J. Math. Tokushima Univ.
Vol. 16 (1982), 95–126

A Remark to Galerkin Method for Nonlinear Periodic Systems with Unknown Parameters

By

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(Received May 15, 1982)

§0. Introduction

In the previous paper [17], we considered a real nonlinear autonomous differential system with unknown parameters, and we have obtained the following results;

1. *The existence of an isolated periodic solution always implies the existence of Galerkin approximations and these Galerkin approximations uniformly converge to an exact periodic solution.*
2. *The existence of a "good" approximate solution always implies the existence of an exact solution and the error bounds for this approximation are given.*

In this paper, we consider a real nonlinear periodic differential system with unknown parameters, and we discuss the problems similar to the ones in the case of autonomous system. And we obtain the same conclusion, that is, the above results 1 and 2 are also true in the case of periodic system. In order to verify the result 2, we give the existence theorem in §1, and we prove the result 1 in §2.

Lastly, in §3, the numerical examples are given. The results show the usefulness of our results.

§1. The existence theorem

In the present section, we consider a real n-dimensional nonlinear periodic differential system with unknown parameters $B_1,\dots, B_d$;

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_d, t), \tag{1.1}$$

where $\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_d, t)$ is periodic in t of period 2π. To the system (1.1), we add the periodic boundary condition

$$\boldsymbol{x}(0)-\boldsymbol{x}(2\pi)=\boldsymbol{0} \tag{1.2}$$

and an appropriate additional condition

(1.3) $$g(u)=0,$$

where $u=u(t)=(x(t), B_1,\dots, B_d)$, and $g(u)$ is a d-dimensional vector-function. Now, we consider the following problem;

"*Look for a 2π-periodic solution $u=u(t)=(x(t), B_1,\dots, B_d)$ satisfying* (1.1)–(1.3)."

For brevity, we rewrite the boundary value problem (1.1)–(1.3) in the following form;

(1.4) $$F(u)=\left[\frac{dx}{dt}-X(x, B_1,\dots, B_d, t), f(u)\right]=0,$$

where

$$f(u)=\begin{pmatrix} x(0)-x(2\pi) \\ g(u) \end{pmatrix}.$$

Let Ω be the domain of the $(x, B_1,\dots, B_d, t)$-space intercepted by two hyperplanes $t=0$ and $t=2\pi$ (the boundary points of Ω on the hyperplanes $t=0$ and $t=2\pi$ are supposed to be included in Ω and to make an open set on each hyperplane).

Put

$$S=\{u(t)=(x(t), B_1,\dots, B_d);\ (u(t), t)\in\Omega \textit{ for } t\in I=[0, 2\pi],\ u(t)\in M\equiv C^1[I]\times R^d\},$$

$$S'=\{u(t)=(x(t), B_1,\dots, B_d);\ (u(t), t)\in\Omega \textit{ for } t\in I,\ u(t)\in C[I]\times R^d\},$$

where

$$C^1[I]=\{x(t)=\mathrm{col}\,[x_1(t),\dots, x_n(t)];\ x_i(t)\ (i=1,\dots, n) \textit{ are } C^1\textit{-class on } I\},$$

$$C[I]=\{x(t)=\mathrm{col}\,[x_1(t),\dots, x_n(t)];\ x_i(t)\ (i=1,\dots, n) \textit{ are continuous on } I\}.$$

Now, we introduce the norms in the product spaces $C[I]\times R^d$ and $N\equiv C[I]\times R^{n+d}$. For any $u(t)=(x(t), B_1,\dots, B_d)\in C[I]\times R^d$, we define

$$\|u(t)\|_\infty=\|x(t)\|_c+\sum_{i=1}^{d}|B_i|,$$

where $\|x(t)\|_c=\sup_{t\in I}\|x(t)\|_n$, and $\|\cdot\|_n$ is the Euclidean norm in R^n.
And for any $n(t)=(\varphi(t), v)\in N$, we define

$$|||n(t)|||=\|\varphi(t)\|_c+\|v\|_{n+d}.$$

In this section, we use the natural induced operator norms, for example, $\|\cdot\|_\infty$ denotes the norm of a continuous linear operator mapping from $C[I]\times R^{n+d}$ with the norm $|||\cdot|||$ to $C[I]\times R^d$ with the norm $\|\cdot\|_\infty$, and $\|\cdot\|_c$ denotes the norm of a continuous linear operator mapping from $C[I]\times \boldsymbol{R}^{n+d}$ with the norm $|||\cdot|||$ to $C[I]$ with the norm $\|\cdot\|_c$ etc..

From the above definition, it is evident that $S\subset S'$ and S and S' are open in M and $C[I]\times R^d$, respectively.

In the system (1.1), we assume that $\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_d, t)$ is defined on Ω and continuously differentiable with respect to $\boldsymbol{u}=(\boldsymbol{x}, B_1,\dots, B_d)$ in Ω, and the d-dimensional vector valued function $\boldsymbol{g}(\boldsymbol{u})$ is defined and continuously Fréchet differentiable on S'. Then, $\boldsymbol{f}(\boldsymbol{u})$ is defined and continuously Fréchet differentiable on S'.

Then the Fréchet differential of $\boldsymbol{F}(\boldsymbol{u})$ at $\boldsymbol{u}=\boldsymbol{u}(t)=(\boldsymbol{x}(t), B_1,\dots, B_d)$ can be written as follows;

$$\boldsymbol{F}'(\boldsymbol{u})\boldsymbol{h}=\left[\frac{d\boldsymbol{h}_1}{dt}-\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}(t), t)\boldsymbol{h}^t, \boldsymbol{f}'(\boldsymbol{u})\boldsymbol{h}\right]$$

for any $\boldsymbol{h}=(\boldsymbol{h}_1(t), h_{n+1},\dots, h_{n+d})\in M=C^1[I]\times \boldsymbol{R}^d$, where

$$\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}(t), t)=(\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}(t), B_1,\dots, B_d, t)\quad \boldsymbol{X}_{B_1}(\boldsymbol{x}(t), B_1,\dots, B_d, t)\cdots \boldsymbol{X}_{B_d}(\boldsymbol{x}(t), B_1,\dots, B_d, t)),$$

$\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}, B_1,\dots, B_d, t)$ denotes the Jacobian matrix of $\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_d, t)$ with respect to $\boldsymbol{x}$ and $\boldsymbol{X}_{B_i}(\boldsymbol{x}, B_1,\dots, B_d, t)$ $(i=1,\dots, d)$ denote the derivatives of $\boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_d, t)$ with respect to $B_i (i=1,\dots, d)$, respectively. Here we have

$$\boldsymbol{f}'(\boldsymbol{u})\boldsymbol{h}=\begin{pmatrix} \boldsymbol{h}_1(0)-\boldsymbol{h}_1(2\pi) \\ \boldsymbol{g}'(\boldsymbol{u})\boldsymbol{h} \end{pmatrix},$$

where $\boldsymbol{f}'(\boldsymbol{u})$ and $\boldsymbol{g}'(\boldsymbol{u})$ are the Fréchet differentials of $\boldsymbol{f}(\boldsymbol{u})$ and $\boldsymbol{g}(\boldsymbol{u})$ at $\boldsymbol{u}=\boldsymbol{u}(t)$, respectively, and where $\boldsymbol{h}^t$ denotes the transpose of $\boldsymbol{h}$.

Now, we introduce a linear operator T mapping from M into $N=C[I]\times \boldsymbol{R}^{n+d}$;

$$T\boldsymbol{h}=\left[\frac{d\boldsymbol{h}_1}{dt}-A(t)\boldsymbol{h}^t, \mathscr{L}\boldsymbol{h}\right], \tag{1.5}$$

where $A(t)$ is an $n\times(n+d)$ matrix whose elements are continuous on I and $\mathscr{L}$ is a linear operator mapping from the product space $C[I]\times \boldsymbol{R}^d$ into $\boldsymbol{R}^{n+d}$. Then, concerning the linear inverse operator T^{-1} of the linear operator T defined by (1.5), we have the following theorem.

Theorem 1 (M. Urabe [6]).
If the $(n+d)\times(n+d)$ matrix $G=\mathscr{L}[\Psi(t)]$ is non-singular, namely,

$$\det G=\det \mathscr{L}[\Psi(t)]=\det(\mathscr{L}\tilde{\boldsymbol{\varphi}}_1, \mathscr{L}\tilde{\boldsymbol{\varphi}}_2,\dots, \mathscr{L}\tilde{\boldsymbol{\varphi}}_{n+d})\neq 0,$$

then the operator T has a linear inverse operator T^{-1}. That is, for any $\boldsymbol{n}=(\boldsymbol{\varphi}(t), \boldsymbol{v})\in N=C[I]\times \boldsymbol{R}^{n+d}$ there exists one and only one solution $\boldsymbol{h}=(\boldsymbol{h}_1(t), h_{n+1},\dots, h_{n+d})\in M$ satisfying the equation $T\boldsymbol{h}=\boldsymbol{n}$. And the solution $\boldsymbol{h}$ can be written as follows;

$$\boldsymbol{h}^t=\mathrm{col}\,[\boldsymbol{h}_1(t),\ h_{n+1},\dots,\ h_{n+d}]=H_1\boldsymbol{\varphi}+H_2\boldsymbol{v},$$

where

$$\begin{cases} H_1\boldsymbol{\varphi}=\Psi(t)\displaystyle\int_0^t \Psi^{-1}(s)\boldsymbol{\phi}(s)ds-\Psi(t)G^{-1}\mathscr{L}[\Psi(t)\displaystyle\int_0^t \Psi^{-1}(s)\boldsymbol{\phi}(s)ds], \\ H_2\boldsymbol{v}=\Psi(t)G^{-1}\boldsymbol{v},\quad \boldsymbol{\phi}(s)=\mathrm{col}\,[\boldsymbol{\varphi}(s),\ \boldsymbol{0}], \end{cases}$$

and by $\Psi(t)$ *we denote the fundamental matrix of the linear homogeneous system*

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix} A(t) \\ \boxed{0} \end{pmatrix}\boldsymbol{z} \qquad (\textit{where } \boxed{0} \textit{ is a } d\times(n+d) \textit{ matrix})$$

with the initial condition $\Psi(0)=E$ *(*$(n+d)\times(n+d)$ *unit matrix) and by* $\mathscr{L}[\Psi(t)]$ *we denote the matrix whose column vectors are* $\mathscr{L}\tilde{\boldsymbol{\varphi}}_i$ $(i=1, 2,\dots, n+d)$. *Here we put* $\tilde{\boldsymbol{\varphi}}_i=(\boldsymbol{\varphi}_{i1}(t),\ \psi_{in+1},\dots,\ \psi_{in+d})=\boldsymbol{\varphi}_i^t\in M$, *where* $\boldsymbol{\varphi}_i=\mathrm{col}\,[\boldsymbol{\varphi}_{i1}(t),\ \psi_{in+1},\dots,\ \psi_{in+d}]$ *are column vectors of the fundamental matrix* $\Psi(t)$.

Now, we introduce a concept of "the isolatedness of a solution of (1.4)."

Definition

Let $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *be a solution of* (1.4). *Then, the solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *of* (1.4) *is called "an isolated solution" if*

$$\det \boldsymbol{f}'(\hat{\boldsymbol{u}}(t))\,[\hat{\Psi}(t)]\neq 0,$$

where $\hat{\Psi}(t)$ *is the fundamental matrix of the linear homogeneous system*

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix} \mathbf{X}_u(\hat{\boldsymbol{u}}(t),\ t) \\ \boxed{0} \end{pmatrix}\boldsymbol{z} \qquad (\textit{where } \boxed{0} \textit{ is a } d\times(n+d) \textit{ matrix})$$

satisfying the initial condition $\hat{\Psi}(0)=E((n+d)\times(n+d)$ *unit matrix).*

We must choose $\boldsymbol{g}(\boldsymbol{u})$ introduced in the beginning of this section as $\det\boldsymbol{f}'(\hat{\boldsymbol{u}}(t))[\hat{\Psi}(t)]\neq 0$. Concerning the choice of $\boldsymbol{g}(\boldsymbol{u})$, see the examples in §3, and also see [17].

Here, we devide the linear inverse operator T^{-1} of the linear operator T defined by (1.5) into several linear operators.

