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On the Conditions for a $\{V, H\}$-manifold to be Locally Minkowskian or Conformally Flat

By

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The present paper is the continuation of the serial papers concerning the Finsler manifold modeled on a Minkowski space ([5], [6], [7]). A Finsler manifold whose tangent spaces at arbitrary points are congruent to a unique Minkowski space is called a Finsler manifold modeled on a Minkowski space. As an example of it, the notion of the $\{V, H\}$-manifold has been introduced in the paper [5].

On the other hand, M. Hashiguchi has defined a notion of a generalized Berwald space [2]. Following his definition, it is a Finsler manifold admitting a linear connection $\Gamma(x)$ with respect to which $\nabla g=0$ holds, where ∇ denotes the covariant derivative with respect to the Finsler connection $(\Gamma^i_{jk}(x), \Gamma^i_{mk}(x)y^m)$.

It has been shown, in the paper [5], that the $\{V, H\}$-manifold is a generalized Berwald space. In the paper [6] it has been proved that a standard generalized Berwald space is a $\{V, H\}$-manifold. And also it has been found, in the paper [7], that a Finsler manifold with a linear connection $\Gamma(x)$ with respect to which $\nabla C=0$ holds good becomes a $\{V, H\}$-manifold under some condition, where C is the tensor given by $C^i_{jk}=\frac{1}{2}g^{im}\dot{\partial}_m g_{jk}$.

Now, the main purpose of the present paper is to find the following two: The one is the condition for the $\{V, H\}$-manifold to be locally Minkowskian and the another is the condition for the $\{V, H\}$-manifold to be locally conformal to a Minkowski space. These will be shown in § 1 and § 2 using the terminology of the theory of G-structures. In section 3, examples of these manifolds will be shown especially in the case where the manifolds admit a Randers metric. The last section is devoted to consideration on the case that a Finsler manifold is globally conformal to a locally Minkowskian manifold.

§1. Let V be an n-dimensional Minkowski space, that is, an n-dimensional linear space on which a Minkowski norm is defined. Here the Minkowski norm on V means a real valued function on V, whose value at $\xi\in V$ we denote by $\|\xi\|$, with properties:

(1) Let $\{e_\alpha\}$ be a fixed base of V, then the norm of any vector $\xi=\xi^\alpha e_\alpha\in V$ can be

represented by $\|\xi\|=f(\xi^1, \xi^2,\dots, \xi^n)$ (For brevity we write $f(\xi^1, \xi^2,\dots, \xi^n)$ as $f(\xi^\alpha)$ or $f(\xi)$). Now, the function $f(\xi)$ is 3-times continuously differentiable at $\xi \neq \mathbf{0}$.
(2) $\|\xi\| \geqq 0$.
(3) $\|\xi\|=0$ if and only if $\xi=\mathbf{0}$.
(4) $\|k\xi\|=k\|\xi\|$ for any $k>0$.
(5) $\|\xi_1+\xi_2\| \leqq \|\xi_1\|+\|\xi_2\|$.

Now we put

$$G=\{T \mid T \in GL(n, R), \quad \|T\xi\|=\|\xi\| \quad \text{for any} \quad \xi \in V\}, \tag{1.1}$$

then G is a Lie group [5].

Let H be a Lie subgroup of G and let M be an n-dimensional C^∞-manifold. We assume here that M admits an H-structure in the sense of a G-structure.

Let $\{U\}$ be a coordinate neighbourhood system and $\{X_\alpha\}$ be an n-frame on U adapted to the H-structure, and y be any vector in $T_p(M)$ with the expression $y=y^i \dfrac{\partial}{\partial x^i}=\xi^\alpha X_\alpha$ where $p \in U$.

