On Relation Modules

By

Motoyoshi Sakuma

(Received April 30, 1979)

Let $\oplus R$ and $\oplus R$ be free modules over a commutative ring $R$ with canonical bases $T'_1, \ldots, T'_m$ and $T_1, \ldots, T_n$ respectively and $m$ and $n$ are positive integers. For any $R$-linear map $f : \oplus R \to \oplus R$, $f(T'_i) = \sum_{j=1}^{n} a_{ji} T_j$, $a_{ji} \in R$ ($i = 1, \ldots, m$), we associate with an $n \times m$ matrix $A$ defined by the coefficients of $T'_i$s:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

Corresponding to $f$, we define an $R$-linear map $'f : \oplus R \to \oplus R$, transpose map of $f$, by

$$'f(T'_i) = \sum_{j=1}^{m} a_{ij} T'_j \quad (i = 1, \ldots, n).$$

The matrix associated with $'f$ is the transposed matrix $A'$ of $A$.

Let $m_1, \ldots, m_n$ be $n$ elements of an $R$-module $M$. We denote by $\text{Rel}(m_1, \ldots, m_n)$ the set of sequences of $n$ elements of $R$, $(r_1, \ldots, r_n) \in \oplus R$, such that

$$r_1 m_1 + \cdots + r_n m_n = 0.$$ 

Obviously $\text{Rel}(m_1, \ldots, m_n)$ is an $R$-module and we call it the relation module of $(m_1, \ldots, m_n)$.

**Lemma 1.** With the same notations as above,

i) $(r_1, \ldots, r_n) \in \text{Rel}(f T_1, \ldots, f T_n)$ if and only if $(r_1, \ldots, r_n) A = 0$.

ii) If $g : \oplus R \to R$ is an $R$-linear map, then $(g(T_1), \ldots, g(T_n)) \in \text{Rel}(f T_1, \ldots, f T_n)$ if and only if $g(\text{Im } f) = 0$.

**Proof.** i) Assume $(r_1, \ldots, r_n) \in \text{Rel}(f T_1, \ldots, f T_n)$, then $r_1 f T_1 + \cdots + r_n f T_n = 0$ so that $(T'_1 \cdots T'_m) A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = 0$. Hence $A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = 0$, and whence $(r_1 \cdots r_n) A = 0$. Con-
verse is clear.

ii) Let \( g(T_i) = s_i \) \((i = 1, \ldots, n)\). Then, we have

\[
g(\text{Im } f) = 0 \iff g(f(T'_i)) = 0 \quad (i = 1, \ldots, m)
\]

\[
\iff g\left( \sum_{j=1}^{n} a_{ji} T_j \right) = 0 \quad (i = 1, \ldots, m)
\]

\[
\iff 0 = \sum_{j=1}^{n} a_{ji} g(T_j) = \sum_{j=1}^{n} a_{ji} s_j \quad (i = 1, \ldots, m).
\]

Hence \( g(\text{Im } f) = 0 \) if and only if \((s_1, \ldots, s_n)A = 0\). Therefore we get ii) in view of i).

**Lemma 2.** \( \text{Rel}(fT_1, \ldots, fT_n) \cong \text{Hom}_R(\oplus^n R/\text{Im } f, R) \).

**Proof.** For any \((r_1, \ldots, r_n) \in \text{Rel}(fT_1, \ldots, fT_n)\), we define an \( R \)-linear map \( g: \oplus^n R \to R, g(T_i) = r_i \) \((i = 1, \ldots, n)\).

By Lemma 1, ii), we have \( g = 0 \) on \( \text{Im } f \), so that \( g \) induces an \( R \)-linear map \( \bar{g}: \oplus^n R/\text{Im } f \to R \). Thus we get a map \( \lambda: \text{Rel}(fT_1, \ldots, fT_n) \to \text{Hom}_R(\oplus^n R/\text{Im } f, R) \) such that

\[
\lambda((r_1, \ldots, r_n)) = \bar{g}.
\]

It is clear that \( \lambda \) is injective. Since \( \text{Hom}(\oplus^n R/\text{Im } f, R) \) is identified with the set of \( R \)-linear maps \( h: \oplus^n R \to R \) which vanish on \( \text{Im } f \), \( \lambda \) is surjective.

Summarizing the above consideration, we get

**Theorem 1.** Let \( f: \oplus^m R \to \oplus^n R \) be an \( R \)-linear map defined by the matrix \( A \):

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1m} \\
\cdots & \cdots & \cdots \\
a_{n1} & \cdots & a_{nm}
\end{pmatrix}
\]

and let \( f': \oplus^m R \to \oplus^n R \) be the transpose of \( f \) corresponding to \( A \). Then,

\( \text{Rel}(fT_1, \ldots, fT_n) \cong (\text{Coker } f)^* \)

where \( M^* = \text{Hom}_R(M, R) \) for an \( R \)-module \( M \).

