On a Submanifold of a Submanifold of a Riemannian Manifold and the Gauss Map

Dedicated to Professor Dr. Makoto Matsumoto on his sixtieth birthday

By

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The fundamental properties of frame bundles of a submanifold of a Riemannian manifold are described by S. Kobayashi and K. Nomizu in [2]. Using the similar method, we will study frame bundles of a submanifold of a submanifold of a Riemannian manifold. The main purpose of this paper is to associate the Gauss (generalized) map to a submanifold of a submanifold of Euclidean space. M. Obata [3] associates the Gauss map to a submanifold of a simply-connected complete $N$-space of constant curvature. We will study the relationship between the Gauss map in the sense of Obata and that of our sense in the forthcoming paper.

§ 1. Euclidean spaces and orthogonal groups

Let $e_1, e_2, \ldots, e_{n+p+q}$ be the natural base for the $(n+p+q)$-dimensional Euclidean space $R^{n+p+q}$. We shall denote by $R^n$ the subspace of $R^{n+p+q}$ spanned by $e_1, e_2, \ldots, e_n$, that is, $R^n = \{e_1, e_2, \ldots, e_n\}$. Similarly we set

\[ R^p = \{e_{n+1}, e_{n+2}, \ldots, e_{n+p}\}, \quad R^q = \{e_{n+p+1}, e_{n+p+2}, \ldots, e_{n+p+q}\}, \]
\[ R^{n+p} = \{e_1, \ldots, e_n, e_{n+p}\}, \quad R^{p+q} = \{e_{n+1}, \ldots, e_{n+p}, e_{n+p+q}\}. \]

Let $O(n+p+q), O(n), O(p), O(q), O(n+p)$ and $O(p+q)$ denote the orthogonal groups of $R^{n+p+q}, R^n, R^p, R^q, R^{n+p}$ and $R^{p+q}$ respectively. We identify $O(n)$ with the subgroup of $O(n+p+q)$ consisting of all elements which induce the identity transformation on the subspace $R^{p+q}$. In other words

\[ O(n) \cong \left( \begin{array}{cc} O(n) & 0 \\ 0 & I_{p+q} \end{array} \right), \]

where $I_{p+q}$ denotes the identity matrix of order $n+p$. Similarly we have

\[ O(p) \cong \left( \begin{array}{cc} I_n & 0 \\ O(p) & I_q \end{array} \right), \quad O(q) \cong \left( \begin{array}{cc} I_{n+p} & 0 \\ 0 & O(q) \end{array} \right), \]
\[ O(n+p) \simeq \begin{pmatrix} O(n+p) & 0 \\ 0 & I(q) \end{pmatrix}, \quad O(p+q) \simeq \begin{pmatrix} I_n & 0 \\ 0 & O(p+q) \end{pmatrix}, \]

where \( I_n, I_{n+p} \) and \( I_q \) are the identity matrices of order \( n, n+p \) and \( q \) respectively. Let \( \mathfrak{o}(n+p+q), \mathfrak{o}(n), \mathfrak{o}(p), \mathfrak{o}(q), \mathfrak{o}(n+p), \mathfrak{o}(n+q) \) and \( \mathfrak{o}(p+q) \) be the Lie algebras of \( O(n+p+q), O(n), O(p), O(q), O(n+p), O(n+q) \) and \( O(p+q) \). Let \( B \) be the Killing-Cartan form of \( \mathfrak{o}(n+p+q) \). It holds

\[ B(X, Y) = 2 \text{trace}(XY). \]

Let \( \mathfrak{g}(n, p, q) \) be the orthogonal complement to \( \mathfrak{o}(n) + \mathfrak{o}(p) + \mathfrak{o}(q) \) in \( \mathfrak{o}(n+p+q) \) with respect to the Killing Cartan form \( B \). Then \( \mathfrak{g}(n, p, q) \) consists of matrices of the form

\[ \begin{pmatrix} 0 & A & B \\ -A^t & 0 & C \\ -B^t & -C & 0 \end{pmatrix}, \]

where \( A \) is a matrix with \( n \) rows and \( p \) columns, \( B \) a matrix with \( n \) rows and \( q \) columns, \( C \) a matrix with \( p \) rows and \( q \) columns, and \( A^t, B^t \) and \( C^t \) are the transposes of \( A, B \) and \( C \) respectively.