Putting

$$\frac{\partial}{\partial x^i}=\mu_i^\alpha(x) X_\alpha \quad \text{or} \quad X_\alpha=\lambda_\alpha^i(x) \frac{\partial}{\partial x^i}, \tag{1.2}$$

we have proved, in the paper [5], that the function

$$F(x, y)=f(\xi^\alpha)=f(\mu_i^\alpha(x) y^i) \tag{1.3}$$

gives M globally a Finsler metric. This Finsler metric is called a $\{V, H\}$*-metric.* When M admits a $\{V, H\}$-metric, we say that M is a $\{V, H\}$*-manifold.* In the $\{V, H\}$-manifold, the tangent Minkowski space $T_p(M)$ at any point $p \in M$ is congruent to the given Minkowski space V.

Let M be a $\{V, H\}$-manifold and G be the Lie group defined by (1.1). As H is a Lie subgroup of G, the manifold M admits the G-structure too. Now we shall prove

Theorem 1. *Let M be a $\{V, H\}$-manifold and G be a Lie group defined by (1.1). The necessary and sufficient condition for M to be locally Minkowskian is that the G-structure admitted by M is integrable.*

PROOF. If the G-structure is integrable, M is covered by the coordinate neighbourhood system $\{U\}$ such that the natural frame $\left\{\dfrac{\partial}{\partial x^i}\right\}$ on each U is adapted to the G-structure ([1], [13]). Let $\{X_\alpha\}$ be the frame on U adapted to the H-structure. Since $H \subset G$, the frame $\{X_\alpha\}$ is, at the same time, adapted to the G-structure. Then (1.2) leads us to $(\mu_i^\alpha(x)) \in G$. Hence we have $F(x^i, y^i)=f(\mu_i^\alpha(x) y^i)=f(y^\alpha)$. Therefore M is locally Minkowskian.

Conversely, we assume that the $\{V, H\}$-manifold M is locally Minkowskian. Then M is covered by a coordinate neighbourhood system $\{U\}$ such that the fundamental function $F(x^i, y^i)=f(\mu_i^\alpha(x)y^i)$ is independent to x^i on each U. Thus we may put $f(\mu_i^\alpha(x)y^i)=g(y^\alpha)$ on each U. Let (x_0^i) be any fixed point in U, then we have $f(\mu_i^\alpha(x_0)y^i)=g(y^\alpha)=f(\mu_i^\alpha(x)y^i)$. Putting $\mu_i^\alpha(x_0)=A_i^\alpha$ and $(A_i^\alpha)^{-1}=(B_\alpha^i)$, we have $f(\mu_i^\alpha(x)y^i)=f(A_i^\alpha y^i)$. On the other hand, we see $f(\mu_i^\alpha(x)y^i)=f(\mu_j^\alpha(x)B_\beta^j A_m^\beta y^m)$. Hence we have $f(\mu_j^\alpha(x)B_\beta^j A_m^\beta y^m)=f(A_m^\alpha y^m)$ for any y^m. Thus we obtain

$$\mu_j^\alpha(x)B_\beta^j \in G. \tag{1.4}$$

Now we consider the transformation of the coordinates on U such that $\bar{x}^\alpha=A_k^\alpha x^k$. Apparently the coordinate systems $\{U, \bar{x}^\alpha\}$ and $\{U, x^i\}$ are mutually equivalent, and $\frac{\partial}{\partial \bar{x}^\beta}=B_\beta^m \frac{\partial}{\partial x^m}=\mu_m^\alpha(x)B_\beta^m X_\alpha$ hold good. Taking account of (1.4), we have that the natural frame $\left\{\frac{\partial}{\partial \bar{x}^\alpha}\right\}$ is adapted to the G-structure, that is, the G-structure under consideration is integrable.

§**2.** Suppose that two distinct Finsler metric functions $F(x, y)$, $F^*(x, y)$ are defined over a Finsler space. The two metrics resulting from these functions are called *conformal* if there exists a factor of proportionality α such that $g_{ij}(x, y)=\alpha g_{ij}^*(x, y)$. By the well-known Knebelman's theorem, the factor of proportionality α depends on x^i alone. For convenience we shall denote it by $g_{ij}=e^{2\sigma}g_{ij}^*$ where $\sigma=\sigma(x)$.

A Finsler manifold M is called *conformally flat* if M is covered by a coordinate neighbourhood system $\{U_\alpha\}$ such that the given Finsler metric g is conformal to a Minkowski metric in each U_α.

