J. Math. Tokushima Univ.
Vol. 17 (1983), 1–9

On Holonomy Mappings Associated with a Non-linear Connection

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(Received May 13, 1983)

Let us consider a manifold provided with a non-linear connection. In the preceding paper [6] we have defined a notion of a holonomy mapping and found a gemetrical significance for the condition that the hv-curvature tensor of a Finsler connection vanishes.

In the present paper, we investigate the holonomy mappings in detail and consider, in connection with the holonomy mappings, the character of Landsberg spaces, Berwald spaces, generalized Berwald spaces and almost Hermitian Finsler spaces.

§**1.** In this paper, we shall treat an n-dimensional C^∞-manifold M and its tangent bundle $T(M)$. In the sequel, we assume that M is connected and provided with a non-linear connection $N(x, y)$. Let $N^i_j(x, y)$ be the components of $N(x, y)$ with respect to a canonical coordinate (x^i, y^i) and put

(1.1) $$\dot{\partial}_k N^i_j = N^i_{kj},^{1)}$$

then $N^i_{kj}(x, y)$ satisfy the law of transformation of a linear connection on a manifold.

Let S be a Finsler tensor field on M, for instance, of type (1, 2), then the covariant derivative $\overset{n}{\nabla}$ with respect to the non-linear connection $N(x, y)$ can be written as

(1.2) $$\overset{n}{\nabla}_k S^h_{ij} = \partial_k S^h_{ij} - \dot{\partial}_m S^h_{ij} N^m_k + N^h_{mk} S^m_{ij} - N^m_{ik} S^h_{mj} - N^m_{jk} S^h_{im}.$$

For the interpretation of a Finsler tensor, there exist many papers, for examples, Cartan [1], Matsumoto [9] and Stavroulakis [11]. We consider, in this paper, fibre tensors defined by the following:

For any point $p=(x)$ in M, the fibre $T_p(M)$ over the point p is a submanifold of $T(M)$. So $T_p(M)$ can be regarded as an n-dimensional manifold. Now, for any

1) The indices $A, B, C, ..$ run over the range $1, 2, \ldots, 2n$ and $i, j, k \ldots$ run over the range $1, 2, \ldots, n$, and the indeces $\bar{i}, \bar{j}, \bar{k} \ldots$ stand for $i+n, j+n, k+n, \ldots$ respectively and ∂_k stands for $\partial/\partial x^k$ and $\dot{\partial}_k$ stands for $\partial/\partial y^k$.

fixed point $p \in M$, we consider a Finsler tensor $S(x, y)$ on the fibre $T_p(M)$ only. That is to say, we consider $S(p, y)$ as the quantity defined on the fibre $T_p(M)$. Since the law of transformation of the canonical coordinate (y^i) in $T_p(M)$ satisfies $\dot{\partial}_j \bar{y}^i = \partial_j \bar{x}^i$, the quantity $S(p, y)$ can be regarded as a tensor field on $T_p(M)$. So, we denote by $S^*(p, y)$ the tensor field on $T_p(M)$, and call it a fibre tensor on $T_p(M)$. However, this fibre tensor $S^*(p, y)$ is neither the so-called vertical lift of $S(x, y)$ [12] nor a tensor field on $T(M)$. This is only a tensor field on the fibre $T_p(M)$. With respect to a canonical coordinate $(Z^A)=(x^i, y^i)$ of $T(M)$, the vertical lift $(S)^v = K(x, y)$ of a tensor field $S(x, y)$ of (1, 2)-type has the following components; $K^{\bar{i}}_{jk} = S^i_{jk}(x, y)$, otherwise vanish.

If M is a Finsler manifold, the fibre tensor $g^*(p, y)$ of the Finsler metric tensor $g(x, y)$ gives $T_p(M)$ a Riemann metric, that is, $g^*(p, y)$ is the so-called tangent Riemann metric of the Finsler manifold.