For any $\boldsymbol{n}=\boldsymbol{n}(t)=(\boldsymbol{\varphi}(t),\ \boldsymbol{v})\in N$, we set

$$\boldsymbol{h}=T^{-1}\boldsymbol{n}=T^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v})=(T_1^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v}),\ T_{n+1}^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v}),\dots,\ T_{n+d}^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v})),$$

that is,

$$\begin{cases} T_1^{-1}\ (\boldsymbol{\varphi}(t),\ \boldsymbol{v})=\boldsymbol{h}_1(t), \\ T_{n+1}^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v})=h_{n+1}, \\ \qquad\vdots \\ T_{n+d}^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v})=h_{n+d} \end{cases}$$

Furthermore, we set

$$H_1\boldsymbol{\varphi}=\begin{pmatrix} H_{11}\boldsymbol{\varphi} \\ H_{1n+1}\boldsymbol{\varphi} \\ \vdots \\ H_{1n+d}\boldsymbol{\varphi} \end{pmatrix}, \quad H_2\boldsymbol{v}=\begin{pmatrix} H_{21}\boldsymbol{v} \\ H_{2n+1}\boldsymbol{v} \\ \vdots \\ H_{2n+d}\boldsymbol{v} \end{pmatrix}.$$

Then, by Theorem 1 we have

$$\begin{cases} T_1^{-1}\ (\boldsymbol{\varphi}(t),\ \boldsymbol{v})=H_{11}\boldsymbol{\varphi}+H_{21}\boldsymbol{v}, \\ T_{n+1}^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v})=H_{1n+1}\boldsymbol{\varphi}+H_{2n+1}\boldsymbol{v}, \\ \vdots \\ T_{n+d}^{-1}(\boldsymbol{\varphi}(t),\ \boldsymbol{v})=H_{1n+d}\boldsymbol{\varphi}+H_{2n+d}\boldsymbol{v}. \end{cases}$$

Therefore we obtain

$$\begin{cases} \|T_1^{-1}\|_c\leq\max\,(\|H_{11}\|_c,\ \|H_{21}\|_c), \\ |T_{n+1}^{-1}|\leq\max\,(|H_{1n+1}|,\ |H_{2n+1}|), \\ \vdots \\ |T_{n+d}^{-1}|\leq\max\,(|H_{1n+d}|,\ |H_{2n+d}|) \end{cases}$$

and

$$\|T^{-1}\|_\infty\leq\|T_1^{-1}\|_c+|T_{n+1}^{-1}|+\cdots+|T_{n+d}^{-1}|.$$

Now, we set $A(t)=\boldsymbol{X}_{\boldsymbol{u}}(\bar{\boldsymbol{u}}(t),\ t)$ and $\mathscr{L}=\boldsymbol{f}'(\bar{\boldsymbol{u}}(t))$ in (1.5), then we have the following theorem.

Theorem 2.

Assume that the boundary value problem (1.4) *possesses an approximate solution* $\boldsymbol{u}=\bar{\boldsymbol{u}}(t)$ *in* S *such that* $\det G=\det\boldsymbol{f}'(\bar{\boldsymbol{u}})\,[\Psi(t)]\neq 0$, *where* $\Psi(t)$ *is the fundamental matrix of the linear homogeneous differential system*

$$\frac{d\boldsymbol{z}}{dt}=\begin{pmatrix} \boldsymbol{X}_{\boldsymbol{u}}(\bar{\boldsymbol{u}}(t),\ t) \\ \boxed{0} \end{pmatrix}\boldsymbol{z} \qquad (\textit{where } \boxed{0} \textit{ is a } d\times(n+d) \textit{ matrix})$$

satisfying the initial condition $\Psi(0)=E((n+d)\times(n+d)$ *unit matrix).*

Let $\mu_1,\ \mu_{n+1},\ \mu_{n+2},\dots,\ \mu_{n+d}$ *and* r *be the positive numbers such that*

$$(1.6)\qquad \begin{cases} \mu_1\geq\max\,(\|H_{11}\|_c,\ \|H_{21}\|_c), \\ \mu_{n+1}\geq\max\,(|H_{1n+1}|,\ |H_{2n+1}|), \\ \vdots \\ \mu_{n+d}\geq\max\,(|H_{1n+d}|,\ |H_{2n+d}|), \end{cases}$$

$$(1.7)\qquad r\geq|||\boldsymbol{F}(\bar{\boldsymbol{u}})|||=\left\|\frac{d\bar{\boldsymbol{x}}}{dt}-\boldsymbol{X}(\bar{\boldsymbol{x}},\ \bar{B}_1,\dots,\ \bar{B}_d,\ t)\right\|_c+\|\boldsymbol{f}(\bar{\boldsymbol{u}})\|_{n+d}.$$

If there exist the positive numbers $\delta_1, \delta_{n+1}, \ldots, \delta_{n+d}$ *and a non-negative number* $\kappa<1$ *such that*

$$D'_\delta=\{\boldsymbol{u}(t);\ \|\boldsymbol{x}(t)-\bar{\boldsymbol{x}}(t)\|_c\le\delta_1,\ |B_1-\bar{B}_1|\le\delta_{n+1},\ldots,\ |B_d-\bar{B}_d|\le\delta_{n+d},\ \boldsymbol{u}(t)\in C[I]\times\boldsymbol{R}^d\}\subset S', \tag{1.8}$$

$$\|\boldsymbol{X}_u(\boldsymbol{u}(t),t)-\boldsymbol{X}_u(\bar{\boldsymbol{u}}(t),t)\|_c+\|\boldsymbol{f}'(\boldsymbol{u})-\boldsymbol{f}'(\bar{\boldsymbol{u}})\|_{n+d} \le\frac{\kappa}{\mu_1+\mu_{n+1}+\cdots+\mu_{n+d}} \quad \textit{on } D'_\delta, \tag{1.9}$$

$$\frac{\mu_1 r}{1-\kappa}\le\delta_1,\ \frac{\mu_{n+1}r}{1-\kappa}\le\delta_{n+1},\ldots,\ \frac{\mu_{n+d}r}{1-\kappa}\le\delta_{n+d}, \tag{1.10}$$

then the boundary value problem (1.4) *has one and only one* 2π*-periodic solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *in*

$$D_\delta=\{\boldsymbol{u}(t);\ \|\boldsymbol{x}(t)-\bar{\boldsymbol{x}}(t)\|_c\le\delta_1,\ |B_1-\bar{B}_1|\le\delta_{n+1},\ldots,\ |B_d-\bar{B}_d|\le\delta_{n+d},\ \boldsymbol{u}(t)\in M\}$$

and for this solution $\hat{\boldsymbol{u}}(t)$ we have

$$\|\hat{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}(t)\|_c\le\frac{\mu_1 r}{1-\kappa},\ |\hat{B}_1-\bar{B}_1|\le\frac{\mu_{n+1}r}{1-\kappa},\ldots,\ |\hat{B}_d-\bar{B}_d|\le\frac{\mu_{n+d}r}{1-\kappa}. \tag{1.11}$$

PROOF. The proof of this theorem is similar to the one of Theorem 2 in [14]. See [14].

By the definition of the isolatedness of a solution, we can easily show that the solution $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ guaranteed in Theorem 2 is an isolated solution. In detail, see [17].

§2. Existence and Uniform Convergence of a Galerkin Approximation

Let $\boldsymbol{x}(t)$ be a continuous periodic vector-function of period 2π, and let its Fourier series be

$$\boldsymbol{x}(t)\sim\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{\infty}(\boldsymbol{c}_{2n-1}\cos nt+\boldsymbol{c}_{2n}\sin nt),$$

where $\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2,\ldots$ are n-dimensional vectors. Then the trigonometric polynomial

$$\boldsymbol{x}_m(t)=\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n-1}\cos nt+\boldsymbol{c}_{2n}\sin nt)$$

is a truncated trigonometric polynomial of the given periodic function $\boldsymbol{x}(t)$. In the sequel we shall denote such a truncation of a periodic function by P_m and write a truncated polynomial $\boldsymbol{x}_m(t)$ of a periodic function $\boldsymbol{x}(t)$ as follows:

$$\boldsymbol{x}_m(t)=P_m\boldsymbol{x}(t).$$

In this section, we use the norm $\|\cdot\|_q$ defined by

$$\|\boldsymbol{x}(t)\|_q=\left[\frac{1}{2\pi}\int_0^{2\pi}\|\boldsymbol{x}(t)\|_n^2 dt\right]^{\frac{1}{2}}.$$

Now, if we put $\boldsymbol{\gamma}_{x_m}=\mathrm{col}\,[\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2,\dots, \boldsymbol{c}_{2m-1}, \boldsymbol{c}_{2m}]$, then it holds that

$$\|\boldsymbol{x}_m\|_q^2=\frac{1}{2\pi}\int_0^{2\pi}\|\boldsymbol{x}_m(t)\|_n^2 dt=\|\boldsymbol{c}_0\|_n^2+\sum_{i=1}^{m}(\|\boldsymbol{c}_{2i-1}\|_n^2+\|\boldsymbol{c}_{2i}\|_n^2)=\|\boldsymbol{\gamma}_{x_m}\|_{n(2m+1)}^2.$$

We owe to Cesari the following proposition concerning continuously differentiable periodic functions.

Proposition.

Let $\boldsymbol{x}(t)$ be a continuously differentiable periodic vector-function of period 2π. Then

$$\text{(2.1)}\qquad \begin{cases} \|\boldsymbol{x}-P_m\boldsymbol{x}\|_c\le\sigma(m)\|\dot{\boldsymbol{x}}\|_q\le\sigma(m)\|\dot{\boldsymbol{x}}\|_c, \\ \|\boldsymbol{x}-P_m\boldsymbol{x}\|_q\le\sigma_1(m)\|\dot{\boldsymbol{x}}\|_q, \end{cases}$$

where $\cdot=\dfrac{d}{dt}$ *and*

$$\sigma(m)=\sqrt{2}\left[\frac{1}{(m+1)^2}+\frac{1}{(m+2)^2}+\cdots\right]^{\frac{1}{2}},$$

$$\sigma_1(m)=\frac{1}{m+1}.$$

Also

$$\frac{\sqrt{2}}{m+1}<\sigma(m)<\frac{\sqrt{2}}{\sqrt{m}}.$$

Here, we introduce the following notations;

For any vector function $\boldsymbol{u}(t)=(\boldsymbol{x}(t), B_1,\dots, B_d)\in M$, we define $\boldsymbol{u}_m(t)=P_m\boldsymbol{u}(t)$ by

$$\boldsymbol{u}_m(t)=P_m\boldsymbol{u}(t)=(P_m\boldsymbol{x}(t), B_1,\dots, B_d)=(\boldsymbol{x}_m(t), B_1,\dots, B_d),$$

and define the norms $\|\cdot\|_q$ and $\|\cdot\|_\infty$ by

$$\|\boldsymbol{u}(t)\|_q=\|\boldsymbol{x}(t)\|_q+|B_1|+\cdots+|B_d|,$$

$$\|\boldsymbol{u}(t)\|_\infty=\|\boldsymbol{x}(t)\|_c+|B_1|+\cdots+|B_d|,$$

respectively.

For any $\boldsymbol{n}(t)=(\boldsymbol{\varphi}(t), \boldsymbol{v})\in N=C[I]\times \boldsymbol{R}^{n+d}$, we define

$$|||\boldsymbol{n}(t)|||_q=\|\boldsymbol{\varphi}(t)\|_q+\|\boldsymbol{v}\|_{n+d},$$

$$|||\boldsymbol{n}(t)||| = \|\boldsymbol{\varphi}(t)\|_c + \|\boldsymbol{v}\|_{n+d}.$$

By Proposition, we have the following results;

$$\begin{cases} \|\boldsymbol{u}(t)-P_m\boldsymbol{u}(t)\|_\infty = \|\boldsymbol{x}(t)-P_m\boldsymbol{x}(t)\|_c \le \sigma(m)\|\dot{\boldsymbol{x}}\|_q = \sigma(m)\|\dot{\boldsymbol{u}}\|_q \le \sigma(m)\|\dot{\boldsymbol{x}}\|_c = \sigma(m)\|\dot{\boldsymbol{u}}\|_\infty, \\ \|\boldsymbol{u}(t)-P_m\boldsymbol{u}(t)\|_q = \|\boldsymbol{x}(t)-P_m\boldsymbol{x}(t)\|_q \le \sigma_1(m)\|\dot{\boldsymbol{x}}\|_q = \sigma_1(m)\|\dot{\boldsymbol{u}}\|_q. \end{cases}$$

For $\boldsymbol{u}_m = P_m\boldsymbol{u}(t) = (P_m\boldsymbol{x}(t), B_1,\dots, B_d) = (\boldsymbol{x}_m(t), B_1,\dots, B_d)$, where

$$\boldsymbol{x}_m(t) = \boldsymbol{c}_0 + \sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n-1}\cos nt + \boldsymbol{c}_{2n}\sin nt),$$

we put

$$\boldsymbol{\gamma}_{\boldsymbol{u}_m} = \mathrm{col}\,[\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2,\dots, \boldsymbol{c}_{2m-1}, \boldsymbol{c}_{2m}, B_1,\dots, B_d].$$

Then we define the norm $|||\cdot|||'$ as follows;

$$|||\boldsymbol{\gamma}_{\boldsymbol{u}_m}|||' = \|\boldsymbol{\gamma}_{\boldsymbol{x}_m}\|_{n(2m+1)} + |B_1| + \cdots + |B_d|.$$

At once, we have the following result.

$$\begin{aligned} \|\boldsymbol{u}_m(t)\|_q &= \|P_m\boldsymbol{u}(t)\|_q = \|\boldsymbol{x}_m(t)\|_q + |B_1| + \cdots + |B_d| = \|\boldsymbol{\gamma}_{\boldsymbol{x}_m}\|_{n(2m+1)} + |B_1| + \cdots + |B_d| \\ &= |||\boldsymbol{\gamma}_{\boldsymbol{u}_m}|||'. \end{aligned}$$

By $\|\cdot\|_{n+d,n}$, we denote the norm of a continuous linear operator which maps from $\boldsymbol{R}^{n+d}$ to $\boldsymbol{R}^d$. Similarly, by $\|\cdot\|_{\infty,d}$, we denote the norm of a continuous linear operator which maps from the product space $C[I]\times\boldsymbol{R}^d$ with the norm $\|\cdot\|_\infty$ to $\boldsymbol{R}^d$ with the norm $\|\cdot\|_d$ and by $\|\cdot\|_{q,d}$, we denote the norm of a continuous linear operator mapping from $C[I]\times\boldsymbol{R}^d$ with the norm $\|\cdot\|_q$ to $\boldsymbol{R}^d$ with the norm $\|\cdot\|_d$.

