The Gauss Map of Kaehler Immersions into Complex Hyperbolic Spaces

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§ 1. Introduction

In the previous paper [2], we studied the Gauss maps of Kaehler immersions in complex projective spaces. We will show in this paper that the similar argument is valid for Kaehler immersions in complex hyperbolic spaces.

It will be agreed that our indices have the following ranges throughout this paper; \(0 \leq A, B, C \leq N, \ 1 \leq a, b, c \leq N, \ 1 \leq i, j, k \leq n, \ n + 1 \leq r, s, t \leq N, \ 0 \leq i^*, j^*, k^* \leq n, n + 1 \leq r^*, s^*, t^* \leq N\) or \(r^*, s^*, t^* = 0\).

Let \(\mathbb{C}^{N+1}\) be a complex \((N+1)\)-space with the indefinite metric \(\langle z, w \rangle = -z^0 \overline{w}^0 + \sum z^a \overline{w}^a\). Put

\[
\mathbb{C}_0^{N+1} = \{z \in \mathbb{C}^{N+1}, \langle z, z \rangle < 0 \}.
\]

Two points \(z_1, z_2 \in \mathbb{C}_0^{N+1}\) are equivalent, provided that there is a non-zero complex number \(c\) such that \(z_1 = cz_2\). The quotient of \(\mathbb{C}_0^{N+1}\) by this equivalent relation is just a complex hyperbolic \(N\)-space \(H^N(C)\). Let \(U(1, N)\) be the group of all linear isometries of \(\mathbb{C}^{N+1}\). Obviously \(H^N(C)\) is the homogeneous space \(U(1, N)/U(1) \times U(N)\). Let \(\pi_c: \mathbb{C}_0^{N+1} \to H^N(C)\) be the natural projection. For \(z \in \mathbb{C}_0^{N+1}\), put \(\pi_c(z) = y\). Then \(\ker (d\pi_c)_z = \{z\}\), where \(\{z\}\) is the complex line in \(\mathbb{C}_0^{N+1}\) determined by \(z\).

Let \(\{z\}^\perp\) be the orthonormal complement of \(\{z\}\) in \(\mathbb{C}_0^{N+1}\). Then it is an \(N\)-subspace with the induced definite Hermitian metric. The restriction of \((d\pi_c)_z\) to \(\{z\}^\perp\) gives an isomorphism of \(\{z\}^\perp\) onto the holomorphic tangent bundle \(T^*_y H^N(C)\) of \(H^N(C)\) at \(y\). Let \(\lambda_z\) be the inverse of this isomorphism. Then for \(v \in T^*_y H^N(C)\), \(\lambda_z(v) = c \cdot \lambda_z(v)\). Set \(\mu_z(z) = \lambda_z(v)\). Thus we have an isomorphism: \(v \to \mu_z\) of \(T^*_y H^N(C)\) onto \(\{z\}^\perp \times \{z\}^\perp\), where \(\{z\}^\perp\) is the dual space of \(\{z\}\). Using this isomorphism, we can regard a linear subspace of \(T^*_y H^N(C)\) as a linear subspace of \(\mathbb{C}_0^{N+1}\).

Let \(f: M \to H^N(C)\) be a holomorphic immersion of a Kaehler \(n\)-space \(M\) to \(H^N(C)\). Since \(df(T^*_x M), x \in M,\) is viewed as a complex \(n\)-subspace of \(\mathbb{C}_0^{N+1}\), we can define the Gauss map \(g\) of \(M\) to a flag manifold \(F_{n,n+1} = U(1, N)/U(1) \times U(n) \times U(N-n)\) by \(g(x) = (f(x)), df(T^*_x M)\). Let \(\pi_1\) (resp. \(\pi_2\)) be the projection of \(F_{n,n+1}\) to \(F_n = U(1, N)/U(1) \times U(1, N-n)\) (resp. \(F_{n+1} = U(1, N)/U(1, n) \times U(N-n)\)). Put
$g_1 = \pi_i g$ ($i = 1, 2$), these will be also called Gauss maps. The map $g_2$ is the same as the Gauss map in the sense of Nishikawa [3]. He showed that his Gauss map is anti-holomorphic (-holomorphic). We show that if an immersion is not totally geodesic, the corresponding maps $g$ and $g_1$ are not $\pm$ holomorphic. But they are always harmonic.