§**2.** The notion of the holonomy mapping has been defined in the paper [6] as follows;

Let p and q be arbitrary two points in M, and c be any piecewise differentiable curve joining the points p and q. Let $\tilde{c}$ be the horizontal lift of the curve c. If we express c by $c = \{x(t)\}$, $\tilde{c}$ is expressed by $\tilde{c} = \{x(t), y(t)\}$ where $y(t)$ satisfies

$$\frac{dy^i}{dt} + N^i_m(x(t), y(t)) \frac{dx^m}{dt} = 0. \tag{2.1}$$

For brevity, we put $p = x(0)$ and $q = x(1)$. For each point $(x(0), y)$ in $T_p(M)$, there exists one and only one horizontal lift $\tilde{c}$ of c passing through the point $(x(0), y)$. And $\tilde{c}$ passes $T_q(M)$ by one point, which we denote by $(x(1), \bar{y})$. Then the correspondence $y \to \bar{y}$ defines a bijective differentiable mapping ψ: $T_p(M) \to T_q(M)$. This mapping ψ is called a holonomy mapping from $T_p(M)$ to $T_q(M)$ along the curve c with respect to the non-linear connection $N(x, y)$.

Let $C^i_{jk}(x, y)$ be a (1, 2) Finsler tensor field which is positively homogeneous of degree -1 with respect to y. The fibre tensor $C^*(p, y)$ of C is a (1, 2) tensor field on $T_p(M)$ and, at the same time, $C^*(p, y)$ is a linear connection on $T_p(M)$, because the transformation of the canonical coordinates of $T_p(M)$ is linear. Thus $\{T_p(M), C^*(p, y)\}$ can be regarded as an affinely connected manifold. With respect to the *hv*-curvature tensor

$$P^i_{jhk} = \dot{\partial}_k N^i_{jh} - \overset{n}{\nabla}_h C^i_{jk}, \tag{2.2}$$

we have shown in the paper [6] that

Theorem 1. *Suppose that M is a connected manifold provided with a non-linear connection $N(x, y)$ and a (1, 2)-tensor field $C(x, y)$ of Finsler type, where $C(x, y)$ is positively homogeneous of degree -1 with respect to y. Let p and q*

be arbitrary two points in M and let c be any piecewise differentiable curve joining p and q. In order that the holonomy mapping from $T_p(M)$ to $T_q(M)$ along c with respect to the non-linear connection $N(x, y)$ is always an affine mapping from the affinely connected space $\{T_p(M), C^*(p, y)\}$ to the affinely connected space $\{T_q(M), C^*(q, y)\}$, it is necessary and sufficient that the hv-curvature tensor $P^i_{jhk} := \dot\partial_k \dot\partial_j N^i_h - \overset{n}{\nabla}_h C^i_{jk}$ vanishes.

Theorem 2. *In order that a connected Finsler manifold $\{M, g\}$ be a Landsberg space, it is necessary and sufficient that, for arbitrary two points p and q, any holonomy mapping from $T_p(M)$ to $T_q(M)$ with respect to the Cartan's (Barwald's) non-linear connection $N^i_j = \overset{*}{\Gamma}{}^i_{mj} y^m = G^i_j$ is always an affine mapping from the Riemann space $\{T_p(M), g^*(p, y)\}$ to the Riemann space $\{T_q(M), g^*(q, y)\}$.*

§**3.** We shall now prove

Theorem 3. *Let $S(x, y)$ be a tensor field of Finsler type on a connected manifold M provided with a non-linear connection $N(x, y)$. Let p and q be arbitrary two points in M. In order that any holonomy mapping from $T_p(M)$ to $T_q(M)$ with respect to the non-linear connection $N(x, y)$ always transfers the fibre tensor field $S^*(p, y)$ on $T_p(M)$ to the fibre tensor field $S^*(q, y)$ on $T_q(M)$, it is necessary and sufficient that $\overset{n}{\nabla} S = 0$ holds good.*