Let D be a closed bounded region of the $\boldsymbol{u}=(\boldsymbol{x}, B_1,\dots, B_d)$-space such that $D\times I\subset\Omega$, and $\boldsymbol{R}$ is the real line, where $I=[0, 2\pi]$.

We set $\boldsymbol{X}(\boldsymbol{u}, t) = \boldsymbol{X}(\boldsymbol{x}, B_1,\dots, B_d, t)$ and

$$\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}, t) = (\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}, B_1,\dots, B_d, t)\quad \boldsymbol{X}_{B_1}(\boldsymbol{x}, B_1,\dots, B_d, t)\cdots\boldsymbol{X}_{B_d}(\boldsymbol{x}, B_1,\dots, B_d, t)).$$

Let $\boldsymbol{X}(\boldsymbol{u}, t)$ be periodic in t of period 2π. We assume that $\boldsymbol{X}(\boldsymbol{u}, t)$ and its first partial derivatives with respect to $\boldsymbol{u}$ are continuously differentiable with respect to $\boldsymbol{u}$ and t in the region $D\times\boldsymbol{R}$, and $\boldsymbol{g}(\boldsymbol{u})$ is defined and continuously Fréchet differentiable on S'.

Lemma 1.

Let K, K_1 and K_2 be non-negative constants such that

$$(2.2)\qquad \begin{cases} K = \max_{D\times\boldsymbol{R}}\|\boldsymbol{X}(\boldsymbol{u}, t)\|_n,\ K_1 = \max_{D\times\boldsymbol{R}}\|\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}, t)\|_{n+d,n}, \\ K_2 = \max_{D\times\boldsymbol{R}}\left\|\dfrac{\partial\boldsymbol{X}(\boldsymbol{u}, t)}{\partial t}\right\|_n. \end{cases}$$

If there exists a 2π*-periodic solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *of* (1.4) *lying in* D, *then*

$$
(2.3)\qquad \begin{cases} \text{(i)} & \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_\infty \leq K\sigma(m), \\ \text{(ii)} & \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}_m\|_q \leq K\sigma_1(m), \\ \text{(iii)} & \|\dot{\hat{\boldsymbol{u}}}-\dot{\hat{\boldsymbol{u}}}_m\|_\infty \leq (KK_1+K_2)\sigma(m). \end{cases}
$$

PROOF. The proof is similar to the one of Lemma 1 in [17]. See [17].

This lemma yields the following corollary.

Corollary 1.1.

If $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *is an isolated periodic solution of* (1.4) *lying inside* D, *then there exists a positive integer* m_0 *such that, for any* $m \geq m_0$,

(i) $\hat{\boldsymbol{u}}_m(t) \in D$;

(ii) *The linear operator* T_m *defined by*

$$
T_m\boldsymbol{h}=\left[\frac{d\boldsymbol{h}_1}{dt}-\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m(t),\, t)\boldsymbol{h}^t,\, \boldsymbol{f}'(\hat{\boldsymbol{u}}_m)\boldsymbol{h}\right]
$$

has a linear inverse operator T_m^{-1} *and there exists a positive constant* M_c *such that*

$$
(2.4)\qquad \|T_m^{-1}\|_q \leq M_c,\ \|T_m^{-1}\|_\infty \leq M_c,
$$

where $\boldsymbol{h}^t$ *denotes the transpose of* $\boldsymbol{h}$, *that is,* $\boldsymbol{h}^t=\mathrm{col}\,[\boldsymbol{h}_1(t),\, h_{n+1},\dots,\, h_{n+d}]$;

(iii) $\frac{d}{dt}\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m(t),\, t)$ *is equibounded, that is, there exists a non-negative constant* K_3 *such that*

$$
(2.5)\qquad \left\|\frac{d}{dt}\boldsymbol{X}_u(\hat{\boldsymbol{u}}_m(t),\, t)\right\|_{n+d,n} \leq K_3 .
$$

PROOF. The proof is also similar to the one of Corollary 1.1 in [17]. See [17].

The Jacobian matrix of the determining equation of Galerkin approximations

We put

$$
\begin{cases} \boldsymbol{x}_m(t)=\boldsymbol{a}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{a}_{2n-1}\cos nt+\boldsymbol{a}_{2n}\sin nt), \\ \boldsymbol{u}_m(t)=(\boldsymbol{x}_m(t),\, B_1,\, B_2,\dots,\, B_d) \end{cases}
$$

and

$$
\boldsymbol{\alpha}=\mathrm{col}\,[\boldsymbol{a}_0,\, \boldsymbol{a}_1,\dots,\, \boldsymbol{a}_{2m-1},\, \boldsymbol{a}_{2m},\, B_1,\, B_2,\dots,\, B_d].
$$

Now, we will determine $\boldsymbol{\alpha}$ satisfying the following system

$$(2.6)\quad \begin{cases} \boldsymbol{F}_0^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s),\ s)\,ds=\mathbf{0}, \\ \boldsymbol{F}_{2n-1}^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s),\ s)\cos ns\,ds-n\boldsymbol{\alpha}_{2n}=\mathbf{0}, \\ \boldsymbol{F}_{2n}^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s),\ s)\sin ns\,ds+n\boldsymbol{\alpha}_{2n-1}=\mathbf{0}, \\ \qquad\qquad (n=1,\ 2,\dots,\ m), \\ \boldsymbol{G}_f(\boldsymbol{\alpha})=\boldsymbol{f}(\boldsymbol{u}_m(t))=\begin{pmatrix} \boldsymbol{x}_m(0)-\boldsymbol{x}_m(2\pi) \\ \boldsymbol{g}(\boldsymbol{u}_m(t)) \end{pmatrix}=\mathbf{0}. \end{cases}$$

The system (2.6) is constructed by $\{n(2m+1)+n+d\}$ equations, but the last equations $\boldsymbol{G}_f(\boldsymbol{\alpha})=\mathbf{0}$ are essentially equivalent to d equations $\boldsymbol{g}(\boldsymbol{u}_m(t))=\mathbf{0}$. Then we will solve the following $\{n(2m+1)+d\}$ equations;

$$(2.6')\quad \begin{cases} \boldsymbol{F}_0^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s),\ s)\,ds=\mathbf{0}, \\ \boldsymbol{F}_{2n-1}^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s),\ s)\cos ns\,ds-n\boldsymbol{\alpha}_{2n}=\mathbf{0}, \\ \boldsymbol{F}_{2n}^{(m)}(\boldsymbol{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi}\boldsymbol{X}(\boldsymbol{u}_m(s),\ s)\sin ns\,ds+n\boldsymbol{\alpha}_{2n-1}=\mathbf{0}, \\ \qquad\qquad (n=1,\ 2,\dots,\ m) \\ \boldsymbol{F}_g^{(m)}(\boldsymbol{\alpha})=\boldsymbol{g}(\boldsymbol{u}_m(t))=\mathbf{0}. \end{cases}$$

A function $\boldsymbol{u}_m(t)$ satisfying (2.6) or (2.6)′ will be denoted as a Galerkin approximation of order m and the system (2.6) (or (2.6′)) will be called the determining equation of Galerkin approximations.

Put $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})=\mathrm{col}\,[\boldsymbol{F}_0^{(m)}(\boldsymbol{\alpha}),\ \boldsymbol{F}_1^{(m)}(\boldsymbol{\alpha}),\dots,\ \boldsymbol{F}_{2m-1}^{(m)}(\boldsymbol{\alpha}),\ \boldsymbol{F}_{2m}^{(m)}(\boldsymbol{\alpha}),\ (F_g)_1,\dots,\ (F_g)_d]$,

then the determining equation (2.6′) can be written briefly as

$$\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})=\mathbf{0},$$

where $\boldsymbol{F}_g^{(m)}(\boldsymbol{\alpha})=\mathrm{col}\,[(F_g)_1,\dots,\ (F_g)_d]$.

Let $J_m(\boldsymbol{\alpha})$ be the Jacobian matrix of $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$. Then the elements of $J_m(\boldsymbol{\alpha})$ are of the following forms;

$$\frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\,ds,$$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\cos ns\,ds, \qquad \text{(the elements with respect to } \boldsymbol{\alpha}_0)$$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\sin ns\ ds,$$

$$\boldsymbol{g}'(\boldsymbol{u}_m(t))(\boldsymbol{u}_{ma_{0i}}(t)) \qquad (1\leq i\leq n),$$

where $\boldsymbol{u}_{ma_{0i}}(t)=(\boldsymbol{x}_{ma_{0i}}(t),\ 0,\dots,\ 0)$, $\boldsymbol{x}_{ma_{0i}}(t)=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\subset i.$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\cos ks\ ds,$$

$$\frac{1}{\pi}\int_0^{2\pi} \boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\cos ks\cos ns\ ds,$$

(the elements with respect to $\boldsymbol{a}_{2k-1}$)

$$\frac{1}{\pi}\int_0^{2\pi} \boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\cos ks\sin ns\ ds+nE_n,$$

$$\boldsymbol{g}'(\boldsymbol{u}_m(t))(\boldsymbol{u}_{ma_{2k-1i}}(t)) \qquad (1\leq i\leq n),$$

where $\boldsymbol{u}_{ma_{2k-1i}}(t)=(\boldsymbol{x}_{ma_{2k-1i}}(t),\ 0,\dots,\ 0)$, $\boldsymbol{x}_{ma_{2k-1i}}(t)=\begin{pmatrix}0\\ \vdots\\ \sqrt{2}\cos kt\\ \vdots\\ 0\end{pmatrix}\subset i$, and E_n is an $n\times n$ unit matrix.

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\sin ks\ ds,$$

$$\frac{1}{\pi}\int_0^{2\pi} \boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\sin ks\cos ns\ ds-nE_n,$$

$$\frac{1}{\pi}\int_0^{2\pi} \boldsymbol{X}_{\boldsymbol{x}}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\sin ks\sin ns\ ds,$$

(the elements with respect to $\boldsymbol{a}_{2k}$)

$$\boldsymbol{g}'(\boldsymbol{u}_m(t))(\boldsymbol{u}_{ma_{2ki}}(t)) \qquad (1\leq i\leq n),$$

where $\boldsymbol{u}_{ma_{2ki}}(t)=(\boldsymbol{x}_{ma_{2ki}}(t),\ 0,\dots,\ 0)$, $\boldsymbol{x}_{ma_{2ki}}(t)=\begin{pmatrix}0\\ \vdots\\ \sqrt{2}\sin kt\\ \vdots\\ 0\end{pmatrix}\subset i.$

$$\frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{X}_{B_i}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\ ds,$$

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \boldsymbol{X}_{B_i}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\cos ns\ ds,$$

(the elements with respect to B_i $(1\leq i\leq d)$)

$$\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \boldsymbol{X}_{B_i}(\boldsymbol{x}_m(s),\ B_1,\dots,\ B_d,\ s)\sin ns\ ds,$$

$$\boldsymbol{g}'(\boldsymbol{u}_m(t))(\boldsymbol{u}_{mB_i}(t)) \qquad (1\le i\le d),$$

where $\boldsymbol{u}_{mB_i}(t)=(\boldsymbol{0},\ 0,\dots,\ 0,\ \underset{\widehat{i}}{1},\ 0,\dots,\ 0)$.

In order to find the basic properties of $J_m(\boldsymbol{\alpha})$, let us consider the auxiliary linear system

$$J_m(\boldsymbol{\alpha})\boldsymbol{\xi}+\boldsymbol{\gamma}=\boldsymbol{0}, \tag{2.7}$$

where

$$\begin{cases} \boldsymbol{\xi}=\mathrm{col}\,[\boldsymbol{v}_0,\ \boldsymbol{v}_1,\dots,\ \boldsymbol{v}_{2m-1},\ \boldsymbol{v}_{2m},\ V_1,\dots,\ V_d], \\ \boldsymbol{\gamma}=\mathrm{col}\,[\boldsymbol{c}_0,\ \boldsymbol{c}_1,\dots,\ \boldsymbol{c}_{2m-1},\ \boldsymbol{c}_{2m},\ -C_1,\dots,\ -C_d]. \end{cases} \tag{2.8}$$

If we put

$$\begin{cases} \boldsymbol{y}(t)=\boldsymbol{v}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{v}_{2n-1}\cos nt+\boldsymbol{v}_{2n}\sin nt), \\ \boldsymbol{\varphi}(t)=\boldsymbol{c}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n-1}\cos nt+\boldsymbol{c}_{2n}\sin nt), \\ \boldsymbol{u}_y(t)=(\boldsymbol{y}(t),\ V_1,\dots,\ V_d),\ \boldsymbol{u}_{\varphi}(t)=(\boldsymbol{\varphi}(t),\ C_1,\dots,\ C_d), \\ \boldsymbol{n}_{\varphi}=\left[\boldsymbol{\varphi}(t),\ \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{w}\end{pmatrix}\right],\ \textit{where}\ \boldsymbol{w}=\mathrm{col}\,[C_1,\dots,\ C_d], \end{cases} \tag{2.9}$$

then the equation $J_m(\boldsymbol{\alpha})\boldsymbol{\xi}+\boldsymbol{\gamma}=\boldsymbol{0}$ are equivalent to

$$\left[\frac{d\boldsymbol{y}}{dt}-P_m[\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}_m(t),\ t)\boldsymbol{u}_y^t],\ \boldsymbol{f}'(\boldsymbol{u}_m(t))\boldsymbol{u}_y(t)\right]=\boldsymbol{n}_{\varphi}=\left[\boldsymbol{\varphi}(t),\ \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{w}\end{pmatrix}\right],$$

or

$$\left[\frac{d\boldsymbol{y}}{dt}-P_m[\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}_m(t),\ t)\boldsymbol{u}_y^t],\ \boldsymbol{g}'(\boldsymbol{u}_m(t))\boldsymbol{u}_y(t)\right]=[\boldsymbol{\varphi}(t),\ \boldsymbol{w}],$$

where $\boldsymbol{u}_y^t$ denotes the transpose of $\boldsymbol{u}_y$.