\section*{§ 2. Frame bundles of a submanifold of a submanifold}

Let \( L \) be an \((n+p+q)\)-dimensional smooth Riemannian manifold with Riemannian metric \( g \). Let \( f_1 \) be an immersion of an \((n+p)\)-dimensional smooth manifold \( N \) into \( L \). Next let \( f_2 \) be an immersion of an \( n \)-dimensional smooth manifold \( M \). We also denote by \( g \) the metric induced on \( N \) as well as the metric induced on \( M \). For any point \( x \) of \( M \) we shall denote \( f_2(x) \in N \) and \( f_1 f_2(x) \in L \) by the same letter \( x \) if there is no danger of confusion. Thus the tangent space \( T_x(N) \) is a subspace of the tangent space \( T_x(L) \) and \( T_x(L) \) is a subspace of \( T_x(N) \).

Let \( O(M), O(N) \) and \( O(L) \) be the bundles of orthogonal frames over \( M, N \) and \( L \) respectively. \( O(N)|M = \{ v \in O(N); \pi(v) \in M \} \) is a principal fibre bundle over \( M \) with structure group \( O(n+p) \). The set of frames \( \{ v \in O(N)|M \} \) of the form \( (Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+p}) \) with \( Y_1, \ldots, Y_n \) tangent to \( M \) forms the principal bundle \( O(N, M) \) over \( M \) with group \( O(n) \times O(p) \). Similarly we have the principal fibre bundle \( O(L, N) \) over \( N \) with group \( O(n+p) \times O(q) \). A frame \( v \in O(L)|M \) is said to be adapted if \( v \) is of the form \( (Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+p}, Y_{n+p+1}, \ldots, Y_{n+p+q}) \) with \( Y_1, \ldots, Y_n \) tangent to \( M \) and \( Y_{n+1}, \ldots, Y_{n+p}, Y_{n+p+1}, \ldots, Y_{n+p+q} \) tangent to \( N \). Thus considered as a linear isomorphism \( \mathbb{R}^{n+q+p} \rightarrow T_x(L), v \) is adapted if and only if \( v \) maps the subspace \( R^n \) onto \( T_x(M) \) and the subspace \( R^{n+p} \) onto \( T_x(N) \), where \( \pi(v) = x \). The set of adapted
frames forms a principal fibre bundle over $M$ with group $O(n) \times O(p) \times O(q)$. The bundle of adapted frames is denoted by $O(L, N, M)$. We define a homomorphism $h_1 : O(L, N, M) \rightarrow O(M, N)$ by

$$h_1(v) = (Y_1, \ldots, Y_n, Y_{n+p})$$

for $v = (Y_1, \ldots, Y_{n+p+q}) \in O(L, N, M)$. Similarly we can define homomorphisms $h_2 : O(L, N, M) \rightarrow O(L, N, M)/O(p)$, $h_3 : O(L, N, M) \rightarrow O(L, N, M)/O(n)$ and $h_0 : O(N, M) \rightarrow O(M)$, where $O(L, N, M)/O(p)$ is the bundle of frames of the form $(Y_1, \ldots, Y_n, Y_{n+p+1}, \ldots, Y_{n+p+q})$ with $Y_1, \ldots, Y_n$ tangent to $M$ and $Y_{n+p+1}, \ldots, Y_{n+p+q}$ normal to $N$, and so on. Corresponding the natural projection $O(n) \times O(p) \times O(q) \rightarrow O(n)$ we obtain a homomorphism $h : O(L, N, M) \rightarrow O(L, N, M)/O(n) \times O(q) = O(M)$. Similarly we have $h' : O(L, N, M) \rightarrow O(L, N, M)/O(n) \times O(q)$ and $h'' : O(L, N, M) \rightarrow O(L, N, M)/O(n) \times O(p)$. The following diagrams illustrate these bundles and homomorphisms:

$$
\begin{array}{c}
O(N, M) \\
\downarrow h_0 \quad O(L, N, M)/O(q) \\
O(L, N, M)/O(p) \times O(q) \\
\downarrow h_1 \quad O(L, N, M)/O(n) \times O(q) \\
O(L, N, M)/O(p) \\
\downarrow h_2 \quad O(L, N, M)/O(n) \times O(p) \\
\end{array}
$$

There are the following natural injections:

$$
\begin{array}{c}
O(N, M) \xrightarrow{k_1} O(N)/M \xrightarrow{k_2} O(N) \\
O(n) \times O(p) \downarrow \quad O(n+p) \downarrow \pi_N \quad O(n+p) \downarrow \pi_N \\
M \quad \xleftarrow{\pi_N} M \quad N
\end{array}
$$

Moreover we have the following commutative diagram:

$$
\begin{array}{c}
O(L, N) \xrightarrow{i_1} O(L, N, M) \xrightarrow{i_2} O(L) \\
O(n) \times O(p) \downarrow \pi_L \quad O(n+p) \downarrow \pi_L \\
M \quad O(L, M) \quad N \quad O(L) \quad M
\end{array}
$$

where $i_1$, $i_2$, $j_1$ and $j_2$ are the natural injections.
§ 3. Canonical forms and connection forms

Let $\theta_M$, $\theta_N$ and $\theta_L$ be the canonical forms of $M$, $N$ and $L$ respectively. $\theta_M$ is an $\mathbb{R}^n$-valued 1-form on $O(M)$, $\theta_N$ is an $\mathbb{R}^{n+p}$-valued 1-form on $O(N)$ and $\theta_L$ is an $\mathbb{R}^{n+p+q}$-valued 1-form on $O(M)$. Put $k = k_2k_1$. Then it holds [2, Chapter 7, Proposition 1.1]

$$
k^*\theta_N = h^*_N \theta_M.
$$

Set $i = i_2i_1$. Then we can prove similarly

**Proposition 1.**

$$(kh_1)^*\theta_N = h^*\theta_M = i^*\theta_L.$$

**Proof.** By definition of $\theta_L$, it follows

$$i^*\theta_L(Y) = (i(v))^{-1} \pi_*h_*i_*(Y) = v^{-1}\pi_*(Y) \quad \text{for} \quad Y \in T_m(O(L, M, N)).$$

Since $\pi_*(Y) \in T_m(M)$, where $x = \pi(v)$, we get $v^{-1}\pi_*(Y) \in \mathbb{R}^n$. Since $h(v) = v \mid \mathbb{R}^n$ and since $\pi_*(Y) = \pi$, we have

$$v^{-1}\pi_*(Y) = v^{-1}(\pi_*(Y)) = (h(v))^{-1} \pi_*(h_*(Y)) = \theta_M(h_*(Y)) = (h^*\theta_M)(Y).$$

Using the equation (1), we get

$$(h_0h_1)^*\theta_M = (kh_1)^*\theta_N.$$ 

q.e.d.

Let $\omega^M$, $\omega^N$ and $\omega^L$ be the Riemannian connection forms on $O(M)$, $O(N)$ and $O(L)$ respectively. The following result is also found in [2, Chapter 7, Proposition 1.2].