PROOF. We represent the curve c by $c = \{c_t \mid c_t = x(t),\ c_0 = x(0) = p,\ c_1 = x(1) = q\}$ and denote by ψ the holonomy mapping along c and denote by Ψ the induced map of ψ and put $\psi(y) = \bar y$. Then the condition under consideration is written as $\Psi(S^*(q, \bar y)) = S^*(p, y)$. On the other hand, we denote by ψ_t the holonomy mapping from $T_p(M)$ to $T_{x(t)}(M)$ along c and put $\Psi_t(S^*(x(t), \psi_t(y)) = \tilde S_t(x(0), y)$. Now we shall calculate the components of

$$\left[\frac{d\tilde S_t(x(0), y)}{dt}\right]_{t=0} = \lim_{t\to 0} \frac{\tilde S_t(x(0), y) - S(x(0), y)}{t}.$$

Without loss of generality, we may assume S is of (1, 2)-type. Then putting $S^*(p, y) = \{S^i_{jk}(x(0), y)\}$, $\tilde S_t(x(0), y) = \{\tilde S_{(t)}{}^i_{jk}(x(0), y)\}$ and $\psi_t(y) = \bar y_t = (\bar y_{(t)}^i)$, we have $S^*(x(t), \psi_t(y)) = \{S^i_{jk}(x(t), \bar y_t)\}$ and $\tilde S_{(t)}{}^i_{jk} = S^a_{bc}(x(t), \bar y_t) \cdot \frac{\partial y^i}{\partial \bar y_{(t)}^a} \cdot \frac{\partial \bar y_{(t)}^b}{\partial y^j} \cdot \frac{\partial \bar y_{(t)}^c}{\partial y^k}$. If we put $\bar y_{(t)}^a = f^a(t, y^1, y^2, \dots, y^n)$ then f^a is a solution of the differential equation $\frac{df^a}{dt} + N^a_m(x(t), \bar y(t)) \frac{dx^m}{dt} = 0$. Hence $\frac{\partial \bar y_{(t)}^a}{\partial t} = -N^a_m(x(t), \bar y_{(t)}^a) \frac{dx^m}{dt}$ and $\left(\frac{\partial \bar y_{(t)}^a}{\partial y^i}\right)_{t=0} = \delta^a_i$ hold. If we put $\left(\frac{dx^i}{dt}\right)_{t=0} = v^i$, we have $\left(\frac{\partial \bar y_{(t)}^a}{\partial t}\right)_{t=0} = -N^a_m(x(0), y) v^m$. And we have

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \bar y_{(t)}^a}{\partial y^i}\right)\right]_{t=0} = \left[-\frac{\partial}{\partial y^i}\{N^a_m(x(t), \bar y_{(t)}^a)\}\frac{dx^m}{dt}\right]_{t=0} = -\dot\partial_j N^a_m(x(0), y) v^m,$$

We have also $\left[\frac{\partial}{\partial t}\left(\frac{\partial y^i}{\partial \bar{y}^a_{(t)}}\right)\right]_{t=0}=\dot{\partial}_a N^i_m(x(0), y)v^m$. Now we see $\left[\frac{d\tilde{S}_{(t)}{}^i_{jk}(x(0), y)}{dt}\right]_{t=0}$ $=\partial_m S^i_{jk}(x(0), y)v^m+\dot{\partial}_h S^i_{jk}(x(0), y)(-N^h_m(x(0), y)v^m)+S^a_{jk}(x(0), y)\dot{\partial}_a N^i_m(x(0), y)v^m+S^i_{ak}(x(0), y)(-\dot{\partial}_j N^a_m(x(0), y)v^m)+S^i_{ja}(x(0), y)(-\dot{\partial}_k N^a_m(x(0), y)v^m)=v^m \overset{n}{\nabla}_m S^i_{jk}(x(0), y)$. Thus we obtain that the condition $\overset{n}{\nabla}_h S^i_{jk}=0$ is necessary.