First, we shall prove the following lemma.

Lemma 2.

Assume that the conditions of Lemma 1 *are satisfied and that* (1.4) *has an isolated periodic solution* $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ *lying inside D. Taking* m_0 *sufficiently large, we consider the differential system*

$$\left[\frac{d\boldsymbol{y}}{dt}-P_m[\boldsymbol{X}_{\boldsymbol{u}}(\hat{\boldsymbol{u}}_m(t),\ t)\boldsymbol{u}_y^t],\ \boldsymbol{f}'(\hat{\boldsymbol{u}}_m(t))\boldsymbol{u}_y(t)\right]=\boldsymbol{n}_{\boldsymbol{\varphi}}=\left[\boldsymbol{\varphi}(t),\ \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{w}\end{pmatrix}\right] \tag{2.10}$$

for $m \geq m_0$, *where* $\hat{\boldsymbol{u}}_m(t) = P_m\hat{\boldsymbol{u}}(t)$ *and where* $\boldsymbol{\varphi}(t)$ *is an arbitrary continuous periodic function of period* 2π *and* $\boldsymbol{w}$ *is an arbitrary vector of* $\boldsymbol{R}^d$. *Then, for any periodic solution* $\boldsymbol{u} = \boldsymbol{u}_y(t)$ *of* (2.10) (*if any exists*), *we have*

$$(2.11) \qquad \|\boldsymbol{u}_y\|_q \leq \frac{M_c(\|\boldsymbol{u}_{\boldsymbol{\varphi}}\|_q + K_1\sigma_1(m)\|\boldsymbol{\varphi}(t)\|_q)}{1 - M_c(K_3 + K_1^2)\sigma_1(m)}$$

and since

$$|||\boldsymbol{n}_{\boldsymbol{\varphi}}|||_q \leq \|\boldsymbol{u}_{\boldsymbol{\varphi}}\|_q \leq \sqrt{d}\,|||\boldsymbol{n}_{\boldsymbol{\varphi}}|||_q, \quad \|\boldsymbol{\varphi}(t)\|_q \leq |||\boldsymbol{n}_{\boldsymbol{\varphi}}|||_q,$$

then we have

$$(2.12) \qquad \|\boldsymbol{u}_y\|_q \leq \frac{M_c(\sqrt{d} + K_1\sigma_1(m))}{1 - M_c(K_3 + K_1^2)\sigma_1(m)} |||\boldsymbol{n}_{\boldsymbol{\varphi}}|||_q .$$

PROOF. For brevity let us put

$$\hat{A}_m(t) = \boldsymbol{X}_{\boldsymbol{u}}(\hat{\boldsymbol{u}}_m(t), t).$$

Then for any periodic solution $\boldsymbol{u} = \boldsymbol{u}_y(t)$ of (2.10) we have

$$T_m\boldsymbol{u}_y = \left[\frac{dy}{dt} - \hat{A}_m(t)\boldsymbol{u}_y^t,\ \boldsymbol{f}'(\hat{\boldsymbol{u}}_m(t))\boldsymbol{u}_y(t)\right] = \left[\boldsymbol{\varphi}(t) + \boldsymbol{\eta}(t), \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{w} \end{pmatrix}\right],$$

namely,

$$\begin{cases} \dfrac{dy}{dt} = \hat{A}_m(t)\boldsymbol{u}_y^t + \boldsymbol{\varphi}(t) + \boldsymbol{\eta}(t), \\ \boldsymbol{f}'(\hat{\boldsymbol{u}}_m(t))\boldsymbol{u}_y(t) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{w} \end{pmatrix}, \text{ where } \boldsymbol{\eta}(t) = -(I - P_m)\hat{A}_m(t)\boldsymbol{u}_y^t . \end{cases}$$

Here I is the identity operator.

Put

$$\boldsymbol{z}(t) = \hat{A}_m(t)\boldsymbol{u}_y^t .$$

Then

$$\dot{\boldsymbol{z}}(t) = \dot{\hat{A}}_m(t)\boldsymbol{u}_y^t + \hat{A}_m(t)\begin{pmatrix} P_m[\hat{A}_m(t)\boldsymbol{u}_y^t] + \boldsymbol{\varphi}(t) \\ \boldsymbol{0} \end{pmatrix},$$

from which, it follows that

$$\|\dot{\boldsymbol{z}}\|_q \leq K_3\|\boldsymbol{u}_y\|_q + K_1[\|P_m[\hat{A}_m(t)\boldsymbol{u}_y^t]\|_q + \|\boldsymbol{\varphi}(t)\|_q].$$

But by Bessel's inequality,

$$\|P_m[\hat{A}_m(t)\boldsymbol{u}_y^t]\|_q \leq \|\hat{A}_m(t)\boldsymbol{u}_y^t\|_q \leq K_1\|\boldsymbol{u}_y\|_q .$$

Therefore, we have

$$\|\dot{z}\|_q \leq (K_3+K_1^2)\|\boldsymbol{u}_y\|_q + K_1\|\varphi(t)\|_q .$$

Since $\|\boldsymbol{\eta}\|_q \leq \sigma_1(m)\|\dot{z}\|_q$ by Proposition (2.1), we have then

$$\|\boldsymbol{\eta}(t)\|_q \leq \sigma_1(m)\,[(K_3+K_1^2)\|\boldsymbol{u}_y\|_q + K_1\|\varphi(t)\|_q] .$$

On the other hand, $\hat{\boldsymbol{u}}(t)$ is an isolated periodic solution of (1.4), so if m_0 is sufficiently large, there exists T_m^{-1} and $\|T_m^{-1}\|_q \leq M_c$ for $m \geq m_0$. Then

$$\|\boldsymbol{u}_y\|_q \leq M_c\{\|\boldsymbol{u}_\varphi(t)\|_q + \|\boldsymbol{\eta}(t)\|_q\},$$

and

$$\|\boldsymbol{u}_y\|_q \leq M_c[\|\boldsymbol{u}_\varphi(t)\|_q + \sigma_1(m)\,[(K_3+K_1^2)\|\boldsymbol{u}_y\|_q + K_1\|\varphi(t)\|_q]] .$$

Since $1-M_c(K_3+K_1^2)\sigma_1(m)>0$ for sufficiently large m, then

$$\begin{aligned}\|\boldsymbol{u}_y\|_q &\leq \frac{M_c(\|\boldsymbol{u}_\varphi\|_q + K_1\sigma_1(m)\|\varphi(t)\|_q)}{1-M_c(K_3+K_1^2)\sigma_1(m)} \\ &\leq \frac{M_c(\sqrt{d}\,\|\|\boldsymbol{n}_\varphi\|\|_q + K_1\sigma_1(m)\|\varphi(t)\|_q)}{1-M_c(K_3+K_1^2)\sigma_1(m)} \\ &\leq \frac{M_c(\sqrt{d}+K_1\sigma_1(m))}{1-M_c(K_3+K_1^2)\sigma_1(m)}\|\|\boldsymbol{n}_\varphi\|\|_q .\end{aligned}$$

Q.E.D.

Let

$$\hat{\boldsymbol{u}}(t) = (\hat{\boldsymbol{x}}(t),\ \hat{B}_1,\dots,\hat{B}_d) \qquad (\text{where } \hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{a}}_0 + \sqrt{2}\sum_{n=1}^{\infty}(\hat{\boldsymbol{a}}_{2n-1}\cos nt + \hat{\boldsymbol{a}}_{2n}\sin nt))$$

be an isolated periodic solution of (1.4) lying inside D, and let us consider the Jacobian matrix $J_m(\hat{\boldsymbol{\alpha}})$ where $\hat{\boldsymbol{\alpha}} = \mathrm{col}\,[\hat{\boldsymbol{a}}_0, \hat{\boldsymbol{a}}_1,\dots, \hat{\boldsymbol{a}}_{2m-1}, \hat{\boldsymbol{a}}_{2m}, \hat{B}_1,\dots, \hat{B}_d]$. Then the lemma above yields the following corollaries.

Corollary 2.1.
There exists a positive integer m_0 such that

$$\det J_m(\hat{\boldsymbol{\alpha}}) \neq 0$$

for any $m \geq m_0$.

PROOF. For $\boldsymbol{u}_y(t)$, $\boldsymbol{n}_\varphi(t)$, $\boldsymbol{\xi}$ and $\boldsymbol{\gamma}$ of the form (2.8) and (2.9), the differential system (2.10) is equivalent to the linear system

$$J_m(\hat{\boldsymbol{\alpha}})\boldsymbol{\xi} + \boldsymbol{\gamma} = \boldsymbol{0} \tag{2.13}$$

as mentioned in the beginning of this section. Now put $\boldsymbol{\gamma}=\boldsymbol{0}$. Then $\boldsymbol{n}_\varphi(t)=\boldsymbol{0}$, and this implies $\boldsymbol{u}_y=\boldsymbol{0}$ by (2.12). Then $\boldsymbol{\xi}=\boldsymbol{0}$ by (2.8) and (2.9). Thus, in (2.13),

$\boldsymbol{\gamma}=\mathbf{0}$ implies $\boldsymbol{\xi}=\mathbf{0}$. That is, $\det J_m(\hat{\boldsymbol{\alpha}})\neq 0$. Q.E.D.

Corollary 2.2.

There is a positive integer m_0 such that, for any $m\geq m_0$, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ exists and

$$|||J_m^{-1}(\hat{\boldsymbol{\alpha}})|||'\leq\frac{M_c(1+K_1\sigma_1(m))}{1-M_c(K_3+K_1^2)\sigma_1(m)}.$$

PROOF. By Corollary 2.1, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ certainly exists for $m\geq m_0$. Further for $\boldsymbol{u}_y(t)$, $\boldsymbol{n}_\varphi$, $\boldsymbol{\xi}$ and $\boldsymbol{\gamma}$ of the form (2.8), (2.9), the differential system (2.10) is equivalent to the linear system (2.13). Hence $\boldsymbol{\xi}=-J_m^{-1}(\hat{\boldsymbol{\alpha}})\boldsymbol{\gamma}$. Since $\|\boldsymbol{u}_y(t)\|_q=|||\boldsymbol{\xi}|||'$, $\|\boldsymbol{u}_\varphi(t)\|_q=|||\boldsymbol{\gamma}|||'$ and $\|\boldsymbol{\varphi}(t)\|_q\leq\|\boldsymbol{u}_\varphi\|_q=|||\boldsymbol{\gamma}|||'$, then

$$\|\boldsymbol{u}_y(t)\|_q\leq\frac{M_c(\|\boldsymbol{u}_\varphi\|_q+K_1\sigma_1(m)\|\boldsymbol{\varphi}(t)\|_q)}{1-M_c(K_3+K_1^2)\sigma_1(m)}$$

and so

$$|||\boldsymbol{\xi}|||'\leq\frac{M_c(1+K_1\sigma_1(m))|||\boldsymbol{\gamma}|||'}{1-M_c(K_3+K_1^2)\sigma_1(m)}.$$

Namely

$$|||J_m^{-1}(\hat{\boldsymbol{\alpha}})|||'\leq\frac{M_c(1+K_1\sigma_1(m))}{1-M_c(K_3+K_1^2)\sigma_1(m)}.$$ Q.E.D.

Lastly, for the difference $J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')$, we shall prove the following lemma.

Lemma 3.