In this case, the tangent Minkowski spaces at arbitrary points of U_α are congruent to a unique Minkowski space.

Because, there exists an admissible coordinate system in each U_α such that $g_{ij}(x, y)=e^{\sigma(x)}g_{ij}^*(y)$ holds good. If we put $h(y)=\sqrt{g_{ij}^*(y)y^iy^j}$, then we have $F(x, y)=e^{\sigma(x)}h(y)$, that is,

$$F(x, e^{-\sigma(x)}y)=h(y). \tag{2.1}$$

Let $V_{(\alpha)}$ be a Minkowski space whose Minkowski norm of $\xi=\xi^\alpha e_\alpha \in V_{(\alpha)}$ is given by $\|\xi\|=h(\xi^\alpha)$ where $\{e_\alpha\}$ is a fixed base of $V_{(\alpha)}$. Now, for any $p\in U_\alpha$, we may define a linear isomorphic mapping φ_p: $T_p(M)\to V_{(\alpha)}$ by $\varphi_p\left(y^i\left(\frac{\partial}{\partial x^i}\right)_p\right)=e^{\sigma(x)}y^ie_i$. Since $F(x^i, y^i)=h(e^{\sigma(x)}y^i)$, this mapping φ_p is an isometry from $T_p(M)$ to $V_{(\alpha)}$. Thus we see that, for any $p\in U_\alpha$, $T_p(M)$ is congruent to the Minkowski space $V_{(\alpha)}$.

Next, let U_α and U_β be the coordinate neighbourhoods mentioned above in a conformally flat Finsler manifold, and $V_{(\alpha)}$ and $V_{(\beta)}$ be the Minkowski spaces defined above in the neighbourhoods U_α and U_β respectively. If $U_\alpha \cap U_\beta \neq \phi$, then for any

$p \in U_\alpha \cap U_\beta$, $T_p(M)$ is congruent to $V_{(\alpha)}$ and, at the same time, to $V_{(\beta)}$. Hence $V_{(\alpha)}$ is congruent to $V_{(\beta)}$. By virtue of the definition of a Finsler manifold modeled on a Minkowski space [5], we obtain

Theorem 2. *If a Finsler manifold is connected and conformally flat, then it is a Finsler manifold modeled on a Minkowski space.*

In what follows we shall treat $\{V, H\}$-manifolds. Since a $\{V, H\}$-manifold is a Finsler manifold modeled on a Minkowski space, the following theorem is evident.

Theorem 3. *A $\{V, H\}$-manifold M is conformally flat if and only if there exists, for any point p of M, such a coordinate neighbourhood U that $p \in U$ and the $\{V, H\}$-metric is conformal to the given Minkowski space V on U.*

Now let G be the Lie group defined by (1.1) in a $\{V, H\}$-manifold. Then we shall prove

Theorem 4. *The necessary and sufficient condition for a $\{V, H\}$-manifold to be conformally flat is that the manifold is covered by a coordinate neighbourhood system $\{U_\alpha, x^i\}$ such that $\left\{e^{-\sigma_\alpha(x)}\frac{\partial}{\partial x^i}\right\}$ is an n-frame adapted to the G-structure for some local scalar field $\sigma_\alpha(x)$ on each U_α.*

Proof. Suppose that M is a conformally flat $\{V, H\}$-manifold. Then M is covered by a coordinate neighbourhood system $\{U_\alpha\}$ such that the $\{V, H\}$-Finsler metric is conformal to the Minkowski space V in each U_α. If we assume that $U \in \{U_\alpha\}$ and $\{X_\alpha\}$ is an n-frame on U adapted to the H-structure and we put $\frac{\partial}{\partial x^i} = \mu_i^\alpha(x) X_\alpha$, then we have, after a suitable coordinate transformation, that $f(\mu_i^\alpha(x) y^i) = e^{\sigma(x)} h(y^i)$ holds on U where $\sigma(x)$ is a local scalar field on U and $h(y^i)$ is the norm function of V. Let $p = (x_0^i)$ be any fixed point in U, then we have

$$f(e^{-\sigma(x)}\mu_i^\alpha(x) y^i) = h(y^i) = f(e^{-\sigma(x_0)}\mu_i^\alpha(x_0) y^i).$$