**Remark.** With the same notations as in Lemma 2, \( \lambda^{-1}: (\text{Coker } f)^* \to \text{Rel}(fT_1, \ldots, fT_n) \) is given by

\[
\bar{g} \mapsto (g(T_1), \ldots, g(T_n)).
\]

**Corollary 1.** Let \( \varphi: \oplus^m R \to \oplus^n R \to \oplus^n R \) be an exact sequence of free \( R \)-
modules. Then,

\[ \text{Rel}(\{gT'_1, \ldots, gT'_m\}) \cong (\text{Im } f)^* \]

where \( T'_1, \ldots, T'_m \) is the canonical base of \( \bigoplus R \).

**Proof.** By Theorem 1, we have

\[ \text{Rel}(\{gT'_1, \ldots, gT'_m\}) \cong (\text{Coker } g)^*. \]

On the other hand

\[ (\text{Coker } g)^* \cong (\bigoplus R/\text{Im } g)^* = (\bigoplus R/\text{Ker } f)^* \cong (\text{Im } f)^*. \]

Thus we get our result.

**Remark.** The isomorphism \( (\text{Im } f)^* \cong \text{Rel}(\{gT'_1, \ldots, gT'_m\}) \) obtained in the Corollary 1 is given by

\[ \phi \longrightarrow ((\phi \cdot f)(T'_1), \ldots, (\phi \cdot f)(T'_m)). \]

**Corollary 2.** Let \( f: \bigoplus R \to \bigoplus R \) be an \( R \)-homomorphism and let \( \text{Coker } f = Ru_1 + \cdots + Ru_n \) where \( u_i \) is the residue of \( T'_i \) modulo \( \text{Im } f \) (\( i = 1, \ldots, n \)). Then, we have the following exact sequence of \( R \)-modules:

\[ 0 \longrightarrow \text{Rel}(fT'_1, \ldots, fT'_m) \longrightarrow \bigoplus R \longrightarrow \text{Rel}(u_1, \ldots, u_n) \longrightarrow 0. \]

**Proof.** Clearly we have

\[ (r_1, \ldots, r_n) \in \text{Rel}(u_1, \ldots, u_n) \iff r_1T'_1 + \cdots + r_nT'_n \in \text{Im } f \]

\[ \iff r_1T'_1 + \cdots + r_nT'_n = r'_1f(T'_1) + \cdots + r'_mf(T'_m) \quad \text{for some elements } \ r'_i \in R \]

\[ \iff \begin{pmatrix} r'_1 \\ \vdots \\ r'_m \end{pmatrix} = A \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \quad \text{for some elements } \ r'_i \in R \ (i = 1, \ldots, m). \]

Now we define a map \( \phi: \text{Hom}(\bigoplus R, R) \to \text{Rel}(u_1, \ldots, u_n) \) by

\[ \phi(g) = A \begin{pmatrix} r'_1 \\ \vdots \\ r'_m \end{pmatrix} \]

where \( g \in \text{Hom}(\bigoplus R, R) \) and \( r'_i = g(T'_i) \ (i = 1, \ldots, m) \). Then, the first part of our proof shows that \( \phi \) is surjective.

Consider an exact sequence

\[ 0 \longrightarrow \text{Ker } \phi \longrightarrow \text{Hom}(\bigoplus R, R) \longrightarrow \text{Rel}(u_1, \ldots, u_n) \longrightarrow 0. \]
Since we have
\[ g \in \ker \phi \iff A \left( \begin{array}{c} gT'_1 \\ \vdots \\ gT'_m \end{array} \right) = 0 \]
\[ \iff g = 0 \text{ on each component of } A \left( \begin{array}{c} T'_1 \\ \vdots \\ T'_m \end{array} \right) \]
\[ \iff g = 0 \text{ on each component of } (T'_1 \ldots T'_m)^* A = (fT_1 \ldots fT_n) \]
\[ \iff g(fT_i) = 0 \quad \text{for } i = 1, \ldots, n. \]

Hence, identifying \( g \) with \( (g(T'_1), \ldots, g(T'_m)) \) we get our assertion. q.e.d.

**Theorem 2.** Let \( \oplus R \xrightarrow{\phi} \oplus R \xrightarrow{f} \oplus R \) be an exact sequence of free \( R \)-modules where \( g \) and \( f \) are \( R \)-homomorphisms. Then,
\[ \Ext^1(Coker f, R) \cong \text{Rel}(\langle gT'_1, \ldots, gT'_m \rangle)/\Im \langle f \rangle \]
where \( T'_1, \ldots, T'_m \) is the canonical base of \( \oplus R \).