Under the conditions of Lemma 2, let K_4 be a positive constant

$$K_4\geq\left[\max_{D\times R}\sum_{k=1}^{n}\sum_{l=1}^{n+d}\left\{\sum_{p=1}^{n}\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2+\sum_{i=1}^{d}\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2\right\}\right]^{\frac{1}{2}}, \tag{2.14}$$

where $X_u^{kl}(\boldsymbol{u},t)$ $(k=1,\dots,n;\ l=1,\dots,n+d)$ are the elements of the matrix

$$\boldsymbol{X}_u(\boldsymbol{u},t)=(\boldsymbol{X}_x(\boldsymbol{x},B_1,\dots,B_d,t)\quad \boldsymbol{X}_{B_1}(\boldsymbol{x},B_1,\dots,B_d,t)\cdots\boldsymbol{X}_{B_d}(\boldsymbol{x},B_1,\dots,B_d,t))$$

and $x_p(p=1,\dots,n)$ are the components of the vector $\boldsymbol{x}$. We assume that there exists a non-negative constant K_6 such that

$$\|\boldsymbol{g}'(\boldsymbol{u}')-\boldsymbol{g}'(\boldsymbol{u}'')\|_{q,d}\leq K_6\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty$$

for any $\boldsymbol{u}',\boldsymbol{u}''\in S'$.
Then, if both

$$\boldsymbol{u}'(t)=(\boldsymbol{x}'(t),B_1',\dots,B_d')\qquad (\textit{where } \boldsymbol{x}'(t)=\boldsymbol{\alpha}_0'+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{\alpha}_{2n-1}'\cos nt+\boldsymbol{\alpha}_{2n}'\sin nt))$$

and

$$\boldsymbol{u}''(t)=(\boldsymbol{x}''(t), B_1'',\dots, B_d'') \qquad (\textit{where } \boldsymbol{x}''(t)=\boldsymbol{a}_0''+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{a}_{2n-1}''\cos nt+\boldsymbol{a}_{2n}''\sin nt))$$

belong to D together with $\theta\boldsymbol{u}'(t)+(1-\theta)\boldsymbol{u}''(t)$ $(0\le\theta\le 1)$, *then*

$$\text{(2.15)} \qquad |||J_m(\boldsymbol{a}')-J_m(\boldsymbol{a}'')|||'\le(K_4+\sqrt{d}K_6)\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty$$
$$\le(K_4+\sqrt{d}K_6)\sqrt{2m+1}\,|||\boldsymbol{a}'-\boldsymbol{a}''|||',$$

where $\boldsymbol{a}'=\mathrm{col}\,[\boldsymbol{a}_0', \boldsymbol{a}_1',\dots, \boldsymbol{a}_{2m-1}', \boldsymbol{a}_{2m}', B_1',\dots, B_d']$ *and*

$$\boldsymbol{a}''=\mathrm{col}\,[\boldsymbol{a}_0'', \boldsymbol{a}_1'',\dots, \boldsymbol{a}_{2m-1}'', \boldsymbol{a}_{2m}'', B_1'',\dots, B_d''].$$

PROOF. Take an arbitrary $\boldsymbol{\xi}=\mathrm{col}\,[\boldsymbol{v}_0, \boldsymbol{v}_1,\dots, \boldsymbol{v}_{2m-1}, \boldsymbol{v}_{2m}, V_1,\dots, V_d]$, and consider

$$\boldsymbol{u}_y(t)=(\boldsymbol{y}(t), V_1,\dots, V_d), \qquad \text{where } \boldsymbol{y}(t)=\boldsymbol{v}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{v}_{2n-1}\cos nt+\boldsymbol{v}_{2n}\sin nt).$$

Put

$$\text{(2.16)} \qquad \boldsymbol{\gamma}'=-J_m(\boldsymbol{a}')\boldsymbol{\xi},\ \boldsymbol{\gamma}''=-J_m(\boldsymbol{a}'')\boldsymbol{\xi},$$

and let

$$\boldsymbol{\gamma}'=\mathrm{col}\,[\boldsymbol{c}_0', \boldsymbol{c}_1',\dots, \boldsymbol{c}_{2m-1}', \boldsymbol{c}_{2m}', -C_1',\dots, -C_d'],\ \boldsymbol{w}'=\mathrm{col}\,[C_1',\dots, C_d'],$$

and

$$\boldsymbol{\gamma}''=\mathrm{col}\,[\boldsymbol{c}_0'', \boldsymbol{c}_1'',\dots, \boldsymbol{c}_{2m-1}'', \boldsymbol{c}_{2m}'', -C_1'',\dots, -C_d''],\ \boldsymbol{w}''=\mathrm{col}\,[C_1'',\dots, C_d''].$$

If we put

$$\boldsymbol{u}_{\varphi'}=(\boldsymbol{\varphi}'(t), C_1',\dots, C_d'),\ \boldsymbol{n}_{\varphi'}=\left[\boldsymbol{\varphi}'(t), \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}'\end{pmatrix}\right], \text{ and}$$

$$\boldsymbol{u}_{\varphi''}=(\boldsymbol{\varphi}''(t), C_1'',\dots, C_d''),\ \boldsymbol{n}_{\varphi''}=\left[\boldsymbol{\varphi}''(t), \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}''\end{pmatrix}\right],$$

$$(\text{where } \boldsymbol{\varphi}'(t)=\boldsymbol{c}_0'+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n-1}'\cos nt+\boldsymbol{c}_{2n}'\sin nt),$$

$$\boldsymbol{\varphi}''(t)=\boldsymbol{c}_0''+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{c}_{2n-1}''\cos nt+\boldsymbol{c}_{2n}''\sin nt)),$$

then by (2.10) and (2.16) we have

$$\left[\frac{d\boldsymbol{y}}{dt}-P_m[\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}'(t), t)\boldsymbol{u}_y^t],\ \boldsymbol{f}'(\boldsymbol{u}'(t))\boldsymbol{u}_y(t)\right]=\boldsymbol{n}_{\varphi'}=\left[\boldsymbol{\varphi}'(t), \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}'\end{pmatrix}\right],$$

$$\left[\frac{d\boldsymbol{y}}{dt}-P_m[\boldsymbol{X}_{\boldsymbol{u}}(\boldsymbol{u}''(t), t)\boldsymbol{u}_y^t],\ \boldsymbol{f}'(\boldsymbol{u}''(t))\boldsymbol{u}_y(t)\right]=\boldsymbol{n}_{\varphi''}=\left[\boldsymbol{\varphi}''(t), \begin{pmatrix}\mathbf{0}\\ \boldsymbol{w}''\end{pmatrix}\right],$$

where $\boldsymbol{u}_y^t$ denotes the transpose of $\boldsymbol{u}_y(t)$.

From this it readily follows that

$$n_{\varphi'}-n_{\varphi''}=\left[\varphi'(t)-\varphi''(t),\begin{pmatrix}\mathbf{0}\\ w'\end{pmatrix}-\begin{pmatrix}\mathbf{0}\\ w''\end{pmatrix}\right]$$

$$=[-P_m[(X_u(u'(t),t)-X_u(u''(t),t))u_y^t],\ [f'(u'(t))-f'(u''(t))]u_y(t)],$$

that is,

$$\begin{cases}\varphi'(t)-\varphi''(t)=-P_m[(X_u(u'(t),t)-X_u(u''(t),t))u_y^t], \text{ and}\\ w'-w''=[g'(u')-g'(u'')]u_y(t)\,.\end{cases}$$

Let us put

$$\varphi(t)=\varphi'(t)-\varphi''(t) \text{ and}$$

$$\gamma=\gamma'-\gamma''=\text{col}\,[c_0'-c_0'',\dots,\ c_{2m}'-c_{2m}'',\ -C_1'+C_1'',\dots,\ -C_d'+C_d''].$$

Then we have

$$\varphi(t)=-P_m[(X_u(u'(t),t)-X_u(u''(t),t))u_y^t] \tag{2.17}$$

and

$$\gamma=-[J_m(\alpha')-J_m(\alpha'')]\xi. \tag{2.18}$$

Now $\|X_u(u'(t),t)-X_u(u''(t),t)\|_{n+d,n}^2\leq\sum_{k=1}^{n}\sum_{l=1}^{n+d}(X_u^{kl}(u'(t),t)-X_u^{kl}(u''(t),t))^2$, where $X_u^{kl}(u'(t),t)$ and $X_u^{kl}(u''(t),t)$ are the elements of $X_u(u'(t),t)$ and $X_u(u''(t),t)$, respectively. Since $u''(t)+\theta(u'(t)-u''(t))\in D$ $(0\leq\theta\leq 1)$ by the assumption, the quantity in the right member of the above inequality estimated successively by means of Schwarz's inequality as follows:

$$[X_u^{kl}(u'(t),t)-X_u^{kl}(u''(t),t)]^2$$

$$=\left[\int_0^1\left\{\sum_{p=1}^{n}\frac{\partial X_u^{kl}(u''+\theta(u'-u''))}{\partial x_p}(x_p'-x_p'')\right.\right.$$

$$\left.\left.+\sum_{i=1}^{d}\frac{\partial X_u^{kl}(u''+\theta(u'-u''))}{\partial B_i}(B_i'-B_i'')\right\}d\theta\right]^2$$

$$=\left[\sum_{p=1}^{n}\int_0^1\frac{\partial X_u^{kl}}{\partial x_p}d\theta\times(x_p'-x_p'')+\sum_{i=1}^{d}\int_0^1\frac{\partial X_u^{kl}}{\partial B_i}d\theta\times(B_i'-B_i'')\right]^2$$

$$\leq\left[\sum_{p=1}^{n}\left\{\int_0^1\frac{\partial X_u^{kl}}{\partial x_p}d\theta\right\}^2+\sum_{i=1}^{d}\left\{\int_0^1\frac{\partial X_u^{kl}}{\partial B_i}d\theta\right\}^2\right]\times\left[\sum_{p=1}^{n}(x_p'-x_p'')^2+\sum_{i=1}^{d}(B_i'-B_i'')^2\right]$$

$$\leq\left[\sum_{p=1}^{n}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2d\theta+\sum_{i=1}^{d}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2d\theta\right]\times\left[\|x'(t)-x''(t)\|_n^2+\sum_{i=1}^{d}(B_i'-B_i'')^2\right]$$

$$\leq\left[\sum_{p=1}^{n}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2 d\theta+\sum_{i=1}^{d}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2 d\theta\right]\times[\|\boldsymbol{x}'-\boldsymbol{x}''\|_c^2+\sum_{i=1}^{d}(B_i'-B_i'')^2]$$

$$\leq\left[\sum_{p=1}^{n}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2 d\theta+\sum_{i=1}^{d}\int_0^1\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2 d\theta\right]\times\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty^2 .$$

Then

$$\sum_{k,l}[X_u^{kl}(\boldsymbol{u}'(t),t)-X_u^{kl}(\boldsymbol{u}''(t),t)]^2$$

$$\leq\int_0^1\left[\sum_{k,l}\left\{\sum_{p=1}^{n}\left(\frac{\partial X_u^{kl}}{\partial x_p}\right)^2+\sum_{i=1}^{d}\left(\frac{\partial X_u^{kl}}{\partial B_i}\right)^2\right\}\right]d\theta\times\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty^2\leq K_4^2\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty^2 .$$

Hence

$$\begin{aligned}&\|(\boldsymbol{X}_u(\boldsymbol{u}'(t),t)-\boldsymbol{X}_u(\boldsymbol{u}''(t),t))\boldsymbol{u}_y^t\|_n\\&\quad\leq\|\boldsymbol{X}_u(\boldsymbol{u}'(t),t)-\boldsymbol{X}_u(\boldsymbol{u}''(t),t)\|_{n+d,n}\|\boldsymbol{u}_y^t\|_{n+d}\\&\quad\leq K_4\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty\|\boldsymbol{u}_y^t\|_{n+d}.\end{aligned}$$

Then by Bessel's inequality it follows from (2.17) that

$$\|\boldsymbol{\varphi}(t)\|_q\leq K_4\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty\|\boldsymbol{u}_y\|_q .$$

Since $|||\boldsymbol{\gamma}|||'=\|\boldsymbol{\varphi}\|_q+\sum_{i=1}^{d}|C_i'-C_i''|$ and $|||\boldsymbol{\xi}|||'=\|\boldsymbol{u}_y\|_q$, from the assumption of Lemma and (2.18) we have

$$\begin{aligned}|||(J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}''))\boldsymbol{\xi}|||'&=|||\boldsymbol{\gamma}|||'=\|\boldsymbol{\varphi}\|_q+\sum_{i=1}^{d}|C_i'-C_i''|\\&\leq K_4\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty\|\boldsymbol{u}_y\|_q+\sqrt{d}\,[\sum_{i=1}^{d}(C_i'-C_i'')^2]\\&\leq K_4\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty|||\boldsymbol{\xi}|||'+\sqrt{d}K_6\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty|||\boldsymbol{\xi}|||',\end{aligned}$$

which implies

$$|||J_m(\boldsymbol{\alpha}')-J_m(\boldsymbol{\alpha}'')|||'\leq(K_4+\sqrt{d}K_6)\|\boldsymbol{u}'-\boldsymbol{u}''\|_\infty.$$

Put $\boldsymbol{\alpha}=\boldsymbol{\alpha}'-\boldsymbol{\alpha}''$, and suppose $\boldsymbol{\alpha}=\mathrm{col}\,[\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_{2m-1},\boldsymbol{a}_{2m},B_1,\dots,B_d]$. Then

$$\boldsymbol{u}(t)=(\boldsymbol{x}_m(t),B_1,\dots,B_d)=\boldsymbol{u}'(t)-\boldsymbol{u}''(t)=(\boldsymbol{x}_m'(t)-\boldsymbol{x}_m''(t),B_1'-B_1'',\dots,B_d'-B_d''),$$

$$\begin{cases}\boldsymbol{x}_m(t)=\boldsymbol{x}_m'(t)-\boldsymbol{x}_m''(t)=\boldsymbol{a}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{a}_{2n-1}\cos nt+\boldsymbol{a}_{2n}\sin nt),\\ B_i=B_i'-B_i''\qquad(i=1,2,\dots,d),\end{cases}$$

and therefore

$$\|\boldsymbol{u}(t)\|_\infty = \|\boldsymbol{x}_m(t)\|_c + \sum_{i=1}^d |B_i| = \|\boldsymbol{x}'_m - \boldsymbol{x}''_m\|_c + \sum_{i=1}^d |B'_i - B''_i|$$

$$\leq \{\sum_k [|a_{0k}| + \sqrt{2}\sum_{n=1}^m \sqrt{a_{2n-1k}^2 + a_{2nk}^2}]^2\}^{\frac{1}{2}} + \sum_{i=1}^d |B'_i - B''_i|$$

$$\leq \{\sum_k (1+2m)[a_{0k}^2 + \sum_{n=1}^m (a_{2n-1k}^2 + a_{2nk}^2)]\}^{\frac{1}{2}} + \sum_{i=1}^d |B'_i - B''_i|$$

$$= (2m+1)^{\frac{1}{2}}[\|\boldsymbol{\alpha}_0\|_n^2 + \sum_{i=1}^m (\|\boldsymbol{\alpha}_{2i-1}\|_n^2 + \|\boldsymbol{\alpha}_{2i}\|_n^2)]^{\frac{1}{2}} + \sum_{i=1}^d |B'_i - B''_i|$$

$$\leq \sqrt{2m+1}\,|||\boldsymbol{\alpha}' - \boldsymbol{\alpha}''|||'.$$

Q.E.D.