Conversely, the above calculation shows us that $\overset{n}{\nabla}_h S^i_{jk}=0$ implies $\left[\frac{d}{dt}\tilde{S}_{(t)}{}^i_{jk}\right]_{t=0}=0$, i.e., $\left[\frac{d}{dt}\Psi_t(S^*(x(t), \psi_t(y)))\right]_{t=0}=0$. Now, on the curve $c=\{x(t)\,|\,x(0)=p\}$, we take arbitrary two points $r=x(s)$ and $r'=x(s+u)$. We denote by φ_u the holonomy mapping from $T_r(M)$ to $T_{r'}(M)$ along c, and by $\Phi_u(S^*_{(s+u)})$ the induced fibre tensor of $S^*_{(s+u)}$ with respect to φ_u. Then it follows from our assumption that $\left[\frac{\partial}{\partial u}\Phi_u(S^*_{(s+u)})\right]_{t=0}=0$ holds. Since $\psi_{s+u}=\varphi_u\circ\psi_s$, it follows also that $\Psi_{s+u}=\Psi_s\circ\Phi_u$. Hence we have

$$0=\left[\frac{\partial}{\partial u}\Phi_u(S^*_{(s+u)})\right]_{u=0}=\left[\frac{\partial}{\partial u}\{\Psi_s^{-1}\circ\Psi_{s+u}(S^*_{(s+u)})\}\right]_{u=0}$$

$$=\lim_{u\to 0}\frac{\Psi_s^{-1}\circ\Psi_{s+u}(S^*_{(s+u)})-\Psi_s^{-1}\circ\Psi_s(S^*_{(s)})}{u}$$

$$=\Psi_s^{-1}\left\{\lim_{u\to 0}\frac{\Psi_{(s+u)}(S^*_{(s+u)})-\Psi_s(S^*_{(s)})}{u}\right\}$$

Since Ψ_s is isomorphic, it follows that

$$\lim_{u\to 0}\frac{\Psi_{(s+u)}(S^*_{(s+u)})-\Psi_s(S^*_{(s)})}{u}=0$$

which implies $\frac{\partial}{\partial s}\Psi_s(S^*_{(s)})=0$. Hence we have $\Psi_s(S^*_{(s)})=\Psi_0(S^*_{(0)})=S^*_{(0)}$. Consequently we have that ψ_s leaves $S^*(x(0), y)$ invariant. Q.E.D.

§**4.** Let M be a Finsler manifold whose Finsler metric function is $F(x, y)$. Let $g(x, y)$ be the Finsler metric tensor, then the fibre tensor $g^*(p, y)$ is a Riemann metric tensor on $T_p(M)$. The condition $\overset{n}{\nabla}_k g_{ij}=0$ implies, from Theorem 3, that, for arbitrary two points p and q in M, the holonomy mapping with respect to the non-linear connection $N(x, y)$ is an isometry from the Riemann space $\{T_p(M), g^*(p, y)\}$ to the Riemann space $\{T_q(M), g^*(q, y)\}$.

Here we adopt the Cartan's non-linear connection as the non-linear connection $N(x, y)$. That is to say, $N^i_j=\overset{*}{\Gamma}{}^i_{mj}y^m=G^i_j$, then $N^i_{kj}=\dot{\partial}_k N^i_j=G^i_{kj}$. And the covariant derivative $\overset{n}{\nabla}_k$ coincides with that of Berwald's $\|k$. On the other hand, it is well-known that $g_{ij\|k}=0$ is the condition for a Finsler manifold to be a Landsberg space. Thus we obtain

