Harmonic Sections of Tangent Bundles

By

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Let \( M \) be an \( m \) dimensional smooth Riemannian manifold with metric \( g \). The tangent bundle \( T(M) \) over \( M \) is endowed with the Riemannian metric \( g^P \), the diagonal lift of \( g \) [3],[5]. Let \( X \) be a vector field on \( M \). Then it is regarded as a mapping \( \phi_X \) of \( M \) to \( T(M) \). The purpose of this paper is to study under what conditions the mapping \( \phi_X \) of Riemannian manifolds is harmonic.

§ 1 is devoted to describe some basic facts on geometry of tangent bundles. We will see in § 2 that the natural projection, \( \pi: T(M) \to M \) is a totally geodesic submersion. In the last section, it is proved that when \( M \) is compact and orientable, \( \phi_X: M \to T(M) \) is harmonic iff the first covariant derivative of \( X \) vanishes.

§ 1. Diagonal lifts of Riemannian metrics to tangent bundles

We will review differential geometry of tangent bundles briefly. For details, compare [5].

Let \( \{ U, x^i \} \) be a coordinate neighborhood, where \( (x^i) \) is a system of local coordinate defined in the open set \( U \). Then we can introduce a system of local coordinates \( (x^i, y^j) \) in the open set \( \pi^{-1}(U) \) of \( T(M) \) in such a way that for each \( p \in U \), \( (x^i(p), y^j)|_{\pi^{-1}(U)} \in T(M) \), where \( \pi: T(M) \to M \) is the natural projection. \( (x^i, y^j) \) are called the induced coordinates in \( \pi^{-1}(U) \).

The Riemannian metric of \( M \) is given locally by

\[
ds_M^2 = \sum_{i=1}^{m} (\theta^i)^2,
\]

where \( \theta^i \) are local 1-forms such that

\[
\theta^i = \sum_{j=1}^{m} \xi_j^i dx^j.
\]

(In the paper, the indices \( i, j, k,... \) run over the range \( \{1,\ldots, m\} \) and the indices \( A, B, C,... \) the range \( \{1,\ldots, m,\ldots, 2m\} \). We also use the notation \( i^* = m + i \).) Let \( \omega^i, \omega^{i*} \) be vertical lifts and horizontal lifts of the local 1-forms \( \theta^i \), i.e.,
\[ \begin{aligned}
&\omega^i = (\theta^i)^\nu = \pi^*\theta^i = \sum_{j=1}^{m} \xi^j_i \cdot \pi dx^j, \\
&\omega^* = (\theta^*^i) = \sum_{j=1}^{m} \tilde{\xi}^j_i \cdot \pi (dy^j + \sum_{k,l=1}^{m} \Gamma^i_{kl}y^k dx^l),
\end{aligned} \]

where \( \Gamma^i_{kl} \) are local components of the Riemannian connection in \( M \). Then the diagonal lift \( g^0 \) of \( g \) is written locally as

\[ ds^2_{T(M)} = \sum_{\alpha=1}^{2m} (\omega^\alpha)^2 = \sum_{i=1}^{m} (\omega^i)^2 + \sum_{i=1}^{m} (\omega^*^i)^2. \]

Let \( X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i} \) be a vector field on \( M \). The vertical lift \( X^V \) and the horizontal lift \( X^H \) of \( X \) are written locally as

\[ \begin{aligned}
X^V &= \sum_{i=1}^{m} X^i \frac{\partial}{\partial y^i}, \\
X^H &= \sum_{j=1}^{m} X^j \left( \frac{\partial}{\partial x^k} - \sum_{k,l=1}^{m} \Gamma^i_{kl}y^k \frac{\partial}{\partial y^l} \right).
\end{aligned} \]

The structure equations in \( M \) are

\[ \begin{aligned}
&d\theta^i = \sum_{j=1}^{m} \theta^j \wedge \theta^j, \\
&d\theta^*^i = \sum_{k=1}^{m} \theta^k \wedge \theta^k - \frac{1}{2} \sum_{k,l=1}^{m} R^i_{kl} \theta^k \wedge \theta^l,
\end{aligned} \]

where \( \theta^j \) are the Riemannian connection forms and \( R^i_{kl} \) are the coefficients of the Riemannian curvature tensor. Let \( \omega^*_B \) be the Riemannian connection forms in \( T(M) \). Then,

\[ d\omega^\alpha = \sum_{B=1}^{2m} \omega^B \wedge \omega^*_B. \]

From the basic properties of vertical lifts [5], it follows

\[ d\omega^i = d(\theta^i)^\nu = (d\theta^i)^\nu = \sum_{j=1}^{m} (\theta^j)^\nu \wedge (\theta^j)^\nu = \sum_{j=1}^{m} \omega^j \wedge \pi^* \theta^j. \]

On the other hand, a direct calculation shows

\[ d\omega^*^i = \sum_{j=1}^{m} \omega^*^j \wedge \pi^* \theta^j + \frac{1}{2} \sum_{j,k=1}^{m} R^i_{jk} \tilde{\xi}^k \cdot \omega^*^j \wedge \omega^k. \]

Comparing with (4), we get

**Proposition 1.** Let \( Y^i = \sum_{j=1}^{m} \xi^j_i y^j \).
\[ \omega^j = \pi^* \theta^j - \frac{1}{2} \sum_{l, k=1}^m R^j_{l k l} Y^l \omega^k, \]
\[ \omega^j_\ast = - \omega^j_\ast = - \frac{1}{2} \sum_{l, k=1}^m R^j_{k l l} Y^l \omega^k, \]
\[ \omega^j_\ast^\ast = \pi^* \theta^j. \]