The existence of a Galerkin approximation

The existence of a Galerkin approximation to an isolated periodic solution is proved by the following theorem.

Theorem 3.
Let

$$\left[\frac{d\boldsymbol{x}}{dt} - \boldsymbol{X}(\boldsymbol{x}, B_1, \dots, B_d, t),\ \boldsymbol{f}(\boldsymbol{u})\right] = \boldsymbol{0} \tag{2.19}$$

be a given boundary value problem, where $\boldsymbol{u}(t) = (\boldsymbol{x}(t), B_1, \dots, B_d)$, $\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix}\boldsymbol{x}(0) - \boldsymbol{x}(2\pi)\\ \boldsymbol{g}(\boldsymbol{u})\end{pmatrix}$ *and where* $\boldsymbol{x}$ *and* $\boldsymbol{X}(\boldsymbol{x}, B_1, \dots, B_d, t)$ *are real n-dimensional vectors and* $\boldsymbol{X}(\boldsymbol{x}, B_1, \dots, B_d, t)$ *is periodic in t of period* 2π, *and where* $\boldsymbol{g}(\boldsymbol{u})$ *is the appropriate d-dimensional vector-function. We assume that* $\boldsymbol{X}(\boldsymbol{x}, B_1, \dots, B_d, t)$ *and its first partial derivatives with respect to* $\boldsymbol{u}$ *are continuously differentiable with respect to* $\boldsymbol{u}$ *and t in the region* $D \times \boldsymbol{R}$, *where D is a closed bounded region of the* $\boldsymbol{u}$*-space such that* $D \times I \subset \Omega$, *and* $\boldsymbol{R}$ *is the real line, where* $I = [0, 2\pi]$. *Moreover we assume that* $\boldsymbol{g}(\boldsymbol{u})$ *is continuously Fréchet differentiable on S' and there exists a positive constant* K_5 *such that* $\|\boldsymbol{g}'(\boldsymbol{u})\|_{\infty,d} \leq K_5$ *for any* $\boldsymbol{u} \in S'$, *and we assume that there exists a non-negative constant* K_6 *such that*

$$\|\boldsymbol{g}'(\boldsymbol{u}') - \boldsymbol{g}'(\boldsymbol{u}'')\|_{q,d} \leq K_6 \|\boldsymbol{u}' - \boldsymbol{u}''\|_\infty$$

for any $\boldsymbol{u}', \boldsymbol{u}'' \in S'$.

If there is an isolated periodic solution $\boldsymbol{u} = \hat{\boldsymbol{u}}(t)$ *of* (2.19) *lying inside D, then there exists a Galerkin approximation* $\boldsymbol{u} = \bar{\boldsymbol{u}}_m(t) = (\bar{\boldsymbol{x}}_m(t), \bar{B}_1, \dots, \bar{B}_d)$ *of any order* $m \geq m_0$ *lying in D provided* m_0 *is sufficiently large.*

Proof. Setting

$$P_m \hat{\boldsymbol{u}}(t) = (P_m \hat{\boldsymbol{x}}(t), \hat{B}_1, \dots, \hat{B}_d) = (\hat{\boldsymbol{x}}_m(t), \hat{B}_1, \dots, \hat{B}_d) = \hat{\boldsymbol{u}}_m(t),$$

we have

(2.20) $$\frac{d\hat{\boldsymbol{x}}_m}{dt}=P_m\frac{d\hat{\boldsymbol{x}}}{dt}=P_m\boldsymbol{X}(\hat{\boldsymbol{u}},t)\qquad (\text{where } \boldsymbol{X}(\hat{\boldsymbol{u}},t)=\boldsymbol{X}(\hat{\boldsymbol{x}},\hat{B}_1,\dots,\hat{B}_d,t))\,.$$

Now let us take a small positive number δ_0 so that

$$U=\left\{\boldsymbol{u}=(\boldsymbol{x},B_1,\dots,B_d)\,;\ \left\|\begin{pmatrix}\boldsymbol{x}-\hat{\boldsymbol{x}}(t)\\ B_1-\hat{B}_1\\ \vdots\\ B_d-\hat{B}_d\end{pmatrix}\right\|_{n+d}\leq\delta_0\quad \textit{for some } t\in\boldsymbol{R}\right\}\subset D.$$

This is possible because $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ lies inside D by the assumption. Then by Lemma 1-(i), $\hat{\boldsymbol{u}}_m(t)\in U\subset D$ for all $t\in\boldsymbol{R}$ and for any $m\geq m_0$ provided m_0 is sufficiently large.

For such m equation (2.20) can be rewritten as follows:

(2.21) $$\frac{d\hat{\boldsymbol{x}}_m}{dt}=P_m\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t),t)+\boldsymbol{R}_m(t),$$

where

$$\boldsymbol{R}_m(t)=P_m[\boldsymbol{X}(\hat{\boldsymbol{u}}(t),t)-\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t),t)]\,.$$

Now

$$\boldsymbol{X}(\hat{\boldsymbol{u}}(t),t)-\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t),t)=-\int_0^1\boldsymbol{X}_{\boldsymbol{u}}(\hat{\boldsymbol{u}}(t)+\theta(\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)),t)\begin{pmatrix}\hat{\boldsymbol{x}}_m-\hat{\boldsymbol{x}}\\ \boldsymbol{0}\end{pmatrix}d\theta,$$

and

$$\|\boldsymbol{X}_{\boldsymbol{u}}(\hat{\boldsymbol{u}}(t)+\theta(\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)),t)\|_{n+d,n}\leq K_1,$$

hence

$$\|\boldsymbol{X}(\hat{\boldsymbol{u}}(t),t)-\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t),t)\|_n\leq K_1\left\|\begin{pmatrix}\hat{\boldsymbol{x}}-\hat{\boldsymbol{x}}_m\\ \boldsymbol{0}\end{pmatrix}\right\|_{n+d}=K_1\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{x}}_m\|_n\,.$$

Then by the proof of Lemma 1-(ii) we have

$$\|\boldsymbol{X}(\hat{\boldsymbol{u}}(t),t)-\boldsymbol{X}(\hat{\boldsymbol{u}}_m(t),t)\|_q\leq K_1\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{x}}_m\|_q\leq KK_1\sigma_1(m)\,.$$

Hence, from Bessel's inequality, we see that

$$\|\boldsymbol{R}_m\|_q\leq KK_1\sigma_1(m)\,.$$

Let us put

$$\hat{\boldsymbol{u}}(t)=(\hat{\boldsymbol{x}}(t),\hat{B}_1,\dots,\hat{B}_d),\ \text{where } \hat{\boldsymbol{x}}(t)=\hat{\boldsymbol{a}}_0+\sqrt{2}\sum_{n=1}^{\infty}(\hat{\boldsymbol{a}}_{2n-1}\cos nt+\hat{\boldsymbol{a}}_{2n}\sin nt)$$

and

$$\boldsymbol{R}_m(t)=\boldsymbol{r}_0^{(m)}+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{r}_{2n-1}^{(m)}\cos nt+\boldsymbol{r}_{2n}^{(m)}\sin nt)\,.$$

Now setting

$$f(\hat{u}_m(t))=\begin{pmatrix} \hat{x}_m(0)-\hat{x}_m(2\pi) \\ g(\hat{u}_m(t)) \end{pmatrix}=\begin{pmatrix} \mathbf{0} \\ v \end{pmatrix}. \tag{2.22}$$

Then we have

$$v=g(\hat{u}_m)-g(\hat{u})=\int_0^1 g'(\hat{u}+\theta(\hat{u}_m-\hat{u}))(\hat{u}_m-\hat{u})\,d\theta,$$

since

$$f(\hat{u}(t))=\begin{pmatrix} \hat{x}(0)-\hat{x}(2\pi) \\ g(\hat{u}) \end{pmatrix}=\mathbf{0}.$$

From this equation and the assumption of theorem, we see that

$$\|v\|_d\leq\|g'(\hat{u}+\theta(\hat{u}_m-\hat{u}))\|_{\infty,d}\|\hat{u}_m-\hat{u}\|_\infty\leq K_5\|\hat{u}_m-\hat{u}\|_\infty .$$

From (2.21) and (2.22), we have

$$\left[\frac{d\hat{x}_m}{dt}-P_mX(\hat{u}_m(t),t),\, f(\hat{u}_m(t))\right]=\left[R_m(t),\begin{pmatrix} \mathbf{0} \\ v \end{pmatrix}\right]. \tag{2.23}$$

Then (2.23) is equivalent to the following system:

$$\begin{cases} F_0^{(m)}(\hat{\alpha})=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} X(\hat{u}_m(t),t)\,dt=-r_0^{(m)}, \\ F_{2n-1}^{(m)}(\hat{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi} X(\hat{u}_m(t),t)\cos nt\,dt-n\hat{\alpha}_{2n}=-r_{2n-1}^{(m)}, \\ F_{2n}^{(m)}(\hat{\alpha})=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{2\pi} X(\hat{u}_m(t),t)\sin nt\,dt+n\hat{\alpha}_{2n-1}=-r_{2n}^{(m)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (n=1,2,\dots,m), \\ F_g^{(m)}(\hat{\alpha})=g(\hat{u}_m(t))=g(\hat{u}_m(t))-g(\hat{u}(t))=v=\begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}, \end{cases} \tag{2.24}$$

where

$$\begin{cases} \hat{\alpha}=\operatorname{col}[\hat{\alpha}_0,\hat{\alpha}_1,\dots,\hat{\alpha}_{2m-1},\hat{\alpha}_{2m},\hat{B}_1,\dots,\hat{B}_d], \\ \rho^{(m)}=\operatorname{col}[r_0^{(m)},r_1^{(m)},\dots,r_{2m-1}^{(m)},r_{2m}^{(m)},-v_1,\dots,-v_d], \\ \hat{u}_m(t)=(\hat{x}_m(t),\hat{B}_1,\dots,\hat{B}_d), \\ \hat{x}_m(t)=\hat{\alpha}_0+\sqrt{2}\sum_{n=1}^{m}(\hat{\alpha}_{2n-1}\cos nt+\hat{\alpha}_{2n}\sin nt). \end{cases}$$

Put $\boldsymbol{F}^{(m)}(\hat{\boldsymbol{\alpha}})=\mathrm{col}\,[\boldsymbol{F}_0^{(m)}(\hat{\boldsymbol{\alpha}}), \boldsymbol{F}_1^{(m)}(\hat{\boldsymbol{\alpha}}),\dots, \boldsymbol{F}_{2m-1}^{(m)}(\hat{\boldsymbol{\alpha}}), \boldsymbol{F}_{2m}^{(m)}(\hat{\boldsymbol{\alpha}}), (F_g)_1,\dots, (F_g)_d]$, then (2.24) can be rewritten as follows:

$$\boldsymbol{F}^{(m)}(\hat{\boldsymbol{\alpha}})=-\boldsymbol{\rho}^{(m)},$$

where, $\boldsymbol{F}_g^{(m)}(\hat{\boldsymbol{\alpha}})=\mathrm{col}\,[(F_g)_1,\dots, (F_g)_d]$.
Then

$$\begin{aligned}|||\boldsymbol{\rho}^{(m)}|||'=|||\boldsymbol{F}^{(m)}(\hat{\boldsymbol{\alpha}})|||'&=\|\boldsymbol{R}_m(t)\|_q+\sum_{i=1}^{d}|v_i|\\ &\le\|\boldsymbol{R}_m(t)\|_q+\sqrt{d}\|\boldsymbol{v}\|_d\\ &\le KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty.\end{aligned}$$

Now, for $m\ge m_0$ and m_0 sufficiently large, let us consider the region

$$V_m=\left\{\boldsymbol{u}=(\boldsymbol{x}, B_1,\dots, B_d);\ \left\|\begin{pmatrix}\boldsymbol{x}-\hat{\boldsymbol{x}}_m(t)\\ B_1-\hat{B}_1\\ \vdots\\ B_d-\hat{B}_d\end{pmatrix}\right\|_{n+d}\le\delta_0-K\sigma(m)\quad\text{for some } t\in\boldsymbol{R}\right\}.$$

For any $\boldsymbol{u}=(\boldsymbol{x}, B_1,\dots, B_d)\in V_m$, then

$$\begin{aligned}\left\|\begin{pmatrix}\boldsymbol{x}-\hat{\boldsymbol{x}}(t)\\ B_1-\hat{B}_1\\ \vdots\\ B_d-\hat{B}_d\end{pmatrix}\right\|_{n+d}&\le\left\|\begin{pmatrix}\boldsymbol{x}-\hat{\boldsymbol{x}}_m(t)\\ B_1-\hat{B}_1\\ \vdots\\ B_d-\hat{B}_d\end{pmatrix}\right\|_{n+d}+\left\|\begin{pmatrix}\hat{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\\ \hat{B}_1-\hat{B}_1\\ \vdots\\ \hat{B}_d-\hat{B}_d\end{pmatrix}\right\|_{n+d}\\ &=\left\|\begin{pmatrix}\boldsymbol{x}-\hat{\boldsymbol{x}}_m(t)\\ B_1-\hat{B}_1\\ \vdots\\ B_d-\hat{B}_d\end{pmatrix}\right\|_{n+d}+\|\hat{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\|_n\\ &\le\delta_0-K\sigma(m)+K\sigma(m)=\delta_0\,.\end{aligned}$$

This implies $\boldsymbol{u}=(\boldsymbol{x}, B_1,\dots, B_d)\in U\subset D$. That is,

$$V_m\subset U\subset D\qquad\text{for any}\quad m\ge m_0.$$

Consider

$$\Omega_m=\left\{\boldsymbol{\alpha};\ |||\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}|||'\le\frac{\delta_0-K\sigma(m)}{\sqrt{2m+1}}\right\},$$

where $\boldsymbol{\alpha}=\mathrm{col}\,[\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1,\dots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, B_1,\dots, B_d]$.