**Proof.** From an exact sequence
\[ 0 \to \Im f \xrightarrow{\phi} \oplus R \to Coker f \to 0, \]
we get a long exact sequence of \( R \)-modules:
\[ 0 \to \Hom(Coker f, R) \to \Hom(\oplus R, R) \xrightarrow{\phi^*} \Hom(\Im f, R) \]
\[ \to \Ext^1(Coker f, R) \to \Ext^1(\oplus R, R) \to \cdots \]
Since \( \Ext^1(\oplus R, R) = 0 \), identifying \( \oplus R \) with \( \oplus R \), we have
\[ \Ext^1(Coker f, R) \cong (\Im f)^*/\phi^*(\oplus R). \]

By the Corollary 1 of Theorem 1, we have
\[ (\Im f)^* \cong \text{Rel}(\langle gT'_1, \ldots, gT'_m \rangle) \]
and \( \phi^*(\oplus R) \) is generated by the restriction to the \( \Im f \) of the projection map
\[ p_i \colon \oplus R \to R \quad (i = 1, \ldots, n), \text{ i.e., } \phi^*(\oplus R) \text{ is generated by} \]
\[ ((p_i f)(T'_1), \ldots, (p_i f)(T'_m)) \quad (i = 1, \ldots, n). \]
Since \( f T_i = \sum_{j=1}^n a_{ji} T_j \) (\( i = 1, \ldots, n \)) and \( (p_i f)(T_i) = a_{ji} \), we have

\[(p_i f)(T_1, \ldots, p_i f)(T_n) = (a_{i1}, \ldots, a_{in}) \quad (i = 1, \ldots, n).\]

Identifying \( (a_{i1}, \ldots, a_{im}) \) with \( a_{i1} T_1 + \cdots + a_{im} T_m = f T_i \) (\( i = 1, \ldots, n \)), we see that \( \phi^*(\bigoplus R) \) is generated by \( f T_1, \ldots, f T_n \), so that \( \phi^*(\bigoplus R) = \text{Im } f \). q.e.d.

Assume \( R \) is a Noetherian local ring and let \( M \) be a finitely generated \( R \)-module with minimal system of generators \( u_1, \ldots, u_n \). Then, it is well known that \( \text{Rel} (u_1, \ldots, u_n) \) is determined uniquely up to isomorphism [1, theorem 26.1]. We call it the relation module of \( M \) and denote it \( \text{Rel} (M) \).

Now, let

\[
\cdots \longrightarrow F_i \overset{d_i}{\longrightarrow} F_{i-1} \longrightarrow \cdots \overset{d_1}{\longrightarrow} F_0 \overset{\varepsilon}{\longrightarrow} M \longrightarrow 0
\]

be a minimal projective (free) resolution of \( M \) with augmentation \( \varepsilon \). For any integer \( n \geq 1 \), the \( n \)-th Syzygy module of \( M \) is defined to be \( \text{Im } d_n \) and is denoted by \( \text{Syz}_n(M) \).

Let \( N \) be a submodule of \( \bigoplus R \), minimally generated by \( m \) elements, \( N = Rv_1 + \cdots + Rv_m \). Take a map \( f: \bigoplus R \rightarrow \bigoplus R \) such that \( f(T_i) = v_i \) (\( i = 1, \ldots, m \)). The submodule \( \text{Im } f \) of \( \bigoplus R \) is called the transpose of \( N \) and is denoted by \( ^t N \).

**Corollary.** If \( M \) is finitely generated over a Noetherian local ring \( R \), then

\[ \text{Ext}^n (M, R) \simeq \text{Rel} \left( \text{Syz}_{n+1}(M) \right) / \left( \text{Syz}_n(M) \right) \]

for \( n \geq 1 \).

**Proof.** We can assume the sequence

\[
\bigoplus R \overset{g}{\longrightarrow} \bigoplus R \overset{f}{\longrightarrow} \bigoplus R \longrightarrow M \longrightarrow 0
\]

is the first three terms of a minimal free resolution of \( M \). Hence, by Theorem 2, we have

\[ \text{Ext}^1 (M, R) \simeq \text{Rel} \left( \text{Im } g \right) / \left( \text{Im } f \right). \]

Since \( \text{Im } g = \text{Syz}_2 M \) and \( \text{Im } f = \text{Syz}_1 M \), we have our corollary in the case \( n = 1 \).

In general we have

\[
\text{Ext}^n (M, R) = \text{Ext}^1 (\text{Syz}_{n-1} M, R) \\
\simeq \text{Rel} \left( \text{Syz}_2 (\text{Syz}_{n-1} M) / \left( \text{Syz}_1 (\text{Syz}_{n-1} M) \right) \right) \\
= \text{Rel} \left( \text{Syz}_{n+1} M / \left( \text{Syz}_n M \right) \right),
\]
which finish our proof. 

q.e.d.

Faculty of Integrated Arts and Sciences
Hiroshima University

References