x(t),\ \ y(t+\pi)\equiv -y(t). \tag{23}$$

Taking account of the fact, we may assume that the Galerkin approximation (19) can be written as follows:

$$(24)\quad \begin{cases} x_m(t) = \sum_{n=1}^{m} [c_{2n-1}\sin(2n-1)t + c_{2n}\cos(2n-1)t], \\ y_m(t) = \sum_{n=1}^{m} [c'_{2n-1}\sin(2n-1)t + c'_{2n}\cos(2n-1)t] \\ \text{and} \\ \omega_m(t) = \omega. \end{cases}$$

Putting

$$Y(x, y, \omega) \equiv -\frac{1}{\lambda^k}\left(\frac{\omega}{2\pi}\right)^2 x + \frac{\omega}{2\pi}\lambda(1-x^2)y,$$

then from $\dot{x} = \lambda^k y$ the Galerkin approximation (24) and the determining equation (22) can be written as follows:

$$(25)\quad \begin{cases} x_m(t) = \sum_{n=1}^{m} [c_{2n-1}\sin(2n-1)t + c_{2n}\cos(2n-1)t], \\ y_m(t) = \sum_{n=1}^{m} [-\frac{(2n-1)}{\lambda^k} c_{2n}\sin(2n-1)t + \frac{(2n-1)}{\lambda^k} c_{2n-1}\cos(2n-1)t] \end{cases}$$

and

$$(26)\quad \begin{cases} F_{2n-1}(\boldsymbol{c}) = \frac{1}{\pi}\int_0^{2\pi} Y(x_m, y_m, \omega)\sin(2n-1)s\,ds + \frac{(2n-1)^2}{\lambda^k} c_{2n-1} = 0, \\ F_{2n}(\boldsymbol{c}) = \frac{1}{\pi}\int_0^{2\pi} Y(x_m, y_m, \omega)\cos(2n-1)s\,ds + \frac{(2n-1)^2}{\lambda^k} c_{2n} = 0, \\ \qquad (n=1, 2, \cdots, m), \\ F_{2m+1}(\boldsymbol{c}) = l(x_m, y_m, \omega) - \beta = 0, \end{cases}$$

where $\boldsymbol{c} = (c_1, c_2, \cdots, c_{2m-1}, c_{2m}, \omega)$.

In practical computations, it is convenient to discrete the determining equation (26) as follows:

$$(27)\quad \begin{cases} F_{2n-1}(\boldsymbol{c}) = \frac{1}{N}\sum_{i=1}^{2N} Y[x_m(t_i), y_m(t_i), \omega]\sin(2n-1)t_i + \frac{(2n-1)^2}{\lambda^k} c_{2n-1} = 0, \\ F_{2n}(\boldsymbol{c}) = \frac{1}{N}\sum_{i=1}^{2N} Y[x_m(t_i), y_m(t_i), \omega]\cos(2n-1)t_i + \frac{(2n-1)^2}{\lambda^k} c_{2n} = 0, \\ \qquad (n=1, 2, \cdots, m), \\ F_{2m+1}(\boldsymbol{c}) = l(x_m, y_m, \omega) - \beta = 0, \end{cases}$$

where $N = 128$.

Now put

(28) $$\boldsymbol{F}(\boldsymbol{c})=\mathrm{col}[F_1(\boldsymbol{c}),\ F_2(\boldsymbol{c}),\cdots,\ F_{2m-1}(\boldsymbol{c}),\ F_{2m}(\boldsymbol{c}),\ F_{2m+1}(\boldsymbol{c})],$$

then the determining equation (26) can be written briefly as

(29) $$\boldsymbol{F}(\boldsymbol{c})=\boldsymbol{0}.$$

Since the function $\boldsymbol{X}(\boldsymbol{u})$ is nonlinear in $\boldsymbol{u}$, $\boldsymbol{F}(\boldsymbol{c})=\boldsymbol{0}$ is also a nonlinear equation in $\boldsymbol{c}$. Hence, for numerical solution of the nonlinear equation (29) the Newton method will be used. Starting from a certain approximation $\boldsymbol{c}=\boldsymbol{c}_0$, we compute the sequence $\{\boldsymbol{c}_p\}$ successively by the iterative process

(30) $$\begin{cases} J(\boldsymbol{c}_p)\,\boldsymbol{h}_p+\boldsymbol{F}(\boldsymbol{c}_p)=\boldsymbol{0}, \\ \boldsymbol{c}_{p+1}=\boldsymbol{c}_p+\boldsymbol{h}_p \qquad (p=0,\ 1,\ 2,\cdots), \end{cases}$$

where $J(\boldsymbol{c})$ is the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{c})$ with respect to $\boldsymbol{c}$.

In order to practise the iterative process (30) on a computer, it suffices to evaluate $\boldsymbol{F}(\boldsymbol{c})$ and $J(\boldsymbol{c})$ for known $\boldsymbol{c}$.

The elements of the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{c})$ are as follows:

$$J_{2n-1,\,2p-1}=\frac{1}{N}\sum_{i=1}^{2N}[Y_x^{(i)}\sin(2p-1)t_i+\frac{(2p-1)}{\lambda^k}Y_y^{(i)}\cos(2p-1)t_i]\sin(2n-1)t_i+\frac{(2n-1)^2}{\lambda^k}\delta_{np},$$

$$J_{2n-1,\,2p}=\frac{1}{N}\sum_{i=1}^{2N}[Y_x^{(i)}\cos(2p-1)t_i-\frac{(2p-1)}{\lambda^k}Y_y^{(i)}\sin(2p-1)t_i]\sin(2n-1)t_i,$$

$$J_{2n,\,2p-1}=\frac{1}{N}\sum_{i=1}^{2N}[Y_x^{(i)}\sin(2p-1)t_i+\frac{(2p-1)}{\lambda^k}Y_y^{(i)}\cos(2p-1)t_i]\cos(2n-1)t_i,$$

$$J_{2n,\,2p}=\frac{1}{N}\sum_{i=1}^{2N}[Y_x^{(i)}\cos(2p-1)t_i-\frac{(2p-1)}{\lambda^k}Y_y^{(i)}\sin(2p-1)t_i]\cos(2n-1)t_i+\frac{(2n-1)^2}{\lambda^k}\delta_{np},$$