**Proposition 2.** Let $\tilde{\omega}$ be the $\mathfrak{o}(n) + \mathfrak{o}(p)$-component of $k^*\omega^N$ with respect to the decomposition $\mathfrak{o}(n+p) = \mathfrak{o}(n) + \mathfrak{o}(p) + \mathfrak{g}(n, p)$, where $\mathfrak{g}(n, p)$ is the orthogonal complement to $\mathfrak{o}(n) + \mathfrak{o}(p)$ in $\mathfrak{o}(n+p)$ with respect to the Killing-Cartan form. Then $\tilde{\omega}$ defines a connection in the bundle $O(N, M)$.

We have the following direct sum decompositions:

$$\mathfrak{o}(n+p+q) = \mathfrak{o}(n) + \mathfrak{o}(p) + \mathfrak{o}(q) + \mathfrak{g}(n, p, q)$$

$$= \mathfrak{o}(n+p) + \mathfrak{o}(q) + \mathfrak{g}(n+p, q)$$

$$= \mathfrak{o}(n) + \mathfrak{o}(p+q) + \mathfrak{g}(n, p+q),$$

$$\mathfrak{g}(n, p, q) = \mathfrak{g}(n+p, q) + \mathfrak{g}(n, p) = \mathfrak{g}(n, p+q) + \mathfrak{g}(p, q).$$

Corresponding to the above, we get the following decompositions of connection forms;
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\[
\begin{align*}
\omega^L &= \omega^L_c + \omega^L_\ell + \omega^L_q + \omega^L_{\bar{g}(n,p,q)} \\
&= \omega^L_c(n+p) + \omega^L_\ell(q) + \omega^L_{\bar{g}(n+p,q)} \\
&= \omega^L_c(n) + \omega^L_\ell(p+q) + \omega^L_{\bar{g}(n,p+q)}.
\end{align*}
\]

Moreover we have

\[
\begin{align*}
\omega^L_c(n+p) &= \omega^L_c(n) + \omega^L_\ell(p) + \omega^L_{\bar{g}(n,p)}, \\
\omega^L_\ell(p+q) &= \omega^L_\ell(p) + \omega^L_q + \omega^L_{\bar{g}(p,q)}, \\
\omega^L_{\bar{g}(n,p,q)} &= \omega^L_{\bar{g}(n+p,q)} + \omega^L_{\bar{g}(n,p)} + \omega^L_{\bar{g}(p,q)}.
\end{align*}
\]

Using Proposition 6.4 in [2, Chapter 2], we can prove the following result as similarly as Proposition 2.

\textbf{Proposition 3.} Put

\[
\begin{align*}
\omega &= i^s(\omega^L_c(n) + \omega^L_\ell(p) + \omega^L_q), \\
\omega' &= i_2^s(\omega^L_c(n+p) + \omega^L_q), \\
\omega'' &= j_2^s(\omega^L_{\bar{g}(n)} + \omega^L_{\bar{g}(p+q)}).
\end{align*}
\]

Then \(\omega, \omega'\) and \(\omega''\) define connections in the bundles \(O(L, N, M), O(L, N)\) and \(O(L, M)\) respectively.

There is the following in [2, Chapter 7, § 1]

\textbf{Proposition 4.} The homomorphism \(h_0: O(N, M) \to O(M)\) maps the connection in \(O(N, M)\) defined by \(\bar{\omega}\) into the Riemannian connection of \(M\), that is

\[
h_0^* \omega^M = \bar{\omega}_\circ(n),
\]

where \(\bar{\omega}_\circ(n)\) denotes the \(\circ(n)\)-component of the \(\circ(n)+\circ(p)\)-valued form \(\bar{\omega}\).

