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On a Submanifold of a Submanifold of a Riemannian Manifold and the Gauss Map

Dedicated to Professor Dr. Makoto Matsumoto on his sixtieth birthday

By

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The fundamental properties of frame bundles of a submanifold of a Riemannian manifold are described by S. Kobayashi and K. Nomizu in [2]. Using the similar method, we will study frame bundles of a submanifold of a submanifold of a Riemannian manifold. The main purpose of this paper is to associate the Gauss (generalized) map to a submanifold of a submanifold of Euclidean space. M. Obata [3] associates the Gauss map to a submanifold of a simply-connected complete N-space of constant curvature. We will study the relationship between the Gauss map in the sense of Obata and that of our sense in the forthcoming paper.

§ 1. Euclidean spaces and orthogonal groups

Let $e_1, e_2, \dots, e_{n+p+q}$ be the natural base for the $(n+p+q)$-dimensional Euclidean space R^{n+p+q}. We shall denote by R^n the subspace of R^{n+p+q} spanned by $e_1, e_2, \dots, e_n$, that is, $R^n = \{e_1, e_2, \dots, e_n\}$. Similarly we set

$$R^p = \{e_{n+1}, e_{n+2}, \dots, e_{n+p}\}, \quad R^q = \{e_{n+p+1}, e_{n+p+2}, \dots, e_{n+p+q}\},$$

$$R^{n+p} = \{e_1, \dots, e_n, \dots, e_{n+p}\}, \quad R^{p+q} = \{e_{n+1}, \dots, e_{n+p}, \dots, e_{n+p+q}\}.$$

Let $O(n+p+q)$, $O(n)$, $O(p)$, $O(q)$, $O(n+p)$ and $O(p+q)$ denote the orthogonal groups of R^{n+p+q}, R^n, R^p, R^q, R^{n+p} and R^{p+q} respectively. We identify $O(n)$ with the subgroup of $O(n+p+q)$ consisting of all elements which induce the identity transformation on the subspace R^{p+q}. In other words

$$O(n) \simeq \begin{pmatrix} O(n) & 0 \\ 0 & I_{p+q} \end{pmatrix},$$

where I_{p+q} denotes the identity matrix of order $n+p$. Similarly we have

$$O(p) \simeq \begin{pmatrix} I_n & & 0 \\ & O(p) & \\ 0 & & I_q \end{pmatrix}, \qquad O(q) \simeq \begin{pmatrix} I_{n+p} & 0 \\ 0 & O(q) \end{pmatrix},$$

$$O(n+p) \simeq \begin{pmatrix} O(n+p) & 0 \\ 0 & I(q) \end{pmatrix}, \qquad O(p+q) \simeq \begin{pmatrix} I_n & 0 \\ 0 & O(p+q) \end{pmatrix},$$

where I_n, I_{n+p} and I_q are the identity matrices of order n, $n+p$ and q respectively. Let $\mathfrak{o}(n+p+q)$, $\mathfrak{o}(n)$, $\mathfrak{o}(p)$, $\mathfrak{o}(q)$, $\mathfrak{o}(n+p)$, $\mathfrak{o}(n+q)$ and $\mathfrak{o}(p+q)$ be the Lie algebras of $O(n+p+q)$, $O(n)$, $O(p)$, $O(q)$, $O(n+p)$, $O(n+q)$ and $O(p+q)$. Let B be the Killing-Cartan form of $\mathfrak{o}(n+p+q)$. It holds

$$B(X, Y) = 2\, trace\,(XY).$$

Let $\mathfrak{g}(n, p, q)$ be the orthogonal complement to $\mathfrak{o}(n)+\mathfrak{o}(p)+\mathfrak{o}(q)$ in $\mathfrak{o}(n+p+q)$ with respect to the Killing Cartan form B. Then $\mathfrak{g}(n, p, q)$ consists of matrices of the form

$$\begin{pmatrix} 0 & A & B \\ -{}^tA & 0 & C \\ -{}^tB & -{}^tC & 0 \end{pmatrix},$$

where A is a matrix with n rows and p columns, B a matrix with n rows and q columns, C a matrix with p rows and q columns, and ${}^tA$, ${}^tB$ and ${}^tC$ are the transposes of A, B and C respectively.