Putting $e^{-\sigma(x_0)}\mu_i^\alpha(x_0) = A_i^\alpha$ and $(A_i^\alpha)^{-1} = (B_\alpha^i)$, we have $f(A_i^\alpha y^i) = f(e^{-\sigma(x)}\mu_j^\alpha(x) B_\beta^j A_i^\beta y^i)$ for any y^i. Hence we obtain

$$e^{-\sigma(x)}\mu_j^\alpha(x) B_\beta^j \in G. \tag{2.2}$$

Now let us consider a coordinate transformation $\bar{x}^\alpha = A_k^\alpha x^k$ in U. It is clear that the coordinate system $\{U, \bar{x}^\alpha\}$ is equivalent to $\{U, x^i\}$. If we put $\sigma(x) = \sigma(x(\bar{x})) = \bar{\sigma}(\bar{x})$, we have

$$e^{-\bar{\sigma}(\bar{x})}\frac{\partial}{\partial \bar{x}^\alpha} = e^{-\sigma(x)} B_\alpha^j \frac{\partial}{\partial x^j} = e^{-\sigma(x)}\mu_j^\beta(x) B_\alpha^j X_\beta.$$

From (2.2) and the fact that $\{X_\alpha\}$ is an n-frame adapted to the G-structure, it follows that $\left\{e^{-\bar{\sigma}(\bar{x})}\frac{\partial}{\partial \bar{x}^{\alpha}}\right\}$ is an n-frame on U adapted to the G-structure.

Conversely, we shall show the condition is sufficient. Let $\{U, x^i\}$ be any coordinate neighbourhood satisfying the given condition and $\left\{e^{-\sigma(x)}\frac{\partial}{\partial x^i}\right\}$ be an n-frame on U adapted to the given G-structure. Let $\{X_\alpha\}$ be an n-frame on U adapted to the H-structure, and put $\frac{\partial}{\partial x^i}=\mu_i^\alpha(x)X_\alpha$, then the $\{V, H\}$-Finsler metric is given by $F(x, y)=f(\mu_i^\alpha(x)y^i)$. From our assumption, both $\{X_\alpha\}$ and $\{e^{-\sigma(x)}\mu_i^\alpha(x)X_\alpha\}$ are n-frames adapted to the G-structure. Hence we have $e^{-\sigma(x)}\mu_i^\alpha(x)\in G$, that is, $f(e^{-\sigma(x)}\mu_i^\alpha(x)y^i)=f(y^i)$. By the homogenity condition of the function $f(y^i)$, we have $f(\mu_i^\alpha(x)y^i)=e^{\sigma(x)}f(y^i)$. Therefore we obtain $g_{ij}(x, y)=e^{2\sigma(x)}g_{ij}^*(y)$. q.e.d.

§3. In this section we assume that a manifold M admits a Randers metric

(3.1) $$F(x, y)=\sqrt{g_{ij}(x)y^iy^j}+v_i(x)y^i$$

where $g_{ij}(x)$ is a Riemannian metric and $v_i(x)$ is a covector field on M.

In the paper [5], the present author has found that the condition for the Randers manifold (M, F) to be a $\{V, H\}$-manifold is that $g^{ij}v_iv_j$ is constant. Hashiguchi and the present author have shown, in the paper [3], that if the vector field $v_i(x)$ is parallel with respect to the Riemannian metric $g_{ij}(x)$, then the Randers manifold (M, F) is a Berwald space. Recently Kikuchi [8] has proved that the converse of this theorem is also true, that is to say, the condition that $v_i(x)$ is parallel with respect to $g_{ij}(x)$ is a necessary and sufficient condition for the Randers manifold (M, F) to be a Berwald space.

Again, Kikuchi has proved, at the same time, that the condition for the Randers manifold (M, F) to be locally Minkowskian is that the Riemannian metric $g_{ij}(x)$ is locally flat and the vector field $v_i(x)$ is parallel with respect to $g_{ij}(x)$.