$$(n,\ p=1,2,\cdots,m),$$

$$J_{2n-1,\,2m+1}=\frac{1}{N}\sum_{i=1}^{2N}Y_\omega^{(i)}\sin(2n-1)t_i,$$

$$J_{2n,\,2m+1}=\frac{1}{N}\sum_{i=1}^{2N}Y_\omega^{(i)}\cos(2n-1)t_i, \qquad (n=1,2,\cdots,m)$$

$$J_{2m+1,\,\hat{n}}=1,$$

$$J_{2m+1,\,p}=0 \qquad (p\neq\hat{n},\ 1\leqslant p\leqslant 2m+1),$$

where δ_{np} is the Kronecker delta,

$$Y_x^{(i)}=Y_x[x_m(t_i),\ y_m(t_i),\ \omega],\quad Y_y^{(i)}=Y_y[x_m(t_i),\ y_m(t_i),\ \omega]$$

and

$$Y_\omega^{(i)} = Y_\omega[x_m(t_i),\ y_m(t_i),\ \omega].$$

The starting value $\boldsymbol{c}_0 = \boldsymbol{c}_0(\bar{\lambda})$ necessary for the Newton method for small $\lambda = \bar{\lambda}$ can be obtained by

$$(31)\qquad \begin{cases} x(t) = 2\cos\dfrac{\omega}{2\pi}t, \\ y(t) = -\dfrac{1}{\lambda^k}\dfrac{\omega}{\pi}\sin\dfrac{\omega}{2\pi}t, \\ \omega = 2\pi, \end{cases}$$

which is a periodic solution to the equations (3) and (4) with sufficiently small $\bar{\lambda}$.(See [6]).

For not small λ, tracing the curve

$$F_n(\boldsymbol{c},\ \lambda) = 0 \qquad (n = 1, 2, 3, \cdots, 2m+1)$$

through the point $\boldsymbol{c}_0(\bar{\lambda})$, we can obtain the starting value $\boldsymbol{c}_0$. (See [8]).

Taking account of the solution (31), we may set $k=0$, $\beta = -0.625$ and $\hat{n} = 1$ in (8) and (9). Numerical results are shown in Tables 1~3.

The computations in the present paper have been carried out by the use of FACOM 230 at Tokushima University.

After having founded an approximate solution $\boldsymbol{u}_m(t)$, it is necessary to verify the existence of the exact periodic solution $\hat{\boldsymbol{u}}(t)$ and to give a posteriori error estimation for $\boldsymbol{u}_m(t)$.

For this purpose we begin with checking the conditions in Theorem 2.

In the present case, from (9) and (8), we have

$$(32)\qquad \| \boldsymbol{f}'(\boldsymbol{u}) - \boldsymbol{f}'(\bar{\boldsymbol{u}}) \| = 0$$

and

$$\boldsymbol{X}_u(x,\ y,\ \omega) = \begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{\lambda}{\pi}xy\omega - \left(\dfrac{\omega}{2\pi}\right)^2 & \dfrac{\omega}{2\pi}\lambda(1-x^2) & \dfrac{\lambda}{2\pi}(1-x^2)y - \dfrac{\omega}{2\pi^2}x \\ 0 & 0 & 0 \end{pmatrix}$$

respectively.

Therefore, for the Galerkin approximation $\bar{\boldsymbol{u}} = \boldsymbol{u}_m(t)$ we have

(33) $\|\boldsymbol{X}_u(x, y, \omega)-\boldsymbol{X}_u(\bar{x}, \bar{y}, \bar{\omega})\|$

$$=\Bigl\{[\frac{\lambda}{\pi}(\bar{x}\bar{y}\bar{\omega}-xy\omega)+\frac{1}{4\pi^2}(\bar{\omega}+\omega)(\bar{\omega}-\omega)]^2$$

$$+\left(\frac{\lambda}{2\pi}\right)^2[(1-x^2)\omega-(1-\bar{x}^2)\bar{\omega}]^2+\left(\frac{1}{2\pi^2}\right)^2\{\pi\lambda[(1-x^2)y-(1-\bar{x}^2)\bar{y}]$$

$$+(\bar{x}\bar{\omega}-x\omega)\}^2\Bigr\}^{\frac{1}{2}}.$$

Then, if we assume that

(34) $$[(x-\bar{x})^2+(y-\bar{y})^2+(\omega-\bar{\omega})^2]^{\frac{1}{2}}\leqslant\delta,$$

then using

$$x=(x-\bar{x})+\bar{x},\ \ y=(y-\bar{y})+\bar{y},\ \ \omega=(\omega-\bar{\omega})+\bar{\omega},$$

we have

(35) $$[\frac{\lambda}{\pi}(\bar{x}\bar{y}\bar{\omega}-xy\omega)+\frac{1}{4\pi^2}(\bar{\omega}+\omega)(\bar{\omega}-\omega)]^2$$

$$\leqslant\Bigl\{\frac{\lambda^2}{\pi^2}\{(|\bar{y}||\bar{\omega}|)^2+|\bar{\omega}|^2(\delta^2+2|\bar{x}|\delta+|\bar{x}|^2)+[\delta^2+\delta(|\bar{x}|+|\bar{y}|)+|\bar{x}||\bar{y}|]^2\}$$

$$+\frac{\lambda}{2\pi^3}\{|\bar{y}||\bar{\omega}|+(|\bar{x}|+\delta)|\bar{\omega}|+(|\bar{x}|+\delta)(|\bar{y}|+\delta)\}(\delta+2|\bar{\omega}|)$$

$$+\frac{1}{16\pi^4}(\delta^2+4|\bar{\omega}|\delta+4|\bar{\omega}|^2)\Bigr\}\delta^2,$$