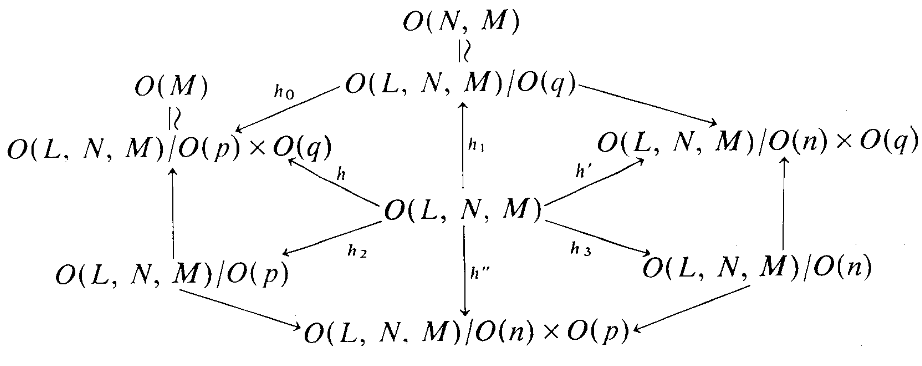
§ 2. Frame bundles of a submanifold of a submanifold

Let L be an $(n+p+q)$-dimensional smooth Riemannian manifold with Riemannian metric g. Let f_1 be an immersion of an $(n+p)$-dimensional smooth manifold N into L. Next let f_2 be an immersion of an n-dimensional smooth manifold M. We also denote by g the metric induced on N as well as the metric induced on M. For any point x of M we shall denote $f_2(x) \in N$ and $f_1 f_2(x) \in L$ by the same letter x if there is no danger of confusion. Thus the tangent space $T_x(N)$ is a subspace of the tangent space $T_x(L)$ and $T_x(M)$ is a subspace of $T_x(N)$.

Let $O(M)$, $O(N)$ and $O(L)$ be the bundles of orthogonal frames over M, N and L respectively. $O(N)|M = \{v \in O(N);\ \pi(v) \in M\}$ is a principal fibre bundle over M with structure group $O(n+p)$. The set of frames $\{v \in O(N)|M$ of the form $(Y_1,\dots, Y_n, Y_{n+1},\dots, Y_{n+p})$ with $Y_1,\dots, Y_n$ tangent to M forms the principal bundle $O(N, M)$ over M with group $O(n) \times O(p)$. Similarly we have the principal fibre bundle $O(L, N)$ over N with group $O(n+p) \times O(q)$. A frame $v \in O(L)|M$ is said to be adapted if v is of the form $(Y_1,\dots, Y_n, Y_{n+1},\dots, Y_{n+p}, Y_{n+p+1},\dots, Y_{n+p+q})$ with $Y_1,\dots, Y_n$ tangent to M and $Y_1,\dots, Y_n, Y_{n+1},\dots, Y_{n+p}$ tangent to N. Thus considered as a linear isomorphism $R^{p+q+p} \to T_x(L)$, v is adapted if and only if v maps the subspace R^n onto $T_x(M)$ and the subspace R^{n+p} onto $T_x(N)$, where $\pi(v) = x$. The set of adapted frames forms a principal fibre bundle over M with group $O(n)\times O(p)\times O(q)$. The bundle of adapted frames is denoted by $O(L, N, M)$. We define a homomorphism $h_1: O(L, N, M)\to O(M, N)$ by