Now, we shall find the condition that the Randers manifold be conformally flat.

First, we assume the Randers metric (3.1) is conformally flat. Then, by virtue of the definition and Theorem 2, we see that M is covered by a coordinate neighbourhood system $\{U, x^i\}$ such that each U admits some local scalar field $\sigma(x)$ with respect to which $F(x, y)=e^{\sigma(x)}(\sqrt{a_{ij}y^iy^j}+b_iy^i)$ holds good on U, where a_{ij} and b_i have constant components. This holds for any y^i, so we have $g_{ij}(x)=e^{2\sigma(x)}a_{ij}$ and $v_i(x)=e^{\sigma(x)}b_i$. The first condition leads us to

$$\left\{\begin{matrix} i \\ jk \end{matrix}\right\}=\sigma_j\delta_k^i+\sigma_k\delta_j^i-\sigma^i g_{jk},$$

where $\sigma_j=\partial_j\sigma$, $\sigma^i=g^{im}\sigma_m$ and $\left\{\begin{matrix} i \\ jk \end{matrix}\right\}$ is the Christoffel's symbol of $g_{ij}(x)$. Denoting the covariant derivative with respect to $\left\{ {i \atop jk} \right\}$ by ;, we find, from these, that

(3.2) $$v_{j;k} = \sigma^m v_m g_{jk} - v_k \sigma_j.$$

From the condition $a_{ij} = e^{-2\sigma(x)} g_{ij}(x)$, we have also

$$\overset{(a)}{R}{}_h{}^i{}_{jk} = R_h{}^i{}_{jk} - \delta^i_j \sigma_{hk} + \delta^i_k \sigma_{hj} - g_{hk}\sigma^i_j + g_{hj}\sigma^i_k,$$

where we denote by $\overset{(a)}{R}{}_h{}^i{}_{jk}$ and $R_h{}^i{}_{jk}$ the curvature tensors for the Riemannian metrics a_{ij} and $g_{ij}(x)$ respectively and we put

$$\sigma_{ij} = \sigma_{i;j} + \sigma_i \sigma_j - \frac{1}{2} g_{ij}\sigma^m \sigma_m \quad \text{and} \quad \sigma^i_j = g^{im}\sigma_{mj}.$$

The tensor a_{ij} being a flat Riemannian metric, we must have

(3.3) $$R_h{}^i{}_{jk} = \delta^i_j \sigma_{hk} - \delta^i_k \sigma_{hj} + g_{hk}\sigma^i_j - g_{hj}\sigma^i_k.$$

Conversely, let us assume that a Randers manifold (M, F) is covered by such a coordinate neighbourhood system $\{U, x^i\}$ that each U admits a local scalar $\sigma(x)$ satisfying (3.2) and (3.3). On putting $a_{ij}(x) = e^{-2\sigma(x)} g_{ij}(x)$, we see, by account of (3.3), that $a_{ij}(x)$ is a flat Riemannian metric on U. Then, after a suitable coordinate transformation $\bar{x} = \bar{x}(x)$ in U, we have that $e^{-2\sigma(\bar{x})} g_{ij}(\bar{x}) = \delta_{ij}$. This gives us $\overline{\left\{ {i \atop jk} \right\}} = \delta^i_k \sigma_j + \delta^i_j \sigma_k - \sigma^i g_{jk}$. Hence the condition (3.2) leads us to $\partial_k v_j(\bar{x}) = v_j(\bar{x})\sigma_k(\bar{x})$. Now, if we put $e^{-\sigma(\bar{x})} v_j(\bar{x}) = b_j(\bar{x})$, then we have

$$\frac{\partial b_j}{\partial \bar{x}^k} = -e^{-\sigma(\bar{x})}\sigma_k(\bar{x}) v_j(\bar{x}) + e^{-\sigma(\bar{x})}\partial_k v_j(\bar{x}) = 0,$$

that is, the components of b_j are constant. Hence, on the coordinate neighbourhood $\{U, \bar{x}^i\}$, we have

$$F(\bar{x}, \bar{y}) = e^{\sigma(\bar{x})}\left(\sqrt{\sum_{k=1}^{n} (\bar{y}^k)^2} + b_k \bar{y}^k \right).$$

Thus we find the Randers manifold (M, F) is conformally flat.