Let \(\omega_\circ(n)\) (resp. \(\omega_\circ(n)+\circ(p)\)) denote the \(\circ(n)\) (resp. \(\circ(n)+\circ(p)\)) component of the \(\circ(n)+\circ(p)+\circ(q)\)-valued form \(\omega\), that is

\[
\omega_\circ(n) = i^s(\omega^L_c(n)), \quad \omega_\circ(n)+\circ(p) = i^s(\omega^L_c(n)+\omega^L_\ell(p)).
\]

Then we have

\textbf{Proposition 5.} The homomorphism \(h_1: O(L, N, M) \to O(M, N)\) maps the connection in \(O(L, N, M)\) defined by \(\omega\) into the connection in \(O(M, N)\) defined by \(\bar{\omega}\). Hence the homomorphism \(h=h_0h_1\) maps the connection in \(O(L, N, M)\) into the Riemannian connection of \(M\) and the following relations are valid:

\[
h_1^*(\bar{\omega}) = \omega_{\circ(n)+\circ(p)}, \quad h^* \omega^M = \omega_{\circ(n)}.
\]
Proof. Since $h_1 : O(L, N, M) \rightarrow O(N, M)$ is a homomorphism such that the induced mapping $h_1 : M \rightarrow M$ is the identity mapping of $M$, from the well known result [2, Chapter 2, Proposition 6.1], it follows that $h_1$ maps the connection defined by $\omega$ into a connection in $O(N, M)$ whose connection form $\tilde{\omega}'$ satisfies $h_1^*(\tilde{\omega}') = \omega_{\sigma(n) + \sigma(p)}$. Since $h_1$ maps $O(L, N, M)$ onto $O(N, M)$, in order to show $\tilde{\omega} = \tilde{\omega}'$, we only prove $h_1^*(\tilde{\omega}) = h_1^*(\tilde{\omega}') = \omega_{\sigma(n) + \sigma(p)}$. We have the following commutative diagram:

$$
\begin{array}{ccc}
O(L, N, M) & \xrightarrow{k_1} & O(N, M) \\
\downarrow i_1 & & \downarrow k \\
O(L, N) & \xrightarrow{k_4} & O(N) 
\end{array}
$$

where $h_4$ is the homomorphism corresponding to the natural projection $O(n + p) \times O(q) \rightarrow O(n + p)$. Corresponding to the decomposition $\sigma(n + p) = \sigma(n) + \sigma(p) + g(n, p)$, $\omega^N$ is written as

$$
\omega^N = \omega^N_{\sigma(n)} + \omega^N_{\sigma(p)} + \omega^N_{g(n, p)}.
$$

Then we have

$$
(5) \quad h_1^* \tilde{\omega} = h_1^* k^* (\omega^N_{\sigma(n)} + \omega^N_{\sigma(p)}) = i_1^* h_1^* (\omega^N_{\sigma(n)} + \omega^N_{\sigma(p)}).
$$

From (3) we get

$$
(6) \quad \omega' = i_1^* (\omega_{\sigma(n) + \sigma(p)} + \omega_{\sigma(q)}) = i_1^* \omega_{\sigma(n)} + i_1^* \omega_{\sigma(p)} + i_1^* \omega_{\sigma(n, p)} + i_1^* \omega_{\sigma(q)}.
$$

Applying Proposition 4, we have

$$
(7) \quad h_4^* \omega^N = i_1^* \omega_{\sigma(n) + \sigma(p)}.
$$

Combining (6) with (7), we obtain

$$
(8) \quad h_4^* (\omega^N_{\sigma(n)} + \omega^N_{\sigma(p)}) = i_1^* \omega_{\sigma(n)} + i_1^* \omega_{\sigma(p)}.
$$

Finally we obtain

$$
(9) \quad h_1^* \tilde{\omega} = i_1^* h_4^* (\omega^N_{\sigma(n)} + \omega^N_{\sigma(p)}) = i_1^* i_1^* (\omega_{\sigma(n)} + \omega_{\sigma(p)}) = \omega_{\sigma(n) + \sigma(p)}.
$$

q. e. d.

The homomorphisms $h_2, h_3, h'$ and $h''$ given in § 3 map the connection defined by $\omega$ in $O(L, N, M)$ into connections in $O(L, N, M)/O(p)$, $O(L, N, M)/O(n)$, $O(L, N, M)/O(n) \times O(q)$ and $O(L, N, M)/O(n) \times O(p)$ respectively. We call those connections as the canonical connections in those bundles respectively.