(36) $$\left(\frac{\lambda}{2\pi}\right)^2[(1-x^2)\omega-(1-\bar{x}^2)\bar{\omega}]^2$$

$$\leqslant\frac{\lambda^2}{4\pi^2}\Bigl\{(1+\delta^2+2|\bar{x}|\delta+|\bar{x}|^2)^2+|\bar{\omega}|^2(\delta+2|\bar{x}|)^2\Bigr\}\delta^2$$

and

(37) $$\left(\frac{1}{2\pi^2}\right)^2\Bigl\{\pi\lambda[(1-x^2)y-(1-\bar{x}^2)\bar{y}]+(\bar{x}\bar{\omega}-x\omega)\Bigr\}^2$$

$$\leqslant\frac{1}{4\pi^4}\Bigl\{\pi^2\lambda^2\{(1+|\bar{x}|^2)^2+(\delta+2|\bar{x}|)^2(\delta+|\bar{y}|)^2\}$$

$$+2\pi\lambda\{(\delta+2|\bar{x}|)(\delta+|\bar{y}|)+(1+|\bar{x}|^2)\}(|\bar{x}|+\delta+|\bar{\omega}|)$$

$$+[|\bar{x}|^2+(\delta+|\bar{\omega}|)^2]\Bigr\}\delta^2.$$

However, for $\lambda=1$, $m=49$ we have from Table 1 that

$$(38)\quad \begin{cases} |\bar{x}(t)| = |\sum_{n=1}^{m} [c_{2n-1}\sin(2n-1)t + c_{2n}\cos(2n-1)t]| \\ \qquad \leqslant \sum_{n=1}^{m} \sqrt{c_{2n-1}^2 + c_{2n}^2} \leqslant 2.91516, \\ |\bar{y}(t)| = |\sum_{n=1}^{m} [-(2n-1)c_{2n}\sin(2n-1)t + (2n-1)c_{2n-1}\cos(2n-1)t]| \\ \qquad \leqslant \sum_{n=1}^{m} (2n-1)\sqrt{c_{2n-1}^2 + c_{2n}^2} \leqslant 3.87783, \\ |\bar{\omega}(t)| < 6.66329. \end{cases}$$

Thus from (33) − (38) we have

$$(39)\quad \| \boldsymbol{X}_u(x, y, \omega) - \boldsymbol{X}_u(\bar{x}, \bar{y}, \bar{\omega}) \| \leqslant \delta[0.154\,\delta^4 + 2.217\,\delta^3 + 18.54\,\delta^2 + 96.4\,\delta + 191.83]^{\frac{1}{2}}.$$

On the other hand, from (9) and (10), we have

$$(40)\quad G = \boldsymbol{f}'(\bar{\boldsymbol{u}})[\Phi(t)] = L_1[E - \Phi(2\pi)] + L_2\int_0^{2\pi} \frac{1}{\pi}\cos t\,\Phi(t)\,dt$$

and from Theorem 1, if $\det G \neq 0$, we have

$$H_1\boldsymbol{\varphi} = \int_0^{2\pi} H_1(t, s)\,\boldsymbol{\varphi}(s)\,ds,$$

where

$$(41)\quad H_1(t, s) = \begin{cases} \Phi(t)\left\{E - G^{-1}[-L_1\Phi(2\pi) + L_2\int_s^{2\pi} \frac{1}{\pi}\cos\xi\,\Phi(\xi)\,d\xi]\right\}\Phi^{-1}(s) \\ \qquad (\text{if } 0 \leqslant s < t \leqslant 2\pi), \\ -\Phi(t)G^{-1}[-L_1\Phi(2\pi) + L_2\int_s^{2\pi} \frac{1}{\pi}\cos\xi\,\Phi(\xi)\,d\xi]\Phi^{-1}(s) \\ \qquad (\text{if } 2\pi \geqslant s \geqslant t \geqslant 0). \end{cases}$$

Hence, we may set

$$(42)\quad \mu = \max(\|H_1\|_c,\ \sup_{t \in I}\|\Phi(t)G^{-1}\|).$$

Let

$$\frac{d\bar{\boldsymbol{u}}}{dt} - \boldsymbol{X}(\bar{\boldsymbol{u}}(t)) = \sum_{n=1}^{\infty} (\boldsymbol{b}_{2n-1}\sin(2n-1)t + \boldsymbol{b}_{2n}\cos(2n-1)t),$$

then inequality (13) is valid if

$$(43)\quad \|\sum_{n=1}^{m'} (\boldsymbol{b}_{2n-1}\sin(2n-1)t + \boldsymbol{b}_{2n}\cos(2n-1)t)\|_c + \|\boldsymbol{f}(\bar{\boldsymbol{u}})\| < r$$

with $m'=m+10$.

Now, we readily see that the conditions (14), (15) and (16) are fulfilled if

$$(44)\qquad \begin{cases} \delta[0.154\,\delta^4+2.217\,\delta^3+18.54\,\delta^2+96.4\,\delta+191.83]^{\frac{1}{2}} \leqq \dfrac{\kappa}{\mu}, \\ \dfrac{\mu r}{1-\kappa} \leqq \delta. \end{cases}$$

In (40), (41) and (42), $\Phi(t)$ is given in Chebyshev series by solving the linear system

$$\frac{dz}{dt} = X_u(\bar{u}(t))z$$

satisfying the initial condition $\Phi(0)=E$.
Let $H_{ij}(t, s)$ and $M_{ij}(t)$ denote the elements of the matrix $H_1(t, s)$ and $\Phi(t)G^{-1}$, respectively. Then we have

$$\|H_1\|_c \leqq [2\pi\cdot\max_p \int_0^{2\pi} \sum_{i,j} H_{ij}^2(t_p, s)\,ds]^{\frac{1}{2}}$$

and

$$\|\Phi(t)\,G^{-1}\| \leqq [\max_p \sum_{i,j} M_{ij}^2(t_p)]^{\frac{1}{2}} \quad (p=0, 2, 4, \cdots, 256),$$

where $t_p=\dfrac{p\pi}{128}$.

By (42), a number slightly greater than the quantity

$$\max([2\pi\cdot\max_p \int_0^{2\pi} \sum_{i,j} H_{ij}^2(t_p,\ s)\,ds]^{\frac{1}{2}},\ [\max_p \sum_{i,j} M_{ij}^2(t_p)]^{\frac{1}{2}})$$

may be taken for the number μ, where the above integral may be evaluated by Simpson's rule.