Then, as is shown in the proof of Lemma 3, for $\boldsymbol{u}(t)=(\boldsymbol{x}(t), B_1,\dots, B_d)$ (where $\boldsymbol{x}(t)=\boldsymbol{\alpha}_0+\sqrt{2}\sum_{n=1}^{m}(\boldsymbol{\alpha}_{2n-1}\cos nt+\boldsymbol{\alpha}_{2n}\sin nt)$) with $\boldsymbol{\alpha}=\mathrm{col}\,[\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1,\dots, \boldsymbol{\alpha}_{2m-1}, \boldsymbol{\alpha}_{2m}, B_1,\dots,$

$B_d] \in \Omega_m$, we have

$$\left\| \begin{pmatrix} \boldsymbol{x}(t) - \hat{\boldsymbol{x}}_m(t) \\ B_1 - \hat{B}_1 \\ \vdots \\ B_d - \hat{B}_d \end{pmatrix} \right\|_{n+d} \le \|\boldsymbol{u}(t) - \hat{\boldsymbol{u}}_m(t)\|_\infty \le \sqrt{2m+1}\, |||\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}|||' \le \delta_0 - K\sigma(m)$$

for any $m \ge m_0$, and hence $\boldsymbol{u}(t) = (\boldsymbol{x}(t), B_1, \dots, B_d) \in V_m \subset D$. Thus, it is proved that $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})$ is well defined for any $\boldsymbol{\alpha} \in \Omega_m$.

From (2.24) we note that a Galerkin approximation is a trigonometric polynomial whose Fourier coefficients satisfy the equation

$$\boldsymbol{F}^{(m)}(\boldsymbol{\alpha}) = \boldsymbol{0}. \tag{2.25}$$

Since $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$ is an approximate solution of the above equation, we shall apply Proposition 2 (M. Urabe [5]) to the above equation in order to prove the existence of an exact solution, namely, the existence of a Galerkin approximation.

Let us take m_0 sufficiently large. Then by Corollary 2.2 of Lemma 2 for any $m \ge m_0$, $J_m^{-1}(\hat{\boldsymbol{\alpha}})$ exists and

$$|||J_m^{-1}(\hat{\boldsymbol{\alpha}})|||' \le \frac{M_c(1 + K_1\sigma_1(m))}{1 - M_c(K_3 + K_1^2)\sigma_1(m)}.$$

This implies that

$$|||J_m^{-1}(\hat{\boldsymbol{\alpha}})|||' \le M' \qquad \text{for any} \quad m \ge m_0, \tag{C.1}$$

where

$$M' = \frac{M_c(1 + K_1\sigma_1(m_0))}{1 - M_c(K_3 + K_1^2)\sigma_1(m_0)}. \tag{2.26}$$

Further by Lemma 3

$$|||J_m(\boldsymbol{\alpha}) - J_m(\hat{\boldsymbol{\alpha}})|||' \le (K_4 + \sqrt{d}K_6)\sqrt{2m+1}\, |||\boldsymbol{\alpha} - \hat{\boldsymbol{\alpha}}|||'.$$

for any $\boldsymbol{\alpha} \in \Omega_m$ provided $m \ge m_0$.

Take an arbitrary number κ such that $0 < \kappa < 1$, and put

$$\delta_1 = \min\left(\frac{\kappa}{(K_4 + \sqrt{d}K_6)M'},\ \delta_0 - K\sigma(m_0)\right).$$

Let us take $m_1 \ge m_0$, so that, for any $m \ge m_1$,

$$\frac{M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1 - \kappa} < \frac{\delta_1}{\sqrt{2m+1}}. \tag{2.27}$$

This is possible because

$$\sqrt{2m+1}\,\sigma_1(m)=\frac{\sqrt{2m+1}}{m+1}\longrightarrow 0 \quad \text{as} \quad m\longrightarrow+\infty,$$

and

$$\sqrt{2m+1}\,\|\hat{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)\|_\infty\longrightarrow 0 \quad \text{as} \quad m\longrightarrow+\infty.$$

By (2.27) we can take a positive number δ_m such that

$$\frac{M'\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa}\leq\delta_m\leq\frac{\delta_1}{\sqrt{2m+1}}.$$

Let us consider the set

$$\Omega_{\delta_m}=\{\boldsymbol{\alpha};\ |||\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}|||'\leq\delta_m\}.$$

For any $\boldsymbol{\alpha}\in\Omega_{\delta_m}$ we have

$$|||\boldsymbol{\alpha}-\hat{\boldsymbol{\alpha}}|||'\leq\frac{\delta_1}{\sqrt{2m+1}}\leq\frac{\delta_0-K\sigma(m_0)}{\sqrt{2m+1}}\leq\frac{\delta_0-K\sigma(m)}{\sqrt{2m+1}} \qquad (m\geq m_1\geq m_0),$$

and consequently,

$$\Omega_{\delta_m}\subset\Omega_m.$$

Then, for any $\boldsymbol{\alpha}\in\Omega_{\delta_m}$, we have

$$\text{(C.2)}\qquad \begin{aligned}|||J_m(\boldsymbol{\alpha})-J_m(\hat{\boldsymbol{\alpha}})|||'&\leq(K_4+\sqrt{d}K_6)\sqrt{2m+1}\,\delta_m\leq(K_4+\sqrt{d}K_6)\delta_1\\&\leq(K_4+\sqrt{d}K_6)\frac{\kappa}{(K_4+\sqrt{d}K_6)M'}\\&\leq\frac{\kappa}{M'}.\end{aligned}$$

Further,

$$\text{(C.3)}\qquad \frac{M'|||\boldsymbol{\rho}^{(m)}|||'}{1-\kappa}\leq\frac{M'\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa}\leq\delta_m.$$

The expressions (C.1)–(C.3) show that the conditions of Proposition 2 (M. Urabe [5]) are all fulfilled. Thus, by that proposition we see that equation $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})=\boldsymbol{0}$ has one and only one solution $\boldsymbol{\alpha}=\bar{\boldsymbol{\alpha}}$ lying in Ω_{δ_m}. This proves the theorem. Q.E.D.

Uniformly convergence of a Galerkin approximation of a periodic solution of (2.19)

Theorem 4.

Assume that the conditions of Theorem 3 are satisfied. Let $\boldsymbol{u}=\hat{\boldsymbol{u}}(t)$ be an isolated periodic solution of (2.19) lying inside D and $\boldsymbol{u}=\bar{\boldsymbol{u}}_m(t)$ be its Galerkin approximation as stated in Theorem 3.

If m_0 is sufficiently large, then for any positive integer $m \geq m_0$,

$$\text{(2.28)} \qquad \|\bar{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty \leq \frac{\sqrt{2m+1}\; M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa} + K\sigma(m),$$

$$\text{(2.29)} \qquad \|\dot{\bar{\boldsymbol{u}}}_m - \dot{\hat{\boldsymbol{u}}}\|_\infty \leq (K_2 + 2KK_1)\sigma(m) + \frac{\sqrt{2m+1}\,M'K_1\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa},$$

where κ is an arbitrary fixed number such that $0<\kappa<1$, K, K_1 and K_2 are the numbers defined in (2.2), *M' is the number defined in* (2.26), *and K_5 is the number such that $\|\boldsymbol{g}'(\boldsymbol{u})\|_{\infty,d} \leq K_5$ for any $\boldsymbol{u} \in S'$.*

PROOF. Put

$$\bar{\boldsymbol{u}}_m(t) = (\bar{\boldsymbol{x}}_m(t), \bar{B}_1, \dots, \bar{B}_d) \text{ (where } \bar{\boldsymbol{x}}_m(t) = \bar{\boldsymbol{\alpha}}_0 + \sqrt{2}\sum_{n=1}^{m}(\bar{\boldsymbol{\alpha}}_{2n-1}\cos nt + \bar{\boldsymbol{\alpha}}_{2n}\sin nt)).$$

As shown in the proof of Theorem 3, $\bar{\boldsymbol{\alpha}} = \mathrm{col}[\bar{\boldsymbol{\alpha}}_0, \bar{\boldsymbol{\alpha}}_1, \dots, \bar{\boldsymbol{\alpha}}_{2m-1}, \bar{\boldsymbol{\alpha}}_{2m}, \bar{B}_1, \dots, \bar{B}_d]$ is a solution of $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha}) = \boldsymbol{0}$ lying in Ω_{δ_m}, and by Proposition 2 (M. Urabe [5]) we have

$$\text{(2.30)} \qquad \|\|\bar{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}\|\|' \leq \frac{M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa},$$

where $\hat{\boldsymbol{\alpha}} = \mathrm{col}[\hat{\boldsymbol{\alpha}}_0, \hat{\boldsymbol{\alpha}}_1, \dots, \hat{\boldsymbol{\alpha}}_{2m-1}, \hat{\boldsymbol{\alpha}}_{2m}, \hat{B}_1, \dots, \hat{B}_d]$ is such that

$$\hat{\boldsymbol{u}}_m(t) = P_m\hat{\boldsymbol{u}}(t) = (P_m\hat{\boldsymbol{x}}(t), \hat{B}_1, \dots, \hat{B}_d) = (\hat{\boldsymbol{x}}_m(t), \hat{B}_1, \dots, \hat{B}_d), \text{ and}$$

$$\hat{\boldsymbol{x}}_m(t) = P_m\hat{\boldsymbol{x}}(t) = \hat{\boldsymbol{\alpha}}_0 + \sqrt{2}\sum_{n=1}^{m}(\hat{\boldsymbol{\alpha}}_{2n-1}\cos nt + \hat{\boldsymbol{\alpha}}_{2n}\sin nt).$$

From (2.15), we have

$$\|\bar{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}_m\|_\infty \leq \sqrt{2m+1}\,\|\|\bar{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}\|\|' \leq \frac{\sqrt{2m+1}\,M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa}.$$

On the other hand, by Lemma 1-(i)

$$\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty \leq K\sigma(m).$$

Thus,

$$\|\bar{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty \leq \|\bar{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}_m\|_\infty + \|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty \leq \frac{\sqrt{2m+1}\,M'\{KK_1\sigma_1(m) + \sqrt{d}K_5\|\hat{\boldsymbol{u}}_m - \hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa} + K\sigma(m).$$

This proves (2.28).