§ 4. The Gauss map

Let $G(n, p)$ be the Grassmann manifold of $n$-planes in $R^{n+p}$. Then we have
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\[ G(n, p) = O(n + p)/O(n) \times O(p). \]

By an \( n \)-frame in \( \mathbb{R}^{n+p} \), we mean an ordered set of \( n \) orthonormal vectors in \( \mathbb{R}^{n+p} \).
Let \( V(n, p) \) be the Stiefel manifold of \( n \)-frames in \( \mathbb{R}^{n+p} \). Then we have

\[ V(n, p) = O(n + p)/O(p). \]

A pair \((U, V)\) of \( n \)-dimensional linear subspace \( U \) and \( p \)-dimensional linear subspace \( V \) of \( \mathbb{R}^{n+p+q} \) such that \( U \cap V = \{0\} \) will be said to be a direct sum pair of type \((n, p)\) in \( \mathbb{R}^{n+p+q} \). Let \( G(n, p, q) \) be the set of direct sum pairs of type \((n, p)\) in \( \mathbb{R}^{n+p+q} \).

The group \( O(n+p+q) \) acts transitively on \( G(n, p, q) \). The elements of \( O(n+p+q) \) which leave invariant the particular pair \( (\mathbb{R}^n, \mathbb{R}^p) \) form the subgroup \( O(n) \times O(p) \times O(q) \). Thus we have

\[ G(n, p, q) = O(n+p+q)/O(p) \times O(p) \times O(q). \]

The homogeneous space \( G(n, p, q) \) is considered as a fibre space over Grassmann manifolds. In fact we have three fibre bundles:

- \( G(n, p, q) \) over \( G(n+p, q) \) with fibre \( G(n, p) \),
- \( G(n, p, q) \) over \( G(n, p+q) \) with fibre \( G(p, q) \),
- \( G(n, p, q) \) over \( G(n+p, q) \) with fibre \( G(n, q) \).

For example the projection of \( G(n, p, q) \) onto \( G(n+p, q) \) maps a direct sum pair \((U, V)\) to the \((n+p)\)-dimensional subspace \( U + V \). We have moreover the following seven principal fibre bundles over \( G(n, p, q) \):

- \( E = O(n+p+q) \) over \( G(n, p, q) \) with group \( O(n) \times O(p) \times O(q) \),
- \( E_1 = V(p+q, n) = O(n+p+q)/O(n) \) over \( G(n, p, q) \) with group \( O(p) \times O(q) \),
- \( E_2 = V(n+q, p) = O(n+p+q)/O(p) \) over \( G(n, p, q) \) with group \( O(n) \times O(q) \),
- \( E_3 = V(n+p, q) = O(n+p+q)/O(q) \) over \( G(n, p, q) \) with group \( O(n) \times O(p) \),
- \( E'_1 = O(n+p+q)/O(p) \times O(q) \) over \( G(n, p, q) \) with group \( O(n) \),
- \( E'_2 = O(n+p+q)/O(n) \times O(q) \) over \( G(n, p, q) \) with group \( O(p) \),
- \( E'_3 = O(n+p+q)/O(n) \times O(p) \) over \( G(n, p, q) \) with group \( O(q) \).