$$h_1(v)=(Y_1,\dots, Y_n,\dots, Y_{n+p})$$

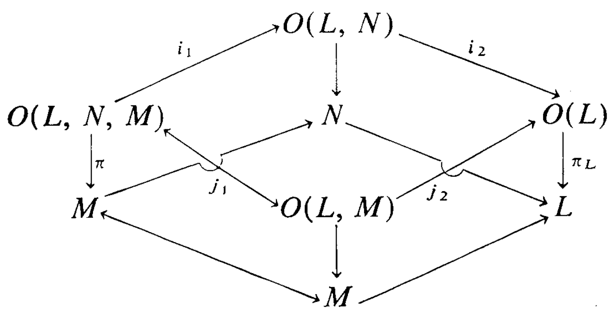
for $v=(Y_1,\dots, Y_{n+p+q})\in O(L, N, M)$. Similarly we can define homomorphisms h_2: $O(L, N, M)\to O(L, N, M)/O(p)$, h_3: $O(L, N, M)\to O(L, N, M)/O(n)$ and h_0: $O(N, M)\to O(M)$, where $O(L, N, M)/O(p)$ is the bundle of frames of the form $(Y_1,\dots, Y_n, Y_{n+p+1},\dots, Y_{n+p+q})$ with $Y_1,\dots, Y_n$ tangent to M and $Y_{n+p+1},\dots, Y_{n+p+q}$ normal to N, and so on. Corresponding the natural projection $O(n)\times O(p)\times O(q)\to O(n)$ we obtain a homomorphism h: $O(L, N, M)\to O(L, N, M)/O(p)\times O(q)=O(M)$. Similarly we have h': $O(L, N, M)\to O(L, N, M)/O(n)\times O(q)$ and h'': $O(L, N, M)\to O(L, N, M)/O(n)\times O(p)$. The following diagrams illustrate these bundles and homomorphisms:



There are the following natural injections:

$$\begin{array}{ccccc} O(N, M) & \xrightarrow{k_1} & O(N)|M & \xrightarrow{k_2} & O(N) \\ {\scriptstyle O(n)\times O(p)}\downarrow & & {\scriptstyle O(n+p)}\downarrow{\scriptstyle \pi_N} & & {\scriptstyle O(n+p)}\downarrow{\scriptstyle \pi_N} \\ M & \longleftrightarrow & M & \longrightarrow & N \end{array}$$

Moreover we have the following commutative diagram:



where i_1, i_2, j_1 and j_2 are the natural injections.

§ 3. Canonical forms and connection forms

Let θ_M, θ_N and θ_L be the canonical forms of M, N and L respectively. θ_M is an R^n-valued 1-form on $O(M)$, θ_N is an R^{n+p}-valued 1-form on $O(N)$ and θ_L is an R^{n+p+q}-valued 1-form on $O(M)$. Put $k=k_2k_1$. Then it holds [2, Chapter 7, Proposition 1.1]

$$k^*\theta_N = h_0^*\theta_M. \tag{1}$$

Set $i=i_2i_1$. Then we can prove similarly

Proposition 1. $\quad (kh_1)^*\theta_N = h^*\theta_M = i^*\theta_L.$

Proof. By definition of θ_L, it follows

$$i^*\theta_L(Y) = (i(v))^{-1}\pi_{L*}i_*(Y) = v^{-1}\pi_*(Y) \qquad \text{for} \quad Y \in T_v(O(L, M, N)).$$

Since $\pi_*(Y) \in T_x(M)$, where $x=\pi(v)$, we get $v^{-1}\pi^*(Y) \in R^n$. Since $h(v)=v\,|\,R^n$ and since $\pi_M h=\pi$, we have

$$v^{-1}\pi^*(Y) = v^{-1}(\pi_{M*}h_*(Y)) = (h(v))^{-1}\pi_{M*}h_*(Y) = \theta_M(h_*(Y)) = (h^*\theta_M)(Y).$$

Using the equation (1), we get

$$(h_0h_1)^*\theta_M = (kh_1)^*\theta_N. \qquad \text{q.e.d.}$$

Let ω^M, ω^N and ω^L be the Riemannian connection forms on $O(M)$, $O(N)$ and $O(L)$ respectively. The following result is also found in [2, Chapter 7, Proposition 1.2].