Consequently we obtain

Theorem 5. *The condition that a Randers manifold* (M, F) *be conformally flat is that* M *is covered by a coordinate neighbourhood system* $\{U\}$ *such that each* U *admits a local scalar field* $\sigma(x)$ *satisfying the conditions* (3.2) *and* (3.3).

Remark. By virtue of (3.2), we see

$$(g^{ij} v_i v_j)_{;k} = 2 g^{ij} v_i v_{j;k} = 2 v^j (\sigma^m v_m g_{jk} - v_k \sigma_j) = 0.$$

Hence the Randers manifold (M, F) satisfying the condition (3.2) is necessarily a $\{V, H\}$-manifold.

§**4.** In connection with the conformally flat Finsler manifold, we introduce the following theorem which has already been proved by Hashiguchi and the present author [4].

Theorem 6. *A Finsler manifold $(M, F(x, y))$ is globally conformal to a locally Minkowskian manifold if and only if M admits a scalar field $\sigma(x)$ and a linear connection $\Gamma(x)$ on M satisfying the conditions*

(1) $\nabla_k F \ :=\partial_k F(x, y)-\dot{\partial}_m F(x, y)\Gamma^m_{rk}(x)y^r=0,$

(2) $R_i{}^h{}_{jk}:=\partial_k\Gamma^h_{ij}-\partial_j\Gamma^h_{ik}+\Gamma^m_{ij}\Gamma^h_{mk}-\Gamma^m_{ik}\Gamma^h_{mj}=0,$

(3) $T^i_{kj} \ :=\Gamma^i_{kj}-\Gamma^i_{jk}=\delta^i_k\partial_j\sigma-\delta^i_j\partial_k\sigma.$

Proof. Let us assume that the manifold M globally admits $\sigma(x)$ and $\Gamma(x)$ satisfying the conditions (1), (2) and (3). If we put $\tilde{F}(x, y)=e^{-\sigma(x)}F(x, y)$ and $\tilde{\Gamma}^i_{kj}=\Gamma^i_{kj}-\delta^i_k\partial_j\sigma$, then $\tilde{F}$ becomes a Finsler metric and $\tilde{\Gamma}^i_{kj}(x)$ becomes a linear connection on M. With respect to these, we have

$$\tilde{\nabla}_k\tilde{F}=-e^{-\sigma}\partial_k\sigma F+e^{-\sigma}\nabla_k F+e^{-\sigma}y^m\dot{\partial}_m F\partial_k\sigma=0,$$

$$\tilde{R}_i{}^h{}_{jk}=R_i{}^h{}_{jk}=0,$$

$$\tilde{T}^i_{kj}=T^i_{kj}-\delta^i_k\partial_j\sigma+\delta^i_j\partial_k\sigma=0.$$

Hence $\tilde{\Gamma}^i_{kj}(x)$ becomes a flat linear metric connection of the Finsler manifold $(M, \tilde{F}(x, y))$. That is, $(M, \tilde{F}(x, y))$ is a locally Minkowskian manifold.

The converse is evident from the fact of $F(x, y)=e^{\sigma(x)}\tilde{F}(x, y)$, where $\tilde{F}(x, y)$ is a local Minkowski metric.

Remark. The condition (1) implies that the Finsler manifold $(M, F(x, y))$ is a generalized Berwald space, namely, a $\{V, H\}$-manifold [6].