By the above way, we obtain

$$\det G=-1.804,\quad r=0.2\times10^{-12} \text{ and } \mu=7.0.$$

Therefore, (44) can be written as

$$(45)\qquad \delta[191.83+96.4\,\delta+18.54\,\delta^2+2.217\,\delta^3+0.154\,\delta^4]^{\frac{1}{2}} \leqq \frac{\kappa}{7.0},$$

$$(46)\qquad \frac{1.4\times10^{-12}}{1-\kappa} \leqq \delta.$$

Since we expect $\kappa \ll 1$, from (46) we suppose

(47) $$\delta \leqslant 10^{-11}.$$

Then (45) is valid if

$$\delta[191.83+96.4\times 10^{-11}+\cdots]^{\frac{1}{2}}\leqslant \frac{\varkappa}{7.0}.$$

This inequality is valid if

$$\delta\times 13.86\leqslant \frac{\varkappa}{7.0},$$

that is,

(48) $$\delta\leqslant \frac{\varkappa}{7.0\times 13.86}\leqslant \frac{\varkappa}{97.02}.$$

Then from (46) and (48) we have

(49) $$\frac{1.4\times 10^{-12}}{1-\varkappa}\leqslant \delta \leqslant \frac{\varkappa}{97.02},$$

which implies

$$1.4\times 97.02\times 10^{-12}\leqslant \varkappa(1-\varkappa) < \varkappa,$$

that is,

$$1.35828\times 10^{-10}\leqslant \varkappa(1-\varkappa) < \varkappa.$$

Hence we suppose

(50) $$\varkappa = 2\times 10^{-10}.$$

Then for this value of $\varkappa$, we have

(51) $$\begin{cases} \dfrac{1.4\times 10^{-12}}{1-\varkappa} = 1.4000\cdots\times 10^{-12}, \\[2ex] \dfrac{\varkappa}{97.02} = 2.061\cdots\times 10^{-12}. \end{cases}$$

Thus taking into account (47), we see that (44) is valid for $\varkappa$ and δ such that

(52) $$\varkappa=2\times 10^{-10},\ 1.5\times 10^{-12}\leqslant \delta\leqslant 2.06\times 10^{-12},$$

in other words, the conditions of Theorem 2 are fulfilled by δ and $\varkappa$ specified in (52).

In conclusion, we thus see that the boundary value problem (3) — (5) possesses a unique exact solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ in the region

$$\left\{[x-\bar{x}(t)]^2+[y-\bar{y}(t)]^2+[\omega-\bar{\omega}]^2\right\}^{\frac{1}{2}}\leqslant 2.06\times 10^{-12}$$

and moreover

$$\left\{[\hat{x}(t)-\bar{x}(t)]^2+[\hat{y}(t)-\bar{y}(t)]^2+[\hat{\omega}-\bar{\omega}]^2\right\}^{\frac{1}{2}} \ll \eta=0.15\times 10^{-11}.$$

The quantity $\eta=0.15\times 10^{-11}$ gives an error bound to the Galerkin approximation $\boldsymbol{u}=\bar{\boldsymbol{u}}(t)$ given in Table 1. Hence, for $\lambda=1$ we obtain $\bar{\omega}=6.663286859323137$ which approximates the exact period $\hat{\omega}$ to eleven significant figures.

Similarly, for $\lambda=2$, 3 we have computed the Galerkin approximations to periodic solutions and their periods. Tables 2, 3 show the results.

For $\lambda \gg 3$, the order of Galerkin approximations become also too high. But the difficulty will be overcome by scaling $\frac{dx}{dt}=\lambda^k y$ with sufficiently large k. Table 5 shows the usefulness of the scaling.

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Table 1

$\lambda=1$, $\bar{\omega}=6.66328\ 68593\ 23137$, $k=0$, $\det G=-1.804$, $\mu=7.0$,
$r=0.2\times10^{-12}$, $\eta=0.15\times10^{-11}$.

n	c_n	c_n'
1	-1.91552 16155 09996	0.625
2	-0.625	-1.91552 16155 09996
3	-0.22973 32965 78206	-0.18246 60735 45665
4	0.06082 20245 15222	-0.68919 98897 34618
5	-0.03103 16568 51100	-0.18301 69194 93757
6	0.03660 33838 98751	-0.15515 82842 55502
7	-0.00085 87105 21505	-0.07618 32347 01645
8	0.01088 33192 43092	-0.00601 09736 50533
9	0.00137 07877 77993	-0.02001 96194 57668
10	0.00222 44021 61963	0.01233 70900 01935
11	0.00059 62770 46804	-0.00265 28340 99400
12	0.00024 11667 36309	0.00655 90475 14848
13	0.00015 61397 31099	0.00052 16467 85355
14	-0.00004 01266 75797	0.00202 98165 04285
15	0.00002 58031 16538	0.00047 63497 79059
16	-0.00003 17566 51937	0.00038 70467 48065
17	0.00000 04690 01390	0.00017 80239 13722
18	-0.00001 04719 94925	0.00000 79730 23632
19	-0.00000 15121 47428	0.00004 26082 55560
20	-0.00000 22425 39766	-0.00002 87308 01138
21	-0.00000 06618 81388	0.00000 49039 66829
22	-0.00000 02335 22230	-0.00001 38995 09144
23	-0.00000 01748 38688	-0.00000 12412 88027
24	0.00000 00539 69045	-0.00000 40212 89828
25	-0.00000 00283 14279	-0.00000 09654 87023
26	0.00000 00386 19481	-0.00000 07078 56964
27	0.00000 00000 78731	-0.00000 03394 69295
28	0.00000 00125 72937	0.00000 00021 25737
29	0.00000 00019 90919	-0.00000 00766 99213
30	0.00000 00026 44800	0.00000 00577 36660
31	0.00000 00008 38098	-0.00000 00076 52180
32	0.00000 00002 46845	0.00000 00259 81052
33	0.00000 00002 16847	0.00000 00026 34353
34	-0.00000 00000 79829	0.00000 00071 55949
35	0.00000 00000 33494	0.00000 00017 93886
36	-0.00000 00000 51254	0.00000 00011 72300
37	-0.00000 00000 00982	0.00000 00006 00159
38	-0.00000 00000 16221	-0.00000 00000 36323
39	-0.00000 00000 02782	0.00000 00001 28863
40	-0.00000 00000 03304	-0.00000 00001 08489
41	-0.00000 00000 01119	0.00000 00000 10822
42	-0.00000 00000 00264	-0.00000 00000 45873
43	-0.00000 00000 00281	-0.00000 00000 05223
44	0.00000 00000 00121	-0.00000 00000 12092
45	-0.00000 00000 00041	-0.00000 00000 03184
46	0.00000 00000 00071	-0.00000 00000 01839
47	0.00000 00000 00003	-0.00000 00000 01019