Let \( \gamma \) be the canonical 1-form of \( O(n+p+q) \), that is, the left invariant \( o(n+p+q) \)-valued 1-form uniquely determined by

\[ \gamma(A) = A \quad \text{for} \quad A \in o(n+p+q). \]

Let \( \omega_k \) be the \( o(n) + o(p) + o(q) \)-component of \( \gamma \) with respect to the decomposition
\( o(n+p+q) = o(n) + o(p) + o(q) + g(n, p, q) \). By the well known result [2, Chapter 2, Theorem 11.1], the form \( \omega_E \) defines a connection in \( E \) which will be called the canonical connection in \( E \) and will be denoted by \( \Gamma_E \). Let \( f_i \) (resp. \( f'_i \)) be the bundle homomorphisms of \( E \) onto \( E_i \) (resp. \( E'_i \)) defined by the natural projections \((i = 1, 2 \text{ and } 3)\). The homomorphisms \( f_i \) (resp. \( f'_i \)) map the connection \( \Gamma_E \) onto connections in \( E_i \) (resp. \( E'_i \)), denoted by \( \Gamma_{E_i} \) (resp. \( \Gamma_{E'_i} \)) such that the connection forms \( \omega_{E_i} \) (resp. \( \omega_{E'_i} \)) are determined by \( f^*_1(\omega_{E_i}) = \gamma_{o(p)} + \gamma_{o(q)} \), \( f^*_2(\omega_{E_i}) = \gamma_{o(n)} + \gamma_{o(q)} \) and \( f^*_3(\omega_{E_i}) = \gamma_{o(n)} + \gamma_{o(p)} \) (resp. \( f'^*_1(\omega_{E'_i}) = \gamma_{o(n)} \), \( f'^*_2(\omega_{E'_i}) = \gamma_{o(p)} \) and \( f'^*_3(\omega_{E'_i}) = \gamma_{o(q)} \)), respectively, where \( \gamma_{o(n)}, \gamma_{o(p)} \) and \( \gamma_{o(q)} \) are the \( o(n) \), \( o(p) \) and \( o(q) \)-components of \( \gamma \) respectively.

Let \( P_1 \) and \( P_2 \) be principal bundles over \( M \) with groups \( G_1 \) and \( G_2 \) respectively. Then \( P_1 \times P_2 \) is a principal fibre bundle over \( M \times M \) with group \( G_1 \times G_2 \). Let \( P_1 + P_2 \) be the restriction of \( P_1 \times P_2 \) to the diagonal \( \Delta M \) of \( M \times M \). Since \( \Delta M \) and \( M \) are diffeomorphic to each other, \( P_1 + P_2 \) is considered as a principal fibre bundle over \( M \). Moreover let \( P_3 \) be a principal fibre bundle over \( M \) with group \( G_3 \). Then we can construct a principal fibre bundle \( P_1 + P_2 + P_3 \) over \( M \) with group \( G_1 \times G_2 \times G_3 \). Now we have the following bundle isomorphisms:

\[
(f_i, f'_i): E \cong E_i + E'_i \quad (i = 1, 2 \text{ and } 3),
\]

\[
(f'_1, f'_2, f'_3): E \cong E'_1 + E'_2 + E'_3.
\]

Let \( N \) be an \((n+p)\)-dimensional manifold immersed in the \((n+p+q)\)-dimensional Euclidean space \( \mathbb{R}^{n+p+q} \). Let \( M \) be an \( n \)-dimensional manifold immersed in \( N \). We have the following principal fibre bundles over \( M \).

\[
P = O(\mathbb{R}^{n+p+q}, N, M) \text{ over } M \text{ with group } O(n) \times O(p) \times O(q),
\]

\[
P_1 = O(\mathbb{R}^{n+p+q}, N, M) \text{ over } M \text{ with group } O(p) \times O(q),
\]

\[
P_2 = O(\mathbb{R}^{n+p+q}, N, M) \text{ over } M \text{ with group } O(n) \times O(q),
\]

\[
P_3 = O(\mathbb{R}^{n+p+q}, N, M) \text{ over } M \text{ with group } O(n) \times O(p),
\]

\[
P'_1 = O(\mathbb{R}^{n+p+q}, N, M) \text{ over } M \text{ with group } O(n),
\]

\[
P'_2 = O(\mathbb{R}^{n+p+q}, N, M) \text{ over } M \text{ with group } O(p),
\]

\[
P'_3 = O(\mathbb{R}^{n+p+q}, N, M) \text{ over } M \text{ with group } O(q).
\]

The canonical connections in \( P, P_1, P_2, P_3, P'_1, P'_2 \) and \( P'_3 \) given as in §3, will be denoted by \( \Gamma_P, \Gamma_{P_1}, \Gamma_{P_2}, \Gamma_{P_3}, \Gamma_{P'_1}, \Gamma_{P'_2}, \Gamma_{P'_3} \) and \( \Gamma_P \).