Theorem 4. *Suppose that M is a connected Finsler manifold. A necessary and sufficient condition for M to be a Landsberg space is that, for arbitrary two points p and q in M, any holonomy mapping from $T_p(M)$ to $T_q(M)$ with respect to the Cartan's non-linear connection G^i_j is always an isometry to the Riemann space $\{T_p(M), g^*(p, y)\}$ to the Riemann space $\{T_q(M), g^*(q, y)\}$.*

Now, for the tensor $C^i_{jk}=\frac{1}{2}g^{im}\dot{\partial}_m g_{jk}$, we see

$$C^i_{jk\|h}=\partial_h C^i_{jk}-\dot{\partial}_m C^i_{jk}G^m_h+G^i_{rh}C^r_{jk}-C^i_{rk}G^r_{jh}-C^i_{jr}G^r_{kh}$$

$$=C^i_{jk|h}+P^i_{rh}C^r_{jk}-C^i_{rk}P^r_{jh}-C^i_{jr}P^r_{kh}$$

where $P^i_{jk}=G^i_{jk}-\overset{*}{\Gamma}{}^i_{jk}$. It is well-known that $P^i_{jr}y^r=0$ and $C^i_{jk|r}y^r=P^i_{jk}$ where $|r$ implies the covariant derivative defined by Cartan. Hence $C^i_{jk\|h}=0$ implies that $C^i_{jk|h}=0$, that is, the space is a Berwald space. Conversely, in a Berwald space, the relation $\overset{*}{\Gamma}{}^i_{jk}=G^i_{jk}$ holds. Therefore $C^i_{jk\|h}=0$ holds good. Consequently we obtain

Theorem 5. *Suppose that M is a connected Finsler manifold provided with the Cartan's non-linear connection G^i_j. Let $C(x, y)$ be the tensor field given by $C^i_{jk}=\frac{1}{2}g^{im}\dot{\partial}_m g_{jk}$. A necessary and sufficient condition for M to be a Berwald space is that any holonomy mapping with respect to G^i_j always leaves the fibre tensor $C^*(x, y)$ invariant.*

§5. In this section we assume that M is a Finsler manifold admitting a linear connection $\Gamma(x)$ and the non-linear connection $N(x, y)$ is given by $N^i_j=\Gamma^i_{mj}(x)y^m$. Then $\overset{n}{\nabla}_k$ is called the h-covariant derivative associated with the linear connection $\Gamma(x)$. We denote it by $\overset{l}{\nabla}_k$. For any (1, 1)-tensor T, $\overset{l}{\nabla}_k$ is written as

$$\overset{l}{\nabla}_k T^i_j=\partial_k T^i_j-\dot{\partial}_m T^i_j \Gamma^m_{r\,k}y^r+T^r_j\Gamma^i_{rk}-T^i_r\Gamma^r_{jk}$$

On the other hand, we know that $\overset{l}{\nabla}_h g_{ij}=0$ is the condition for M to be a generalized Berwald space with respect to the linear connection $\Gamma(x)$ ([2], [4], [5]). Hence we obtain

Theorem 6. *Suppose that M is a connected Finsler manifold provided with a linear connection $\Gamma(x)$. A necessary and sufficient condition for M to be a generalized Berwald space with respect to $\Gamma(x)$ is that, for arbitrary two points p and q in M, any holonomy mapping from $T_p(M)$ to $T_q(M)$ with respect to the non-linear connection $\Gamma^i_{mj}(x)y^m$ is always an isometry from the Riemann space $\{T_p(M), g^*(p, y)\}$ to the Riemann space $\{T_q(M), g^*(q, y)\}$.*

If we pay attention to the theorem shown in the paper [7], we can prove easily

Theorem 7. *Suppose that M is a connected Finsler manifold provided with a linear connection $\Gamma(x)$ and admits such a point p that the Riemann space $\{T_p(M), g^*(p, y)\}$ is irreducible. A necessary and sufficient condition for M to be a generalized Berwald space is that any holonomy mapping with respect to $\Gamma^i_{mj}(x)y^m$ always leaves the fibre tensor $C^*(x, y)$ invariant.*