n	c_n	c'_n
48	0.00000 00000 00022	0.00000 00000 00118
49	0.00000 00000 00004	-0.00000 00000 00208
50	0.00000 00000 00004	0.00000 00000 00196
51	0.00000 00000 00002	-0.00000 00000 00014
52	0.00000 00000 00000	0.00000 00000 00078
53	0.00000 00000 00000	0.00000 00000 00010
54	0.00000 00000 00000	0.00000 00000 00020
55	0.00000 00000 00000	0.00000 00000 00005
56	0.00000 00000 00000	0.00000 00000 00003
57	0.00000 00000 00000	0.00000 00000 00002
58	0.00000 00000 00000	0.00000 00000 00000
⋮	⋮	⋮
98	0.00000 00000 00000	0.00000 00000 00000

Table 2

$\lambda=2$, $\bar{\omega}=7.62987\ 44796\ 74841$, $k=0$, $\det G=-1.606$, $\mu=30.0$, $r=0.5\times10^{-11}$, $\eta=0.16\times10^{-9}$.

n	c_n	c_n'
1	-1.94963 22632 90607	0.625
2	-0.625	-1.94963 22632 90607
3	-0.40646 31111 04402	0.01572 18547 84519
4	-0.00524 06182 61506	-1.21938 93333 13205
5	-0.13828 96494 40829	-0.22756 28036 51265
6	0.04551 25607 30253	-0.69144 82472 04146
7	-0.04671 00464 28654	-0.26198 65540 57315
8	0.03742 66505 79616	-0.32697 03250 00580
9	-0.01303 56531 11866	-0.20300 65620 90038
10	0.02255 62846 76671	-0.11732 08780 06790
11	-0.00169 49891 43109	-0.12722 96744 59984
12	0.01156 63340 41817	-0.01864 48805 74203
13	0.00125 55813 69035	-0.06754 84793 94603
14	0.00519 60368 76508	0.01632 25577 97450
15	0.00144 32350 20311	-0.03018 47927 11115
16	0.00201 23195 14074	0.02164 85253 04671
17	0.00098 00173 11256	-0.01049 05283 68133
18	0.00061 70899 04008	0.01666 02942 91350
19	0.00053 73056 32816	-0.00179 66855 33484
20	0.00009 45623 96499	0.01020 88070 23502
21	0.00025 25728 75676	0.00115 43373 53188
22	-0.00005 49684 45390	0.00530 40303 89189
23	0.00010 13179 61002	0.00160 57419 78619
24	-0.00006 98148 68636	0.00233 03131 03051
25	0.00003 21222 51644	0.00123 11753 87116
26	-0.00004 92470 15485	0.00080 30562 91096
27	0.00000 52729 19343	0.00074 76622 35212
28	-0.00002 76911 93897	0.00014 23688 22252
29	-0.00000 27175 13819	0.00038 50290 18250
30	-0.00001 32768 62698	-0.00007 88079 00756
31	-0.00000 36453 39449	0.00016 79925 82206
32	-0.00000 54191 15555	-0.00011 30055 22907
33	-0.00000 26240 41483	0.00005 77548 71738
34	-0.00000 17501 47628	-0.00008 65933 68946
35	-0.00000 14963 88471	0.00001 04880 71122
36	-0.00000 02996 59175	-0.00005 23735 96479
37	-0.00000 07258 65965	-0.00000 52499 82737
38	0.00000 01418 91425	-0.00002 68570 40687
39	-0.00000 02995 08752	-0.00000 77169 78559
40	0.00000 01978 71245	-0.00001 16808 41315
41	-0.00000 00979 44097	-0.00000 59191 15523
42	0.00000 01443 68671	-0.00000 40157 07977
43	-0.00000 00172 93935	-0.00000 35735 96559
44	0.00000 00831 06897	-0.00000 07436 39213
45	0.00000 00076 39996	-0.00000 18286 80229
46	0.00000 00406 37338	0.00000 03437 99816
47	0.00000 00109 96522	-0.00000 07942 39893