§ 2. Riemannian submersion

Let \( N \) be an \( n \)-dimensional Riemannian manifold with metric \( ds_N^2 \). We assume \( n > m \). Let \( f: N \to M \) be a smooth mapping. If for every point \( p \) of \( N \), we can choose local 1-forms \( \omega^1, \ldots, \omega^m \) in a neighborhood of \( p \) in \( N \) and \( \theta^1, \ldots, \theta^m \) in a neighborhood of \( f(p) \) in \( M \) such that \( ds_N^2 = \sum_{a=1}^n (\omega^a)^2 \), \( ds_M^2 = \sum_{i=1}^m (\theta^i)^2 \) and

\[ f^* \theta^i = \omega^i, \quad i = 1, \ldots, m, \]

\( f: N \to M \) is called a Riemannian submersion. (In this section, the indices \( a, b, c \) run from 1 to \( n \) and \( \alpha, \beta \) from \( m+1 \) to \( n \).) Let \( \omega^g \) be the connection forms in \( N \), i.e.,

\[ d\omega^a = \sum_{b=1}^n \omega^b \wedge \omega^g_b. \]

Then we can put

\[ f^* \theta^j - \omega^j = \sum_{a=m+1}^n L^j_a \omega^a, \]
\[ \omega^j_\ast = \sum_{\beta=m+1}^n L^j_\beta \omega^\beta. \]

\( L^j_a, L^j_\beta \) are called the structure tensors of the Riemannian submersion \( f \). If \( \sum_{a=m+1}^n L^i_a = 0 \) (resp. \( L^i_\beta = 0 \)), \( f \) is said to be minimal (resp. totally geodesic) [2].

Now we will return to the natural projection \( \pi: T(M) \to M \). Since we have \( \pi^* \theta^i = \omega^i \), it is a Riemannian submersion. Moreover, Proposition 1 implies

**Proposition 2.** The natural projection \( \pi: T(M) \to M \) is a totally geodesic Riemannian submersion with structure tensors

\[ L^j_{ik} = \frac{1}{2} \sum_{l=1}^m R^j_{ik l} Y^l, \quad L^j_{i k \ast} = 0. \]

§ 3. Sections of tangent bundles

Let \( \phi_X: M \to T(M) \) be a section of the tangent bundle. We can put locally
\[ X = \sum_{i=1}^{n} X^i e_i \] with respect to the dual base \( \{ e_i \} \) of \( \{ \theta^i \} \). Define \( F^A_i \) by

\[ \phi^*_A(\omega^A) = \sum_{i=1}^{n} F^A_i \theta^i. \]

Then it holds

\[ \phi^*_A(\omega^A) = \phi^*_{X^A} \pi^*(\theta^i) = \theta^i. \]

By a calculation, we get

\[ \phi^*_A(\omega^A) = \sum_{k=1}^{n} X^i_k \theta^k, \]

where \( X_k^i \) are components of the first covariant differential of \( X \) given by

\[ \sum_{k=1}^{n} X^i_k \theta^k = dX^i + \sum_{j=1}^{n} X^{ij} \theta^j. \]

Thus it is evident

\[ F^i_j = \delta^i_j, \quad F^*_{ij} = X^i_j. \]

The fundamental tensor \( F^A_{ij} \) of the mapping \( \phi_X \) is defined to be

\[ \sum_{j=1}^{m} F^A_{ij} \theta^j = dF^i_A + \sum_{k=1}^{m} F^A_k \omega^k_B - \sum_{j=1}^{m} F^i_A \theta^j. \]

If \( \sum_{i=1}^{m} F^A_i = 0 \), \( \phi_X \) is called a harmonic mapping [1]. Using (5) and (9) we obtain

**Proposition 3.** The components \( F^A_{ij} \) of the fundamental tensor of the mapping \( \phi_X: M \to T(M) \) are given by

\[
\begin{align*}
F^A_{ij} &= \frac{1}{2} \sum_{l,h} \left( R^A_{ih} X^j_l + R^A_{lj} X^h_i \right) X^i, \\
F^A_{ij} &= X^i_j + \frac{1}{2} \sum_{l,h} R^i_{lj} X^l,
\end{align*}
\]

where \( X^i_{ij} \) are the components of the second covariant differential of the vector field \( X \).

**Proposition 4.** \( \phi_X: M \to T(M) \) is a harmonic mapping iff

\[ \sum_{i=1}^{m} X^i_l = 0, \quad \sum_{j=1}^{m} R^i_{ij} X^l = 0. \]

If \( M \) is compact and orientable, we have the following integral formula [4, p. 39]

\[
\int_M \left\{ \frac{m}{2} \sum_{i,k=1}^{m} X^i_l X^k + \sum_{i,j=1}^{m} (X^i_j)^2 \right\} dV = 0,
\]
where \( dV \) is the Riemannian volume element. Hence, \( \sum_{i=1}^{m} X^i_T = 0 \) \( (k = 1, \ldots, m) \) imply \( X^i_j = 0 \) \( (i, j = 1, \ldots, m) \). Thus we get

**Proposition 5.** Assume that \( M \) is compact and orientable. \( \phi_X : M \to T(M) \) is harmonic iff \( X \) has the vanishing covariant derivative.

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References