§ 2. Kaehler immersions

In the sequel, we denote $H^N(C)$ by $H^N$ simply. Let $FH^N$ be the bundle of unitary frames of $H^N$. Let $F^*H^N$ be the set of elements $b = (y, e_0, e_1, \ldots, e_n)$ such that $e_0 \in C_0^{n+1}$; $\|e_0\| = -1$, $d\pi_i(e_0) = 0$ and $(y, e_1, \ldots, e_n) \in FH^N$. Then $F^*H^N$ is a principal bundle over $H^N$ with group $U(1) \times U(N)$. Since $T_y H^N \cong \pi_i^{-1}(y) \times \{\pi_i^{-1}(y)\}^\perp$, each $e_a$ is regarded as a linear map $e_a: \pi_i^{-1}(y) \rightarrow \{\pi_i^{-1}(y)\}^\perp$. By identifying moreover $e_a$ with $e_i(e_0)$, we view $F^*H^N$ as a subspace of $H^N \times U(1, N)$.

Let $\phi^b_a$ be the Maurer-Cartan forms of the group $U(1, N)$. Then they satisfy the following relations:

$$\phi^b_a - \phi^0_a = 0, \quad \phi^b_a + \phi^b_a = 0$$

and

$$d\phi^b_a = \sum \phi^c_b \wedge \phi^a_c.$$ 

Let $\rho: H^N \times U(1, N) \rightarrow U(1, N)$ be the natural projection. Denote by $\theta^a$ (resp. $\theta^b_a$) the restriction of $\rho^*\phi^b_a$ (resp. $\rho^*\phi^a_b - \delta^a_b \rho^*\phi^0_b$) to $F^*H^N$. Then the Kaehler metric on $H^N$ is given locally by $d\sigma^2 = 2 \sum \theta^a \theta^a$ and the structure equations in $H^N$ are written as

$$d\theta^a = \sum \theta^b \wedge \theta^a$$

$$d\theta^a_b = \sum \theta^c_b \wedge \theta^a_c - \theta^a_b \wedge \theta^c_b = \sum \theta^b \wedge \theta^c.$$ 

Let $M$ be a Kaehler $n$-manifold isometrically immersed into $H^N$ by a holomorphic mapping $f : M \rightarrow H^N$. Let $FM$ be the bundle of unitary frames on $M$. Let denote by $F^*M$ the set of elements $b = (x, e_0, \ldots, e_n) \in FM$ and $(f(x), e_0, df(e_1), \ldots, df(e_n), e_{n+1}, \ldots, e_n) \in F^*H^N$. Then $F^*M$ is a principal bundle over $M$ with group $U(1) \times U(n) \times U(N - n)$. The natural immersion $\tilde{f}: F^*M \rightarrow F^*H^N$ is defined by $\tilde{f}(b) = (f(x), e_0, df(e_1), \ldots, df(e_n), e_{n+1}, \ldots, e_n) \in F^*H^N$. On putting $\omega^a = f^*\omega^a$ and $\omega^b_a = f^*\omega^b_a$, we have $\omega^a = 0$ and we can put $\omega^k_\ell = \sum A^k_{\ell j} \omega^j$, where $A^k_{\ell j}$ are regarded as the components of the second fundamental form of $f: M \rightarrow H^N$. The Kaehler metric on $M$ is given locally by $ds^2 = 2 \sum \theta^a \theta^a$. From (1), we obtain the following structure equations in $M$

$$d\omega^i = \sum \omega^k \wedge \omega^k_i$$

$$d\omega^j_i = \sum \omega^j_\ell \wedge \omega^k_\ell - \sum \delta^i_{\ell k} \delta^j_\ell + \delta^i_{\ell j} \delta^k_\ell + \sum A^k_{j \ell} \omega^k \wedge \omega^i.$$
Let $-2i \sum R_{ij} \omega^i \wedge \bar{\omega}^k$ be the Ricci form of $M$ (in this paper, $i, j, k$ run from $\bar{1}$ to $\bar{n}$). Then it holds