Proposition 2. *Let $\tilde{\omega}$ be the $\mathfrak{o}(n)+\mathfrak{o}(p)$-component of $k^*\omega^N$ with respect to the decomposition $\mathfrak{o}(n+p)=\mathfrak{o}(n)+\mathfrak{o}(p)+\mathfrak{g}(n, p)$, where $\mathfrak{g}(n, p)$ is the orthogonal complement to $\mathfrak{o}(n)+\mathfrak{o}(p)$ in $\mathfrak{o}(n+p)$ with respect to the Killing-Cartan form. Then $\tilde{\omega}$ defines a connection in the bundle $O(N, M)$.*

We have the following direct sum decompositions:

$$\begin{aligned}\mathfrak{o}(n+p+q) &= \mathfrak{o}(n)+\mathfrak{o}(p)+\mathfrak{o}(q)+\mathfrak{g}(n, p, q)\\ &= \mathfrak{o}(n+p)+\mathfrak{o}(q)+\mathfrak{g}(n+p, q)\\ &= \mathfrak{o}(n)+\mathfrak{o}(p+q)+\mathfrak{g}(n, p+q),\end{aligned}$$

$$\mathfrak{g}(n, p, q) = \mathfrak{g}(n+p, q)+\mathfrak{g}(n, p) = \mathfrak{g}(n, p+q)+\mathfrak{g}(p, q).$$

Corresponding to the above, we get the following decompositions of connection forms;

$$
(2)\qquad
\begin{cases}
\omega^L = \omega^L_{\mathfrak{o}(n)} + \omega^L_{\mathfrak{o}(p)} + \omega^L_{\mathfrak{o}(q)} + \omega^L_{\mathfrak{g}(n,p,q)} \\
\quad = \omega^L_{\mathfrak{o}(n+p)} + \omega^L_{\mathfrak{o}(q)} + \omega^L_{\mathfrak{g}(n+p,q)} \\
\quad = \omega^L_{\mathfrak{o}(n)} + \omega^L_{\mathfrak{o}(p+q)} + \omega^L_{\mathfrak{g}(n,p+q)} .
\end{cases}
$$

Moreover we have

$$
(3)\qquad
\begin{cases}
\omega^L_{\mathfrak{o}(n+p)} = \omega^L_{\mathfrak{o}(n)} + \omega^L_{\mathfrak{o}(p)} + \omega^L_{\mathfrak{g}(n,p)} , \\
\omega^L_{\mathfrak{o}(p+q)} = \omega^L_{\mathfrak{o}(p)} + \omega^L_{\mathfrak{o}(q)} + \omega^L_{\mathfrak{g}(p,q)} , \\
\omega^L_{\mathfrak{g}(n,p,q)} = \omega^L_{\mathfrak{g}(n+p,q)} + \omega^L_{\mathfrak{g}(n,p)} = \omega^L_{\mathfrak{g}(n,p+q)} + \omega^L_{\mathfrak{g}(p,q)} .
\end{cases}
$$

Using Proposition 6.4 in [2, Chapter 2], we can prove the following result as similarly as Proposition 2.

Proposition 3. *Put*

$$
(4)\qquad
\begin{cases}
\omega = i^*(\omega^L_{\mathfrak{o}(n)} + \omega^L_{\mathfrak{o}(p)} + \omega^L_{\mathfrak{o}(q)}) , \\
\omega' = i_2^*(\omega^L_{\mathfrak{o}(n+p)} + \omega^L_{\mathfrak{o}(q)}), \\
\omega'' = j_2^*(\omega^L_{\mathfrak{o}(n)} + \omega^L_{\mathfrak{o}(p+q)}) .
\end{cases}
$$

Then ω, ω' and ω'' define connections in the bundles $O(L, N, M)$, $O(L, N)$ and $O(L, M)$ respectively.

There is the following in [2, Chapter 7, § 1]

Proposition 4. *The homomorphism h_0: $O(N, M) \to O(M)$ maps the connection in $O(N, M)$ defined by $\tilde{\omega}$ into the Riemannian connection of M, that is*

$$
h_0^* \omega^M = \tilde{\omega}_{\mathfrak{o}(n)} ,
$$

where $\tilde{\omega}_{\mathfrak{o}(n)}$ denotes the $\mathfrak{o}(n)$-component of the $\mathfrak{o}(n)+\mathfrak{o}(p)$-valued form $\tilde{\omega}$.