Since $\bar{\boldsymbol{\alpha}}$ is a solution of $\boldsymbol{F}^{(m)}(\boldsymbol{\alpha})=\mathbf{0}$, for $\bar{\boldsymbol{u}}_m(t)$ we have

$$\left[\frac{d\bar{\boldsymbol{x}}_m}{dt}-P_m\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t),\ \boldsymbol{f}(\bar{\boldsymbol{u}}_m(t))\right]=\mathbf{0}$$

$$(\text{where } \boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)=\boldsymbol{X}(\bar{\boldsymbol{x}}_m(t),\,\bar{B}_1,\dots,\,\bar{B}_d,\,t)).$$

This can be rewritten as follows:

$$\left[\frac{d\bar{\boldsymbol{x}}_m}{dt}-\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t),\ \boldsymbol{f}(\bar{\boldsymbol{u}}_m(t))\right]=[\boldsymbol{\eta}_m(t),\,\mathbf{0}]\,, \tag{2.31}$$

where $\boldsymbol{\eta}_m(t)=-(I-P_m)\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)$, and I is the identity operator.
Since

$$\frac{d}{dt}\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)=\boldsymbol{X}_{\boldsymbol{x}}(\bar{\boldsymbol{u}}_m(t),\,t)\frac{d\bar{\boldsymbol{x}}_m}{dt}+\frac{\partial\boldsymbol{X}}{\partial t}(\bar{\boldsymbol{u}}_m(t),\,t)$$

$$=\boldsymbol{X}_{\boldsymbol{x}}(\bar{\boldsymbol{u}}_m(t),\,t)P_m\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)+\frac{\partial\boldsymbol{X}}{\partial t}(\bar{\boldsymbol{u}}_m(t),\,t)\,,$$

by Bessel's inequality we have

$$\left\|\frac{d}{dt}\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)\right\|_q\le K_1\|P_m\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)\|_q+K_2\le KK_1+K_2\,.$$

Then, by Proposition we have

$$\|\boldsymbol{\eta}_m(t)\|_c=\|(I-P_m)\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)\|_c\le\sigma(m)\left\|\frac{d}{dt}\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)\right\|_q\le(KK_1+K_2)\sigma(m)\,.$$

On the other hand, $\hat{\boldsymbol{u}}(t)=(\hat{\boldsymbol{x}}(t),\,\hat{B}_1,\dots,\,\hat{B}_d)$ satisfies

$$\left[\frac{d\hat{\boldsymbol{x}}}{dt}-\boldsymbol{X}(\hat{\boldsymbol{u}}(t),\,t),\ \boldsymbol{f}(\hat{\boldsymbol{u}})\right]=\mathbf{0}\qquad\text{where } \boldsymbol{X}(\hat{\boldsymbol{u}}(t),\,t)=\boldsymbol{X}(\hat{\boldsymbol{x}}(t),\,\hat{B}_1,\dots,\,\hat{B}_d,\,t). \tag{2.32}$$

Now, we have

$$\frac{d\bar{\boldsymbol{u}}_m}{dt}-\frac{d\hat{\boldsymbol{u}}}{dt}=\left(\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt},\,0,\,0,\dots,\,0\right).$$

By (2.31) and (2.32), we have

$$\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}=[\boldsymbol{X}(\bar{\boldsymbol{u}}_m(t),\,t)-\boldsymbol{X}(\hat{\boldsymbol{u}}(t),\,t)]+\boldsymbol{\eta}_m(t)$$

$$=\int_0^1\boldsymbol{X}_{\boldsymbol{u}}(\hat{\boldsymbol{u}}(t)+\theta(\bar{\boldsymbol{u}}_m(t)-\hat{\boldsymbol{u}}(t)))\begin{pmatrix}\bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\\ \bar{B}_1-\hat{B}_1\\ \vdots\\ \bar{B}_d-\hat{B}_d\end{pmatrix}d\theta+\boldsymbol{\eta}_m(t)\,,$$

and consequently

$$\left\|\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_n \leq \|\boldsymbol{X}_{\boldsymbol{u}}\|_{n+d,n}\left\|\begin{pmatrix} \bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t) \\ \bar{B}_1-\hat{B}_1 \\ \vdots \\ \bar{B}_d-\hat{B}_d \end{pmatrix}\right\|_{n+d}+\|\boldsymbol{\eta}_m(t)\|_n .$$

Since

$$\left\|\begin{pmatrix} \bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t) \\ \bar{B}_1-\hat{B}_1 \\ \vdots \\ \bar{B}_d-\hat{B}_d \end{pmatrix}\right\|_{n+d} \leq \|\bar{\boldsymbol{x}}_m(t)-\hat{\boldsymbol{x}}(t)\|_n+\sum_{i=1}^{d}|\bar{B}_i-\hat{B}_i| ,$$

then we have

$$\begin{aligned}\left\|\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_c &\leq K_1[\|\bar{\boldsymbol{x}}_m-\hat{\boldsymbol{x}}\|_c+\sum_{i=1}^{d}|\bar{B}_i-\hat{B}_i|]+\|\boldsymbol{\eta}_m(t)\|_c \\ &\leq K_1\|\bar{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty+(KK_1+K_2)\sigma(m) .\end{aligned}$$

Thus, by (2.28), then we have

$$\left\|\frac{d\bar{\boldsymbol{u}}_m}{dt}-\frac{d\hat{\boldsymbol{u}}}{dt}\right\|_\infty=\left\|\frac{d\bar{\boldsymbol{x}}_m}{dt}-\frac{d\hat{\boldsymbol{x}}}{dt}\right\|_c \leq K_1\|\bar{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty+(KK_1+K_2)\sigma(m)$$

$$\leq \frac{\sqrt{2m+1}M'K_1\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa}+KK_1\sigma(m)+(KK_1+K_2)\sigma(m)$$

$$=(K_2+2KK_1)\sigma(m)+\frac{\sqrt{2m+1}M'K_1\{KK_1\sigma_1(m)+\sqrt{d}K_5\|\hat{\boldsymbol{u}}_m-\hat{\boldsymbol{u}}\|_\infty\}}{1-\kappa} .$$

This proves (2.29). Q. E. D.

§ 3. Numerical Examples

In this section, we apply our results in §2 to a Duffing type differential equation appearing in the electrical engineering. So far, it is difficult to apply the usual Galerkin method (that is, the Galerkin method with no parameters) to the case of Example 2. But, our Galerkin method (that is, the Galerkin method with unknown parameters) can be applicable to such problems. And we get the high accurate computational result and the sharp error estimate.

Example 1. We consider the differential equation

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+x^3=B\cos t \qquad (k=0.2) . \tag{3.1}$$

We rewrite (3.1) in the following form;

$$(3.2)\qquad \begin{cases} \dfrac{dx}{dt} = y, \\[2ex] \dfrac{dy}{dt} = -x^3 - ky + B\cos t. \end{cases}$$

In this case, we take B and $y_0(=y(0))$ as unknown parameters $B_1,\dots, B_d$ $(d=2)$ in Theorem 2 in §2, and we adopt

$$\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} x(0)-x(2\pi) \\ y(0)-y(2\pi) \\ \boldsymbol{g}(\boldsymbol{u}) \end{pmatrix} = \begin{pmatrix} x(0)-x(2\pi) \\ y(0)-y(2\pi) \\ x(0)-x_0 \\ y(0)-y_0 \end{pmatrix}, \quad \left(\boldsymbol{u}=\boldsymbol{u}(t)=(\boldsymbol{x}(t), B, y_0),\ \boldsymbol{x}(t)=\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\right)$$

as the boundary and additional conditions, where x_0 is given. That is, put

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$

then the 2-dimensional vector-function $\boldsymbol{g}(\boldsymbol{u})$ is written in the form

$$\boldsymbol{g}(\boldsymbol{u}) = \begin{pmatrix} x(0)-x_0 \\ y(0)-y_0 \end{pmatrix} = L_1 \begin{pmatrix} \boldsymbol{x}(0) \\ B \\ y_0 \end{pmatrix} - \boldsymbol{\beta}.$$

We put

$$x_m(t) = \sum_{n=1}^{m} [a_{2n-1}\sin(2n-1)t + a_{2n}\cos(2n-1)t] \qquad (m=10).$$

Then the results of numerical computations are as follows;

$$x_0 = -0.626506 \text{ (given)},$$

$$\begin{cases} \bar{B}=0.45999\ 98475\ 38564,\ \bar{y}_0=0.20284\ 57065\ 35160,\ \det G=4.1, \\ r=0.1\times 10^{-13},\ \mu_1=6.0,\ \mu_2=3.0,\ \mu_3=4.0,\ \kappa=0.4\times 10^{-11}, \\ \delta_1=0.601\times 10^{-13},\ \delta_2=0.301\times 10^{-13},\ \delta_3=0.401\times 10^{-13}. \end{cases}$$

Thus, applying Theorem 2 in §2, we get the following error estimates;

$$\{(\hat{x}(t)-\bar{x}_m(t))^2 + (\hat{y}(t)-\bar{y}_m(t))^2\}^{\frac{1}{2}} \le \frac{\mu_1 r}{1-\kappa} \le \delta_1 = 0.601\times 10^{-13},$$

$$\begin{cases} |\hat{B}-\bar{B}| \leq \dfrac{\mu_2 r}{1-\kappa} \leq \delta_2 = 0.301 \times 10^{-13}, \\ |\hat{y}_0 - \bar{y}_0| \leq \dfrac{\mu_3 r}{1-\kappa} \leq \delta_3 = 0.401 \times 10^{-13}. \end{cases}$$

Example 2. In equation (3.1), we set $x = x(t, x_0)$, where x_0 is a initial value of $x(t)$. Setting $x_1(t) = x(t, x_0)$ and $x_3(t) = x_{x_0}(t, x_0)$, we have the following system:

$$(3.3) \qquad \begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = -x_1^3 - kx_2 + B\cos t, \\ \dfrac{dx_3}{dt} = x_4, \\ \dfrac{dx_4}{dt} = -3x_1^2 x_3 - kx_4 + B_{x_0}\cos t, \end{cases}$$

where $x_{x_0}(t, x_0)$ is the partial derivative of $x(t, x_0)$ with respect to x_0 and B_{x_0} is the derivative of $B(x_0)$ with respect to x_0.

In this example, we take (B, x_0, B_{x_0}) as unknown parameters $B_1, \dots, B_d$ $(d=3)$ in Theorem 2 in §2, and we take

$$g(u) = \begin{pmatrix} x_1(0) - x_0 \\ x_3(0) - 1.0 \\ B_{x_0} \end{pmatrix} = 0 \quad \left(u = u(t) = (x(t), B, x_0, B_{x_0}),\ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} \right)$$

as the additional conditions, namely, we set

$$(3.4) \qquad f(u) = \begin{pmatrix} x(0) - x(2\pi) \\ g(u) \end{pmatrix} = \begin{pmatrix} x(0) - x(2\pi) \\ x_1(0) - x_0 \\ x_3(0) - 1.0 \\ B_{x_0} \end{pmatrix}.$$

In the system (3.3), the case of $B_{x_0} = 0$ is the case that in equation (3.1), the stable periodic solution consists with the unstable periodic solution. Then, since the Jacobian matrix of the determining equation of the usual Galerkin approximation is singular, it can not apply to this problem. But, in the case of our Galerkin method, the Jacobian matrix of the determining equation is non-singular.

We put

$$\begin{cases} (x_1)_m(t)=\sum_{n=1}^{m}[a_{2n-1}\sin(2n-1)t+a_{2n}\cos(2n-1)t], \\ (x_3)_m(t)=\sum_{n=1}^{m}[c_{2n-1}\sin(2n-1)t+c_{2n}\cos(2n-1)t]. \end{cases} \qquad (m=8)$$

Then the results of numerical computations are as follows;
$\bar{B}=0.46139\ 11552\ 41294$, $\bar{x}_0=-0.65391\ 94080\ 07279$, $\bar{B}_{x_0}=-0.34\times 10^{-16}$.

$$\begin{cases} \det G=58.0,\ r=0.11\times 10^{-10},\ \mu_1=20.0,\ \mu_2=3.0,\ \mu_3=3.0,\ \mu_4=10.0,\ \kappa=0.4\times 10^{-6}, \\ \delta_1=0.221\times 10^{-9},\ \delta_2=0.331\times 10^{-10},\ \delta_3=0.331\times 10^{-10},\ \delta_4=0.111\times 10^{-9}. \end{cases}$$

Thus, applying Theorem 2 in §2, we get the following error estimates;

$$\begin{cases} \|\hat{\boldsymbol{x}}(t)-\bar{\boldsymbol{x}}_m(t)\|_c\le \dfrac{\mu_1 r}{1-\kappa}\le\delta_1=0.221\times 10^{-9}, \\ |\hat{B}-\bar{B}|\le\dfrac{\mu_2 r}{1-\kappa}\le\delta_2=0.331\times 10^{-10}, \\ |\hat{x}_0-\bar{x}_0|\le\dfrac{\mu_3 r}{1-\kappa}\le\delta_3=0.331\times 10^{-10}, \\ |\hat{B}_{x_0}-\bar{B}_{x_0}|\le\dfrac{\mu_4 r}{1-\kappa}\le\delta_4=0.111\times 10^{-9}. \end{cases}$$

In this example, we set $B_{x_0}=0$ in (3.3), then the problem (3.3)–(3.4) is equivalent to

$$(3.5)\qquad \begin{cases} \dfrac{dx_1}{dt}=x_2, \\ \dfrac{dx_2}{dt}=-x_1^3-kx_2+B\cos t, \\ \dfrac{dx_3}{dt}=x_4, \\ \dfrac{dx_4}{dt}=-3x_1^2x_3-kx_4 \end{cases}$$

and

$$(3.6)\qquad \boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix} \boldsymbol{x}(0)-\boldsymbol{x}(2\pi) \\ \boldsymbol{g}(\boldsymbol{u}) \end{pmatrix}=\begin{pmatrix} \boldsymbol{x}(0)-\boldsymbol{x}(2\pi) \\ x_1(0)-x_0 \\ x_3(0)-1.0 \end{pmatrix}\quad \left(\boldsymbol{g}(\boldsymbol{u})=\begin{pmatrix} x_1(0)-x_0 \\ x_3(0)-1.0 \end{pmatrix}\right).$$

In this case, B and x_0 are the unknown parameters, then we set

$$\boldsymbol{u}=\boldsymbol{u}(t)=(\boldsymbol{x}(t),\ B,\ x_0).$$

We can apply our Galerkin method to (3.5)–(3.6), and we get the same results in the case of (3.3)–(3.4).

Furthermore, we can take B as the only one unknown parameter and we can take

$$g(\boldsymbol{u})=x_3(0)-1.0=0 \qquad (\boldsymbol{u}=\boldsymbol{u}(t)=(\boldsymbol{x}(t),\ B))$$

as the additional condition. In this case, of course, we can get the same results of the above two cases.

Acknowledgement

The author expresses his hearty gratitude to Professor Y. Shinohara for his constant advice and encouragement.

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