n	c_n	c'_n
48	0.00000 00168 98721	0.00000 05168 36536
49	0.00000 00081 05351	-0.00000 02733 52579
50	0.00000 00055 78624	0.00000 03971 62182
51	0.00000 00046 98848	-0.00000 00514 94574
52	0.00000 00010 09698	0.00000 02396 41269
53	0.00000 00023 11581	0.00000 00222 32599
54	-0.00000 00004 19483	0.00000 01225 13780
55	0.00000 00009 67085	0.00000 00341 47661
56	-0.00000 00006 20867	0.00000 00531 89653
57	0.00000 00003 21721	0.00000 00263 03610
58	-0.00000 00004 61467	0.00000 00183 38106
59	0.00000 00000 59472	0.00000 00158 73305
60	-0.00000 00002 69039	0.00000 00035 08865
61	-0.00000 00000 23343	0.00000 00081 13085
62	-0.00000 00001 33001	-0.00000 00014 23896
63	-0.00000 00000 35455	0.00000 00035 23048
64	-0.00000 00000 55921	-0.00000 00022 33641
65	-0.00000 00000 26543	0.00000 00012 17209
66	-0.00000 00000 18726	-0.00000 00017 25299
67	-0.00000 00000 15549	0.00000 00002 36313
68	-0.00000 00000 03527	-0.00000 00010 41783
69	-0.00000 00000 07719	-0.00000 00000 90494
70	0.00000 00000 01312	-0.00000 00005 32585
71	-0.00000 00000 03259	-0.00000 00001 44982
72	0.00000 00000 02042	-0.00000 00002 31410
73	-0.00000 00000 01098	-0.00000 00001 12325
74	0.00000 00000 01539	-0.00000 00000 80141
75	-0.00000 00000 00210	-0.00000 00000 67887
76	0.00000 00000 00905	-0.00000 00000 15774
77	0.00000 00000 00074	-0.00000 00000 34724
78	0.00000 00000 00451	0.00000 00000 05715
79	0.00000 00000 00118	-0.00000 00000 15101
80	0.00000 00000 00191	0.00000 00000 09353
81	0.00000 00000 00090	-0.00000 00000 05243
82	0.00000 00000 00065	0.00000 00000 07270
83	0.00000 00000 00053	-0.00000 00000 01045
84	0.00000 00000 00013	0.00000 00000 04399
85	0.00000 00000 00026	0.00000 00000 00359
86	-0.00000 00000 00004	0.00000 00000 02252
87	0.00000 00000 00011	0.00000 00000 00600
88	-0.00000 00000 00007	0.00000 00000 00980
89	0.00000 00000 00004	0.00000 00000 00468
90	-0.00000 00000 00005	0.00000 00000 00341
91	0.00000 00000 00Q01	0.00000 00000 00284
92	-0.00000 00000 00003	0.00000 00000 00069
93	0.00000 00000 00000	0.00000 00000 00145
94	-0.00000 00000 00002	-0.00000 00000 00022
95	0.00000 00000 00000	0.00000 00000 00064
96	-0.00000 00000 00001	-0.00000 00000 00038
97	0.00000 00000 00000	0.00000 00000 00022
98	0.00000 00000 00000	-0.00000 00000 00030

Table 3

$\lambda=3$, $\bar{\omega}=8.85909\ 54997\ 19845$, $k=0$, $\det G=-1.403$, $\mu=600.0$, $r=0.6\times10^{-10}$, $\eta=0.37\times10^{-7}$.

n	c_n	c_n'
1	-1.97854 07842 87074	0.625
2	-0.625	-1.97854 07842 87074
3	-0.49273 44392 79166	0.25468 95133 33519
4	-0.08489 65044 44506	-1.47820 33178 37499
5	-0.22412 42025 06231	0.02773 33275 22736
6	-0.00554 66655 04547	-1.12062 10125 31154
7	-0.11651 26942 87701	-0.11996 44704 17452
8	0.01713 77814 88207	-0.81558 88600 13909
9	-0.06253 87093 45118	-0.19497 90942 94923
10	0.02166 43438 10547	-0.56284 83841 06065
11	-0.03323 02164 65749	-0.21487 49333 28140
12	0.01953 40848 48013	-0.36553 23811 23235
13	-0.01699 93643 11778	-0.20019 73820 54092
14	0.01539 97986 19546	-0.22099 17360 53120
15	-0.00810 92660 93343	-0.16832 83043 18147
16	0.01122 18869 54543	-0.12163 89914 00138
17	-0.00339 51603 43064	-0.13140 06649 20476
18	0.00772 94508 77675	-0.05771 77258 32087
19	-0.00103 05458 19186	-0.09658 88431 90912
20	0.00508 36233 25837	-0.01958 03705 64541
21	0.00004 89776 23001	-0.06731 72133 25987
22	0.00320 55815 86952	0.00102 85300 83013
23	0.00045 70007 82810	-0.04457 03759 01716
24	0.00193 78424 30509	0.01051 10180 04623
25	0.00053 84848 62453	-0.02796 18784 87365
26	0.00111 84751 39495	0.01346 21215 61334
27	0.00048 00053 10159	-0.01648 05921 03246
28	0.00061 03923 00120	0.01296 01433 74288
29	0.00037 68471 37533	-0.00895 10919 20997
30	0.00030 86583 42103	0.01092 85669 88465
31	0.00027 34176 70377	-0.00428 12169 74882
32	0.00013 81037 73383	0.00847 59477 81672
33	0.00018 70834 56343	-0.00156 91015 11701
34	0.00004 75485 30658	0.00617 37540 59335
35	0.00012 18154 19029	-0.00012 71109 74582
36	0.00000 36317 42131	0.00426 35396 66018
37	0.00007 56898 80790	0.00053 71010 34798
38	-0.00001 45162 44184	0.00280 05255 89226
39	0.00004 47852 14173	0.00075 69555 09067
40	-0.00001 94091 15617	0.00174 66233 52746
41	0.00002 50378 33711	0.00074 62138 82979
42	-0.00001 82003 38609	0.00102 65511 82146
43	0.00001 29961 47719	0.00063 39634 27708
44	-0.00001 47433 35528	0.00055 88343 51902
45	0.00000 60181 47110	0.00049 26233 61203
46	-0.00001 09471 85805	0.00027 08166 19932
47	0.00000 22172 06054	0.00035 87443 69095