Let $\omega_{\mathfrak{o}(n)}$ (resp. $\omega_{\mathfrak{o}(n)+\mathfrak{o}(p)}$) denote the $\mathfrak{o}(n)$ (resp. $\mathfrak{o}(n)+\mathfrak{o}(p)$) component of the $\mathfrak{o}(n)+\mathfrak{o}(p)+\mathfrak{o}(q)$-valued form ω, that is

$$
\omega_{\mathfrak{o}(n)} = i^* \omega^L_{\mathfrak{o}(n)}, \quad \omega_{\mathfrak{o}(n)+\mathfrak{o}(p)} = i^*(\omega^L_{\mathfrak{o}(n)} + \omega^L_{\mathfrak{o}(p)}) .
$$

Then we have

Proposition 5. *The homomorphism h_1: $O(L, N, M) \to O(M, N)$ maps the connection in $O(L, N, M)$ defined by ω into the connection in $O(M, N)$ defined by $\tilde{\omega}$. Hence the homomorphism $h = h_0 h_1$ maps the connection in $O(L, N, M)$ into the Riemannian connection of M and the following relations are valid:*

$$
h_1^*(\tilde{\omega}) = \omega_{\mathfrak{o}(n)+\mathfrak{o}(p)}, \quad h^*(\omega^M) = \omega_{\mathfrak{o}(n)} .
$$

Proof. Since $h_1: O(L, N, M) \to O(N, M)$ is a homomorphism such that the induced mapping $h_1: M \to M$ is the identity mapping of M, from the well known result [2, Chapter 2, Proposition 6.1], it follows that h_1 maps the connection defined by ω into a connection in $O(N, M)$ whose connection form $\tilde{\omega}'$ satisfies $h_1^*(\tilde{\omega}') = \omega_{\mathfrak{o}(n)+\mathfrak{o}(p)}$. Since h_1 maps $O(L, N, M)$ onto $O(N, M)$, in order to show $\tilde{\omega} = \tilde{\omega}'$, we only prove $h_1^*(\tilde{\omega}) = h_1^*(\tilde{\omega}') = \omega_{\mathfrak{o}(n)+\mathfrak{o}(p)}$. We have the following commutative diagram:

$$\begin{array}{ccc} O(L, N, M) & \xrightarrow{k_1} & O(N, M) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle k} \\ O(L, N) & \xrightarrow[k_4]{} & O(N), \end{array}$$

where h_4 is the homomorphism corresponding to the natural projection $O(n+p) \times O(q) \to O(n+p)$. Corresponding to the decomposition $\mathfrak{o}(n+p) = \mathfrak{o}(n) + \mathfrak{o}(p) + \mathfrak{g}(n, p)$, ω^N is written as

$$\omega^N = \omega^N_{\mathfrak{o}(n)} + \omega^N_{\mathfrak{o}(p)} + \omega^N_{\mathfrak{g}(n,p)}.$$

Then we have

$$h_1^*\tilde{\omega} = h_1^* k^*(\omega^N_{\mathfrak{o}(n)} + \omega^N_{\mathfrak{o}(p)}) = i_1^* h_4^*(\omega^N_{\mathfrak{o}(n)} + \omega^N_{\mathfrak{o}(p)}). \tag{5}$$

From (3) we get

$$\omega' = i_1^*(\omega^L_{\mathfrak{o}(n+p)} + \omega_{\mathfrak{o}(q)})^L = i_1^*\omega^L_{\mathfrak{o}(n)} + i_1^*\omega^L_{\mathfrak{o}(p)} + i^*\omega^L_{\mathfrak{g}(n,p)} + i_1^*\omega^L_{\mathfrak{o}(q)}. \tag{6}$$