§**6.** In this section we assume that the dimension of the manifold M is $2n$ and M admits an almost complex structure $f(x)$. In this case, the fibre tensor $f^*(p)$ in $T_p(M)$ is independent to y, and satisfies $f^i_m(p)f^m_j(p)=-\delta^i_j$. So, $f^*(p)$ is a complex structure on $T_p(M)$. Now we show

Theorem 8. *Let M be a $2n$-dimensional connected manifold and admit an almost complex structure $f(x)$ and a non-linear connection $N(x, y)$. For arbitrary two points p and q in M, any holonomy mapping from the complex space $\{T_p(M), f^*(p)\}$ to the complex space $\{T_q(M), f^*(q)\}$ is a holomorphic mapping, if and only if $\overset{n}{\nabla} f=0$ holds.*

PROOF. The condition for the holonomy mapping ψ to be holomorphic is given by $(d\psi)_{(p,y)}\circ f^*(p, y)=f^*(q, \psi(y))\circ d\psi_{(p,y)}$. This can be reduced to $f^*(p)=\Psi(f^*(q))$. Hence, due to Theorem 3, we obtain that $\overset{n}{\nabla} f=0$ is the necessary and sufficient condition. Q. E. D.

Now, we assume moreover that M admits a Finsler metric $F(x, y)$. Here we put

$$\varphi_{\theta}{}^i_j=\cos\theta\delta^i_j+\sin\theta f^i_j. \tag{6.1}$$

If the relation

$$F(x, \varphi_\theta y)=F(x, y) \tag{6.2}$$

holds for any θ, M is called an almost Hermitian Finsler manifold. With respect to this manifold, we have shown in the paper [8] that

Theorem 9. *The tangent space at any point of an almost Hermitian Finsler manifold is an n-dimensional complex Banach space.*

Now we show

Theorem 10. *In an almost Hermitian Finsler manifold M, let us put*

$$\tilde{g}_{ij}(x, y)=\frac{1}{2}(g_{ij}(x, y)+g_{lm}(x, y)f^l_i(x)f^m_j(x)). \tag{6.3}$$

Then the tangent space $T_p(M)$ at any point $p\in M$ admits a Kähler structure $\{\tilde{g}^*(p, y), f^*(p)\}$.

PROOF. The fibre tensor $f^*(p)$ is a complex structure on $T_p(M)$ and satisfies $\tilde{g}_{lm} f^l_i f^m_j = \tilde{g}_{ij}$. So, $\{\tilde{g}^*(p, y), f^*(p)\}$ gives $T_p(M)$ an Hermitian structure. Moreover it follows that

$$\dot{\partial}_k(\tilde{g}_{im} f^m_j) + \dot{\partial}_i(\tilde{g}_{jm} f^m_k) + \dot{\partial}_j(\tilde{g}_{km} f^m_i)$$

$$= \frac{1}{2}\{\dot{\partial}_k(g_{im} f^m_j - g_{mj} f^m_i) + \dot{\partial}_i(g_{jm} f^m_k - g_{mk} f^m_j) + \dot{\partial}_j(g_{km} f^m_i - g_{mi} f^m_k)\}$$

$$= 0$$

Hence, $\{\tilde{g}^*(p, y), f^*(p)\}$ is a Kähler structure on $T_p(M)$. Q. E. D.

Putting together the above theorems, we obtain

Theorem 11. *Suppose that the manifold M admits a non-linear connection $N(x, y)$ and an almost Hermitian Finsler metric $\{g(x, y), f(x)\}$. In order that any holonomy mapping for arbitrary two points p and q always transfers the Kähler structure $\{\tilde{g}^*(p, y), f^*(p)\}$ on $T_p(M)$ to the Kähler structure $\{\tilde{g}^*(q, y), f^*(q)\}$ on $T_q(M)$, it is necessary and sufficient that*

$$(\delta^p_i \delta^q_j + f^p_i f^q_j) \overset{n}{\nabla}_k g_{pq} = 0 \quad and \quad \overset{n}{\nabla}_k f^i_j = 0$$

hold good.