n	c_n	c'_n
48	-0.00000 76328 58917	0.00010 42086 84536
49	0.00000 03170 34088	0.00024 75426 53074
50	-0.00000 50518 90879	0.00001 55346 70310
51	-0.00000 05058 12582	0.00016 25051 77028
52	-0.00000 31863 76020	-0.00002 57964 41679
53	-0.00000 07587 31929	0.00010 13867 23985
54	-0.00000 19129 57056	-0.00004 02127 92228
55	-0.00000 07393 64232	0.00005 97187 10603
56	-0.00000 10857 94738	-0.00004 06650 32787
57	-0.00000 06123 95901	0.00003 26951 98856
58	-0.00000 05735 99980	-0.00003 49065 66360
59	-0.00000 04620 68069	0.00001 60603 61116
60	-0.00000 02722 09510	-0.00002 72620 16043
61	-0.00000 03264 03734	0.00000 64220 59850
62	-0.00000 01052 79670	-0.00001 99106 27775
63	-0.00000 02185 26679	0.00000 12646 95680
64	-0.00000 00200 74535	-0.00001 37671 80749
65	-0.00000 01393 21303	-0.00000 11731 81770
66	0.00000 00180 48950	-0.00000 90558 84710
67	-0.00000 00845 45234	-0.00000 20632 25015
68	0.00000 00307 94403	-0.00000 56645 30673
69	-0.00000 00485 46800	-0.00000 21461 14751
70	0.00000 00311 03112	-0.00000 33497 29219
71	-0.00000 00260 05528	-0.00000 18646 61301
72	0.00000 00262 62835	-0.00000 18463 92490
73	-0.00000 00125 88816	-0.00000 14660 35502
74	0.00000 00200 82678	-0.00000 09189 83593
75	-0.00000 00050 61757	-0.00000 10753 95874
76	0.00000 00143 38612	-0.00000 03796 31741
77	-0.00000 00011 56300	-0.00000 07461 17788
78	0.00000 00096 89841	-0.00000 00890 35098
79	0.00000 00006 36902	-0.00000 04923 59138
80	0.00000 00062 32394	0.00000 00503 15244
81	0.00000 00012 74978	-0.00000 03090 84399
82	0.00000 00038 15857	0.00000 01032 73244
83	0.00000 00013 35643	-0.00000 01836 44039
84	0.00000 00022 12579	0.00000 01108 58385
85	0.00000 00011 48616	-0.00000 01019 51262
86	0.00000 00011 99427	0.00000 00976 32378
87	0.00000 00008 89117	-0.00000 00513 85870
88	0.00000 00005 90642	0.00000 00773 53145
89	0.00000 00006 40892	-0.00000 00218 40508
90	0.00000 00002 45399	0.00000 00570 39372
91	0.00000 00004 36697	-0.00000 00058 02350
92	0.00000 00000 63762	0.00000 00397 39446
93	0.00000 00002 83066	0.00000 00019 97193
94	-0.00000 00000 21475	0.00000 00263 25133
95	0.00000 00001 74678	0.00000 00050 68579
96	-0.00000 00000 53353	0.00000 00165 94442
97	0.00000 00001 02167	0.00000 00056 36941
98	-0.00000 00000 58113	0.00000 00099 10155
99	0.00000 00000 55976	0.00000 00050 38717

n	c_n	c'_n
100	-0.00000 00000 50896	0.00000 00055 41618
101	0.00000 00000 27988	0.00000 00040 26107
102	-0.00000 00000 39862	0.00000 00028 26819
103	0.00000 00000 11965	0.00000 00029 86161
104	-0.00000 00000 28992	0.00000 00012 32350
105	0.00000 00000 03430	0.00000 00020 90166
106	-0.00000 00000 19906	0.00000 00003 60156
107	-0.00000 00000 00652	0.00000 00013 90528
108	-0.00000 00000 12996	-0.00000 00000 69772
109	-0.00000 00000 02243	0.00000 00008 80452
110	-0.00000 00000 08078	-0.00000 00002 44535
111	-0.00000 00000 02550	0.00000 00005 28596
112	-0.00000 00000 04762	-0.00000 00002 83097
113	-0.00000 00000 02276	0.00000 00002 97718
114	-0.00000 00000 02635	-0.00000 00002 57184
115	-0.00000 00000 01803	0.00000 00001 53613
116	-0.00000 00000 01336	-0.00000 00002 07389
117	-0.00000 00000 01323	0.00000 00000 68518
118	-0.00000 00000 00586	-0.00000 00001 54791
119	-0.00000 00000 00915	0.00000 00000 21610
120	-0.00000 00000 00182	-0.00000 00001 08890
121	-0.00000 00000 00601	-0.00000 00000 01812
122	0.00000 00000 00015	-0.00000 00000 72770
123	-0.00000 00000 00376	-0.00000 00000 11609
124	0.00000 00000 00094	-0.00000 00000 46291
125	-0.00000 00000 00224	-0.00000 00000 14072
126	0.00000 00000 00113	-0.00000 00000 27942
127	-0.00000 00000 00125	-0.00000 00000 13011
128	0.00000 00000 00102	-0.00000 00000 15850
129	-0.00000 00000 00064	-0.00000 00000 10595
130	0.00000 00000 00082	-0.00000 00000 08268
131	-0.00000 00000 00029	-0.00000 00000 07961
132	0.00000 00000 00061	-0.00000 00000 03765
133	-0.00000 00000 00010	-0.00000 00000 05630
134	0.00000 00000 00042	-0.00000 00000 01265
135	0.00000 00000 00000	-0.00000 00000 03780
136	0.00000 00000 00028	0.00000 00000 00000
137	0.00000 00000 00004	-0.00000 00000 02416
138	0.00000 00000 00018	0.00000 00000 00542
139	0.00000 00000 00005	-0.00000 00000 01467
140	0.00000 00000 00011	0.00000 00000 00693
141	0.00000 00000 00005	-0.00000 00000 00838
142	0.00000 00000 00006	0.00000 00000 00654
143	0.00000 00000 00004	-0.00000 00000 00442
144	0.00000 00000 00003	0.00000 00000 00538
145	0.00000 00000 00003	-0.00000 00000 00206
146	0.00000 00000 00001	0.00000 00000 00408
147	0.00000 00000 00002	-0.00000 00000 00073
148	0.00000 00000 00000	0.00000 00000 00291
149	0.00000 00000 00001	-0.00000 00000 00006
150	0.00000 00000 00000	0.00000 00000 00199

Table 4. *Comparison of periods*

λ	Urabe et al.'s results	Krogdahl's results	Strasberg's results
1	6.687	6.66328	6.66328 6860
2	7.6310	7.62986	7.62987 4480
3	8.8613	8.85946	8.85909 5500

λ	Our results	error bound
1	6.66328 68593 23137	0.15×10^{-11}
2	7.62987 44796 74841	0.16×10^{-9}
3	8.85909 54997 19845	0.37×10^{-7}

Table 5

$\lambda = 3$, $\bar{\omega} = 8.85909\ 54997\ 20826$, $k = 10$, $r = 0.4 \times 10^{-10}$.