Applying Proposition 4, we have

$$h_4^*\omega^N = i_1^*\omega^L_{\mathfrak{o}(n+p)}. \tag{7}$$

Combining (6) with (7), we obtain

$$h_4^*(\omega^N_{\mathfrak{o}(n)} + \omega^N_{\mathfrak{o}(p)}) = i_2^*(\omega^L_{\mathfrak{o}(n)} + \omega^N_{\mathfrak{o}(p)}).$$

Finally we obtain

$$h_1^*\tilde{\omega} = i_1^* h_4^*(\omega^N_{\mathfrak{o}(n)} + \omega^N_{\mathfrak{o}(p)}) = i_1^* i_2^*(\omega^L_{\mathfrak{o}(n)} + \omega^N_{\mathfrak{o}(p)}) = \omega_{\mathfrak{o}(n)+\mathfrak{o}(p)}. \qquad \text{q.e.d.}$$

The homomorphisms h_2, h_3, h' and h'' given in § 3 map the connection defined by ω in $O(L, N, M)$ into connections in $O(L, N, M)/O(p)$, $O(L, N, M)/O(n)$, $O(L, N, M)/O(n) \times O(q)$ and $O(L, N, M)/O(n) \times O(p)$ respectively. We call those connections as the canonical connections in those bundles respectively.

§ 4. The Gauss map

Let $G(n, p)$ be the Grassmann manifold of n-planes in R^{n+p}. Then we have

$$G(n,\ p)=O(n+p)/O(n)\times O(p)\,.$$

By an n-frame in R^{n+p}, we mean an ordered set of n orthonormal vectors in R^{n+p}. Let $V(n,\ p)$ be the Stiefel manifold of n-frames in R^{n+p}. Then we have

$$V(n,\ p)=O(n+p)/O(p)\,.$$

A pair $(U,\ V)$ of n-dimensional linear subspace U and p-dimensional linear subspace V of R^{n+p+q} such that $U\cap V=\{0\}$ will be said to be a direct sum pair of type $(n,\ p)$ in R^{n+p+q}. Let $G(n,\ p,\ q)$ be the set of direct sum pairs of type $(n,\ p)$ in R^{n+p+q}. The group $O(n+p+q)$ acts transitively on $G(n,\ p,\ q)$. The elements of $O(n+p+q)$ which leave invariant the particular pair $(R^n,\ R^p)$ form the subgroup $O(n)\times O(p)\times O(q)$. Thus we have

$$G(n,\ p,\ q)=O(n+p+q)/O(p)\times O(p)\times O(q)\,.$$

The homogeneous space $G(n,\ p,\ q)$ is considered as a fibre space over Grassmann manifolds. In fact we have three fibre bundles:

$$G(n,\ p,\ q)\ \text{over}\ G(n+p,\ q)\ \text{with fibre}\ G(n,\ p),$$

$$G(n,\ p,\ q)\ \text{over}\ G(n,\ p+q)\ \text{with fibre}\ G(p,\ q),$$

$$G(n,\ p,\ q)\ \text{over}\ G(n+p,\ q)\ \text{with fibre}\ G(n,\ q)\,.$$

For example the projection of $G(n,\ p,\ q)$ onto $G(n+p,\ q)$ maps a direct sum pair $(U,\ V)$ to the $(n+p)$-dimensional subspace $U+V$. We have moreover the following seven principal fibre bundles over $G(n,\ p,\ q)$:

$E=O(n+p+q)$ over $G(n,\ p,\ q)$ with group $O(n)\times O(p)\times O(q)$,

$E_1=V(p+q,\ n)=O(n+p+q)/O(n)$ over $G(n,\ p,\ q)$ with group $O(p)\times O(q)$,

$E_2=V(n+q,\ p)=O(n+p+q)/O(p)$ over $G(n,\ p,\ q)$ with group $O(n)\times O(q)$,

$E_3=V(n+p,\ q)=O(n+p+q)/O(q)$ over $G(n,\ p,\ q)$ with group $O(n)\times O(p)$,

$E_1'=O(n+p+q)/O(p)\times O(q)$ over $G(n,\ p,\ q)$ with group $O(n)$,

$E_2'=O(n+p+q)/O(n)\times O(q)$ over $G(n,\ p,\ q)$ with group $O(p)$,

$E_3'=O(n+p+q)/O(n)\times O(p)$ over $G(n,\ p,\ q)$ with group $O(q)$.