Now, we consider a $\{V, H\}$-manifold and assume that the Lie group G defined by (1.1) is totally disconnected, that is, the connected component containing the identity e is $\{e\}$. Let us assume moreover that the $\{V, G\}$-metric is conformally flat. Now, on each neighbourhood U defined in Theorem 4, we have, by virtue of (2.2), that $e^{-\sigma(x)}\mu^\alpha_j(x)B^j_\beta=g^\alpha_\beta$, that is, $\mu^\alpha_j(x)=e^{\sigma(x)}g^\alpha_\beta e^{-\sigma(x_0)}\mu^\beta_j(x_0)$ where g^α_β is an element of G. As G is totally disconnected, g^α_β does not depend on x^i, and the Lie algebra of the Lie group G is $\{0\}$. Hence the G-connection is given uniquely by $\Gamma^i_{kj}=\lambda^i_\alpha\partial_j\mu^\alpha_k$ where $(\lambda^i_\alpha)=(\mu^\alpha_i)^{-1}$. Then we have, on the neighbourhood U, that

$$\Gamma^i_{kj}=e^{-\sigma(x)}e^{\sigma(x_0)}\overset{-1}{g^\gamma_\alpha}\lambda^i_\gamma(x_0)\partial_j(e^{\sigma(x)}g^\alpha_\beta e^{-\sigma(x_0)}\mu^\beta_k(x_0))=\delta^i_k\partial_j\sigma.$$

This leads us, for the curvature tensor and the torsion tensor, to $R_i{}^h{}_{jk}=0$ and $T^i_{kj}=\delta^i_k\partial_j\sigma-\delta^i_j\partial_k\sigma$. Of course, $\nabla_k F=0$ holds good [5]. On the other hand, the connection $\Gamma^i_{kj}(x)$ is a global one on M, and so is the torsion tensor $T^i_{kj}(x)$. Let $\{U_1, x^i\}$ and $\{\bar{U}_2, \bar{x}^a\}$ be the coordinate neighbourhoods, σ_1 and $\bar{\sigma}_2$ be the local scalar fields defined in Theorem 4 on U_1 and $\bar{U}_2$ respectively. If we assume $U_1\cap\bar{U}_2\neq\phi$, then the torsion tensor satisfies, in $U_1\cap\bar{U}_2$, $\frac{\partial x^i}{\partial\bar{x}^a}\bar{T}^a_{bc}=T^i_{jk}\frac{\partial x^j}{\partial\bar{x}^b}\frac{\partial x^k}{\partial\bar{x}^c}$, that is,

$$\frac{\partial x^i}{\partial\bar{x}^b}\partial_c\bar{\sigma}_2-\frac{\partial x^i}{\partial\bar{x}^c}\partial_b\bar{\sigma}_2=\frac{\partial x^i}{\partial\bar{x}^b}\partial_c\sigma_1-\frac{\partial x^i}{\partial\bar{x}^c}\partial_b\sigma_1.$$

This can be rewritten as $\frac{\partial x^i}{\partial\bar{x}^b}\partial_c(\bar{\sigma}_2-\sigma_1)=\frac{\partial x^i}{\partial\bar{x}^c}\partial_b(\bar{\sigma}_2-\sigma_1)$, which leads us to $\partial_c(\bar{\sigma}_2-\sigma_1)=0$, that is, $\bar{\sigma}_2-\sigma_1=k$ (const.). Hence the factors e^{σ_1} and $e^{\bar{\sigma}_2}$ of the proportionality of local conformal transformations in U_1 and $\bar{U}_2$ have the relation $e^{\bar{\sigma}_2}=e^k e^{\sigma_1}$. Now, we consider the coordinate transformation $\bar{\bar{x}}^a=e^k\bar{x}^a$, then, in the coordinate neighbourhood $\{\bar{U}_2, \bar{\bar{x}}^a\}$, we have $e^{\bar{\bar{\sigma}}_2}=e^{-k}e^{\bar{\sigma}_2}=e^{\sigma_1}$. Hence the scalar $\sigma_1(x)$ can be considered as a global scalar field on M if M is connected. Therefore, by virtue of Theorem 6, we obtain

Theorem 7. *Let V be a Minkowski space, G be the Lie Group defined by* (1.1) *and let M be a connected $\{V, G\}$-manifold. If the Lie group G is totally disconnected and the $\{V, G\}$-metric is conformally flat, then the $\{V, G\}$-manifold M is globally conformal to a locally Minkowskian Manifold.*

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