We show lastly

Theorem 12. *Suppose that the manifold M admits an almost Hermitian Finsler metric $\{g(x, y), f(x)\}$ and the non-linear connection $N(x, y)$ satisfing the conditions $N^i_j(x, \varphi_\theta y) = \dot{\partial}_m N^i_j(x, y) \varphi_{\theta r}^{\ m} y^r$ and $\overset{n}{\nabla}_k f^i_j = 0$. Then, for arbitrary two points p and q in M, any holonomy mapping from $T_p(M)$ to $T_q(M)$ with respect to the non-linear connection $N(x, y)$ carries every complex line passing through the origine of $T_p(M)$ to a complex line passing through the origine of $T_q(M)$.*

PROOF. Since the non-linear connection satisfies

$$N(x, ky) = kN(x, y) \quad (\forall k > 0),$$

the holonomy mapping ψ always carries every half real line starting from the origine to a half real line starting from the origine. That is,

$$\psi(ky) = k\psi(y) \quad (\forall k > 0). \tag{6.4}$$

Now, we see

$$\frac{d}{dt}(\varphi^{\ i}_{\theta m} y^m) + N^i_m(x, \varphi_\theta y) \frac{dx^m}{dt}$$

$$= \cos\theta \frac{dy^i}{dt} + \sin\theta \frac{df^i_j}{dt} y^j + \sin\theta f^i_j \frac{dy^j}{dt} + \dot{\partial}_l N^i_m(x, y) \varphi^{\ l}_{\theta r} y^r \frac{dx^m}{dt}$$

$$=\varphi^i_{\theta m}\frac{dy^m}{dt}+\sin\theta\{-\dot{\partial}_m N^i_k(x, y)f^m_j+\dot{\partial}_j N^m_k(x, y)f^i_m\}\frac{dx^k}{dt}y^j$$

$$+\dot{\partial}_l N^i_m(x, y)\varphi^l_{\theta r}y^r\frac{dx^m}{dt}$$

$$=-\varphi^i_{\theta m}N^m_r(x, y)\frac{dx^r}{dt}-\sin\theta\dot{\partial}_m N^i_r(x, y)f^m_j y^j\frac{dx^r}{dt}+\sin\theta f^i_m N^m_r(x, y)\frac{dx^r}{dt}$$

$$+\dot{\partial}_1 N^i_r(x, y)\varphi^1_{\theta m}y^m\frac{dx^r}{dt}$$

$$=\frac{dx^r}{dt}\{-\cos\theta N^i_r(x, y)-\sin\theta f^i_m N^m_r(x, y)-\sin\theta\dot{\partial}_m N^i_r(x, y)f^m_j y^j$$

$$+\sin\theta f^i_m N^m_r(x, y)+\cos\theta\dot{\partial}_l N^i_r(x, y)y^l+\sin\theta\dot{\partial}_l N^i_r(x, y)f^l_m y^m\}$$

$$=0.$$

Thus we have

$$\psi((\varphi_\theta y)_p)=(\varphi_\theta(\psi(y)))_q. \tag{6.5}$$

Because of the fact $\varphi^i_{\pi j}=-\delta^i_j$, it follows that

$$\psi(-y)=-\psi(y). \tag{6.6}$$

Due to (6.4) and (6.6), we have $\psi(ky)=k\psi(y)\,(^\forall k)$. Moreover we have (6.5), i.e., $\psi((\varphi_\theta y)_p)=(\varphi_\theta(\psi(y)))_q$. Hence ψ carries every complex line passing through the origine of $T_p(M)$ to a complex line passing through the origine of $T_q(M)$. Q. E. D.

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