n	c_n	c'_n
1	-1.97854 07842 87304	0.00001 05844 29880
2	-0.625	-0.00003 35067 61914
3	-0.49273 44392 79225	0.00000 43131 89272
4	-0.08489 65044 44559	-0.00002 50335 02986
5	-0.22412 42025 06275	0.00000 04696 66337
6	-0.00554 66655 04587	-0.00001 89778 15247
7	-0.11651 26942 87737	-0.00000 20316 08840
8	0.01713 77814 88183	-0.00001 38120 68960
9	-0.06253 87093 45148	-0.00000 33019 88083
10	0.02166 43438 10534	-0.00000 95318 86808
11	-0.03323 02164 65771	-0.00000 36389 25864
12	0.01953 40848 48008	-0.00000 61903 22971
13	-0.01699 93643 11795	-0.00000 33903 60244
14	0.01539 97986 19545	-0.00000 37425 14455
15	-0.00810 92660 93354	-0.00000 28506 54614
16	0.01122 18869 54545	-0.00000 20599 67000
17	-0.00339 51603 43071	-0.00000 22252 81799
18	0.00772 94508 77678	-0.00000 09774 54755
19	-0.00103 05458 19191	-0.00000 16357 40541
20	0.00508 36233 25841	-0.00000 03315 95295
21	0.00004 89776 22998	-0.00000 11400 22919
22	0.00320 55815 86955	0.00000 00174 18247
23	0.00045 70007 82808	-0.00000 07548 03230
24	0.00193 78424 30512	0.00000 01780 05013
25	0.00053 84848 62453	-0.00000 04735 36867
26	0.00111 84751 39496	0.00000 02279 82211
27	0.00048 00053 10159	-0.00000 02791 00274
28	0.00061 03923 00121	0.00000 02194 81166
29	0.00037 68471 37533	-0.00000 01515 87528
30	0.00030 86583 42104	0.00000 01850 76242
31	0.00027 34176 70377	-0.00000 00725 02785
32	0.00013 81037 73384	0.00000 01435 40920
33	0.00018 70834 56344	-0.00000 00265 72872
34	0.00004 75485 30658	0.00000 01045 53067
35	0.00012 18154 19029	-0.00000 00021 52636
36	0.00000 36317 42131	0.00000 00722 03419
37	0.00007 56898 80790	0.00000 00090 95853
38	-0.00001 45162 44184	0.00000 00474 27147
39	0.00004 47852 14173	0.00000 00128 19108
40	-0.00001 94091 15617	0.00000 00295 79220
41	0.00002 50378 33711	0.00000 00126 37198
42	-0.00001 82003 38609	0.00000 00173 84734
43	0.00001 29961 47719	0.00000 00107 36226
44	-0.00001 47433 35528	0.00000 00094 63909
45	0.00000 60181 47110	0.00000 00083 42620
46	-0.00001 09471 85805	0.00000 00045 86303
47	0.00000 22172 06054	0.00000 00060 75367

n	c_n	c'_n
48	-0.00000 76328 58917	0.00000 00017 64783
49	0.00000 03170 34088	0.00000 00041 92157
50	-0.00000 50518 90879	0.00000 00002 63081
51	-0.00000 05058 12582	0.00000 00027 52039
52	-0.00000 31863 76020	-0.00000 00004 36865
53	-0.00000 07587 31929	0.00000 00017 16993
54	-0.00000 19129 57056	-0.00000 00006 81007
55	-0.00000 07393 64233	0.00000 00010 11342
56	-0.00000 10857 94738	-0.00000 00006 88666
57	-0.00000 06123 95901	0.00000 00005 53696
58	-0.00000 05735 99980	-0.00000 00005 91146
59	-0.00000 04620 68069	0.00000 00002 71984
60	-0.00000 02722 09511	-0.00000 00004 61685
61	-0.00000 03264 03735	0.00000 00001 08758
62	-0.00000 01052 79670	-0.00000 00003 37188
63	-0.00000 02185 26680	0.00000 00000 21418
64	-0.00000 00200 74535	-0.00000 00002 33148
65	-0.00000 01393 21305	-0.00000 00000 19868
66	0.00000 00180 48951	-0.00000 00001 53362
67	-0.00000 00845 45237	-0.00000 00000 34941
68	0.00000 00307 94404	-0.00000 00000 95929
69	-0.00000 00485 46805	-0.00000 00000 36345
70	0.00000 00311 03115	-0.00000 00000 56728
71	-0.00000 00260 05534	-0.00000 00000 31578
72	0.00000 00262 62841	-0.00000 00000 31269
73	-0.00000 00125 88823	-0.00000 00000 24827
74	0.00000 00200 82689	-0.00000 00000 15563
75	-0.00000 00050 61763	-0.00000 00000 18212
76	0.00000 00143 38630	-0.00000 00000 06429
77	-0.00000 00011 56304	-0.00000 00000 12636
78	0.00000 00096 89872	-0.00000 00000 01508
79	0.00000 00006 36906	-0.00000 00000 08338
80	0.00000 00062 32442	0.00000 00000 00852
81	0.00000 00012 75000	-0.00000 00000 05234
82	0.00000 00038 15929	0.00000 00000 01749
83	0.00000 00013 35701	-0.00000 00000 03110
84	0.00000 00022 12681	0.00000 00000 01877
85	0.00000 00011 48739	-0.00000 00000 01727
86	0.00000 00011 99564	0.00000 00000 01654
87	0.00000 00008 89351	-0.00000 00000 00870
88	0.00000 00005 90812	0.00000 00000 01310
89	0.00000 00006 41309	-0.00000 00000 00370
90	0.00000 00002 45585	0.00000 00000 00967
91	0.00000 00004 37406	-0.00000 00000 00099
92	0.00000 00000 63918	0.00000 00000 00674
93	0.00000 00002 84240	0.00000 00000 00034
94	-0.00000 00000 21450	0.00000 00000 00448
95	0.00000 00001 76640	0.00000 00000 00086
96	-0.00000 00000 53668	0.00000 00000 00284
97	0.00000 00001 05863	0.00000 00000 00097
98	-0.00000 00000 59313	0.00000 00000 00174