Let γ be the canonical 1-form of $O(n+p+q)$, that is, the left invariant $\mathfrak{o}(n+p+q)$-valued 1-form uniquely determined by

$$\gamma(A)=A \qquad \text{for} \quad A\in\mathfrak{o}(n+p+q)\,.$$

Let ω_E be the $\mathfrak{o}(n)+\mathfrak{o}(p)+\mathfrak{o}(q)$-component of γ with respect to the decomposition

$\mathfrak{o}(n+p+q)=\mathfrak{o}(n)+\mathfrak{o}(p)+\mathfrak{o}(q)+\mathfrak{g}(n, p, q)$. By the well known result [2, Chapter 2, Theorem 11.1], the form ω_E defines a connection in E which will be called the canonical connection in E and will be denoted by Γ_E. Let f_i (resp. f'_i) be the bundle homomorphisms of E onto E_i (resp. E'_i) defined by the natural projections ($i=1$, 2 and 3). The homomorphisms f_i (resp. f'_i) map the connection Γ_E onto connections in E_i (resp. E'_i), denoted by Γ_{E_i} (resp. $\Gamma_{E'_i}$) such that the connection forms ω_{E_i} (resp. $\omega_{E'_i}$) are determined by $f^*_1(\omega_{E_1})=\gamma_{\mathfrak{o}(p)}+\gamma_{\mathfrak{o}(q)}$, $f^*_2(\omega_{E_2})=\gamma_{\mathfrak{o}(n)}+\gamma_{\mathfrak{o}(q)}$ and $f^*_3(\omega_{E_3})=\gamma_{\mathfrak{o}(n)}+\gamma_{\mathfrak{o}(p)}$ (resp. $f'^*_1(\omega_{E'_1})=\gamma_{\mathfrak{o}(n)}$, $f'^*_2(\omega_{E'_2})=\gamma_{\mathfrak{o}(p)}$ and $f'^*_3(\omega_{E'_3})=\gamma_{\mathfrak{o}(q)}$), respectively, where $\gamma_{\mathfrak{o}(n)}$, $\gamma_{\mathfrak{o}(p)}$ and $\gamma_{\mathfrak{o}(q)}$ are the $\mathfrak{o}(n)$-, $\mathfrak{o}(p)$- and $\mathfrak{o}(q)$-components of γ respectively.

Let P_1 and P_2 be principal bundles over M with groups G_1 and G_2 respectively. Then $P_1\times P_2$ is a principal fibre bundle over $M\times M$ with group $G_1\times G_2$. Let P_1+P_2 be the restriction of $P_1\times P_2$ to the diagonal ΔM of $M\times M$. Since ΔM and M are diffeomorphic to each other, P_1+P_2 is considered as a principal fibre bundle over M. Moreover let P_3 be a principal fibre bundle over M with group G_3. Then we can construct a principal fibre bundle $P_1+P_2+P_3$ over M with group $G_1\times G_2\times G_3$. Now we have the following bundle isomorphisms:

$$(f_i, f'_i):\ E\simeq E_i+E'_i \quad (i=1,\ 2 \text{ and } 3),$$

$$(f'_1, f'_2, f'_3):\ E\simeq E'_1+E'_2+E'_3.$$

Let N be an $(n+p)$-dimensional manifold immersed in the $(n+p+q)$-dimensional Euclidean space R^{n+p+q}. Let M be an n-dimensional manifold immersed in N. We have the following principal fibre bundles over M.