We now define a bundle map \( g: P \rightarrow E \). The bundle \( O(\mathbb{R}^{n+p+q}) \) of orthonormal frames over \( \mathbb{R}^{n+p+q} \) is trivial, that is, \( O(\mathbb{R}^{n+p+q}) = \mathbb{R}^{n+p+q} \times O(n+p+q) \). Let \( \rho: O(\mathbb{R}^{n+p+q}) \rightarrow O(n+p+q) \) be the natural projection. Let \( i: P \rightarrow O(\mathbb{R}^{n+p+q}) \) be the natural injection. Then we define
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\[ g(v) = \rho i(v) \quad \text{for} \quad v \in P. \]

Since \( g \) commutes with the right translation by \( O(n) \times O(p) \times O(q) \), \( g \) is a bundle map of \( P \) into \( E \). The bundle map \( g \) induces bundle maps \( g_i: P_i \rightarrow E_i \) and \( g'_i: P'_i \rightarrow E'_i \) \((i = 1, 2 \text{ and } 3)\). It induces also a mapping \( g: M \rightarrow G(n, p, q) \). Summing up, we have the following commutative diagram:

\[
\begin{array}{ccc}
P_i & \xrightarrow{g_i} & E_i \\
\downarrow h_i & & \downarrow f_i \\
P_i + P'_i = P & \xrightarrow{f'_i} & E = E_i + E'_i \\
\downarrow h'_i & & \downarrow f'_i \\
P'_i & \xrightarrow{g'_i} & E'_i \\
\end{array}
\]

where \( h_i: P \rightarrow P_i \) and \( h'_i: P \rightarrow P'_i \) are the natural homomorphisms as given in § 3. Now we have the following fundamental relationship of connections.

**Proposition 6.** The bundle maps \( g, g_1, g_2, g_3, g'_1, g'_2 \) and \( g'_3 \) map the connections \( \Gamma_P, \Gamma_{P_1}, \Gamma_{P_2}, \Gamma_{P_3}, \Gamma_{E_1}, \Gamma_{E_2} \) and \( \Gamma_{E_3} \) upon the connections \( \Gamma_E, \Gamma_{E_1}, \Gamma_{E_2}, \Gamma_{E_3}, \Gamma_{E'_1}, \Gamma_{E'_2} \) and \( \Gamma_{E'_3} \) respectively.

**Proof.** Since each \( f_i \) (resp. \( f'_i \)) maps \( \Gamma_E \) upon \( \Gamma_{E_i} \) (resp. \( \Gamma_{E'_i} \)) and since each \( h_i \) (resp. \( h'_i \)) maps \( \Gamma_P \) upon \( \Gamma_{P_i} \) (resp. \( \Gamma_{P'_i} \)), it suffices to prove that \( g \) maps \( \Gamma_P \) upon \( \Gamma_E \).

The flat Riemannian connection of \( R^{n+p+q} \) is given by the form \( \rho^*(\gamma) \) on \( O(R^{n+p+q}) \). The connection form \( \omega \) is the \( o(n)+o(p)+o(q) \)-component of \( i^*\rho^*(\gamma) = g^*(\gamma) \). On the other hand, \( \omega_E \) is the \( o(n)+o(p)+o(q) \)-component of \( \gamma \). Hence we obtain that \( \omega \) is equal to \( g^*(\omega_E) \).

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**References**