$$P=O(R^{n+p+q}, N, M) \text{ over } M \text{ with group } O(n)\times O(p)\times O(q),$$

$$P_1=O(R^{n+p+q}, N, M)/O(n) \text{ over } M \text{ with group } O(p)\times O(q),$$

$$P_2=O(R^{n+p+q}, N, M)/O(p) \text{ over } M \text{ with group } O(n)\times O(q),$$

$$P_3=O(R^{n+p+q}, N, M)/O(q) \text{ over } M \text{ with group } O(n)\times O(p),$$

$$P'_1=O(R^{n+p+q}, N, M)/O(p)\times O(q) \text{ over } M \text{ with group } O(n),$$

$$P'_2=O(R^{n+p+q}, N, M)/O(n)\times O(q) \text{ over } M \text{ with group } O(p),$$

$$P'_3=O(R^{n+p+q}, N, M)/O(n)\times O(p) \text{ over } M \text{ with group } O(q).$$

The canonical connections in P, P_1, P_2, P_3, P'_1, P'_2 and P'_3 given as in § 3, will be denoted by Γ_P, Γ_{P_1}, Γ_{P_2}, Γ_{P_3}, $\Gamma_{P'_1}$, $\Gamma_{P'_2}$ and Γ_{P_3}.

We now define a bundle map $g: P\rightarrow E$. The bundle $O(R^{n+p+q})$ of orthonormal frames over R^{n+p+q} is trivial, that is, $O(R^{n+p+q})=R^{n+p+q}\times O(n+p+q)$. Let ρ: $O(R^{n+p+q})\rightarrow O(n+p+q)$ be the natural projection. Let $i: P\rightarrow O(R^{n+p+q})$ be the natural injection. Then we define

$$g(v)=\rho i(v) \qquad \text{for} \quad v\in P.$$

Since g commutes with the right translation by $O(n)\times O(p)\times O(q)$, g is a bundle map of P into E. The bundle map g induces bundle maps g_i: $P_i\to E_i$ and g'_i: $P'_i\to E'_i$ ($i=1$, 2 and 3). It induces also a mapping $\tilde{g}$: $M\to G(n, p, q)$. Summing up, we have the following commutative diagram:

$$\begin{array}{ccc} P_i & \xrightarrow{\;g_i\;} & E_i \\ \uparrow{\scriptstyle h_i} & & \uparrow{\scriptstyle f_i} \\ P_i+P'_i=P & \longrightarrow & E=E_i+E'_i \\ \downarrow{\scriptstyle h'_i} & & \downarrow{\scriptstyle f'_i} \\ P'_i & \xrightarrow[\;g'_i\;]{} & E'_i \end{array} \qquad (i=1,\ 2 \text{ and } 3)\,,$$

where h_i: $P\to P_i$ and h'_i: $P\to P'_i$ are the natural homomorphisms as given in §3. Now we have the following fundamental relationship of connections.

Proposition 6. *The bundle maps g, g_1, g_2, g_3, g'_1, g'_2 and g'_3 map the connections Γ_P, Γ_{P_1}, Γ_{P_2}, Γ_{P_3}, $\Gamma_{P'_1}$, $\Gamma_{P'_2}$ and $\Gamma_{P'_3}$ upon the connections Γ_E, Γ_{E_1}, Γ_{E_2}, Γ_{E_3}, $\Gamma_{E'_1}$, $\Gamma_{E'_2}$ and $\Gamma_{E'_3}$ respectively.*

PROOF. Since each f_i (resp. f'_i) maps Γ_E upon Γ_{E_i} (resp. $\Gamma_{E'_i}$) and since each h_i (resp. h'_i) maps Γ_P upon Γ_{P_i} (resp. $\Gamma_{P'_i}$), it suffices to prove that g maps Γ_P upon Γ_E. The flat Riemannian connection of R^{n+p+q} is given by the form $\rho^*(\gamma)$ on $O(R^{n+p+q})$. The connection form ω is the $o(n)+o(p)+o(q)$-component of $i^*\rho^*(\gamma)=g^*(\gamma)$. On the other hand, ω_E is the $o(n)+o(p)+o(q)$-component of γ. Hence we obtain that ω is equal to $g^*(\omega_E)$. q.e.d.

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