$$\sum A^i_j A^k_i = -(n+1) \delta_{jk} - R_{jk}. \tag{2}$$

§ 3. The Gauss map

Let $\tilde{g}$ be the composition of the natural immersion $f : F^* M \to F^* H^n \subset H^n \times U(1, N)$ and the natural projection $\rho : H^n \times U(1, N) \to U(1, N)$. Then $\tilde{g}$ is viewed as a bundle map

$$\begin{array}{ccc}
F^* M & \longrightarrow & U(1, N) \\
\downarrow & & \downarrow \\
M & \longrightarrow & F_{n,n+1}. \\
\end{array}$$

The corresponding base map $g$ is just the Gauss map given in §1. Let $g_1$ and $g_2$ be also as in §1.

We introduce a quadratic differential form

$$d \sum^2 = -2 \sum \phi^\natural \bar{\phi}^\natural - 2 \sum \phi_\natural \bar{\phi}_\natural + 2 \sum \phi^i \bar{\phi}_i$$

and a complex structure $J$ given by $J \phi^\natural = i \phi^\natural, J \phi_i^\natural = i \phi_i^\natural$ on the flag manifold $F_{n,n+1}$. Then $(d \sum^2, J)$ gives a Hermitian structure of $F_{n,n+1}$. From the definition of the Gauss map $g$, we can see

$$g^*(\phi^\natural) = \omega^i, \quad g^*(\phi_i^\natural) = 0, \quad g^*(\phi_i^\natural) = -\sum A_i^k \bar{\omega}^j. \tag{3}$$

A Kaehler metric $d \sum^2$ (resp. $d \sum^3$) on $F_n$ (resp. $F_{n+1}$) is given by $d \sum^1 = -2 \sum \phi^\natural \bar{\phi}^\natural + 2 \sum \phi^i \bar{\phi}_i$ (resp. $d \sum^1 = -2 \sum \phi_\natural \bar{\phi}_\natural + 2 \sum \phi_i^\natural \bar{\phi}_i$) and a complex structure $J_1$ (resp. $J_2$) is given by $J_1 \phi_i^\natural = i \phi_i^\natural$ (resp. $J_2 \phi_i^\natural = i \phi_i^\natural$) on $F_n$ (resp. $F_{n+1}$). We have also

$$g_1^*(\phi^\natural) = \omega^i, \quad g_1^*(\phi_i^\natural) = -\sum A_i^k \bar{\omega}^j, \tag{4}$$

$$g_2^*(\phi^\natural) = 0, \quad g_2^*(\phi_i^\natural) = -\sum A_i^k \bar{\omega}^j. \tag{5}$$

Using (2), we can show

$$g^*(d \sum^2) = g_1^*(d \sum^2) = ds^2 + \sum A_i^j A_k^i \omega^j \bar{\omega}^k = -nds^2 - \sum R_{jk} \omega^j \bar{\omega}^k, \tag{6}$$

$$g_2^*(d \sum^2) = \sum A_i^j A_k^i \omega^j \bar{\omega}^k = -(n+1)ds^2 - \sum R_{jk} \omega^j \bar{\omega}^k. \tag{7}$$

Nishikawa [3] proved that $g_2$ is anti-holomorphic and that $g_2$ is a constant map iff the immersion $f$ is totally geodesic. From (6) it follows

**Proposition 1.** If the immersion $f$ is totally geodesic, $g$ and $g_1$ are isometric and holomorphic. Conversely if $g$ is isometric or holomorphic, or if $g_1$ is isometric
or holomorphic, h is totally geodesic.

**Proposition 2.** Let \( f : M \to H^N \) be a holomorphic and isometric immersion. The following are equivalent: (1) \( M \) Einstein; (2) \( g \) a homothety or constant map; (3) \( g_1 \) a homothety or constant map; (4) \( g_2 \) a homothety or constant map.

### §4. The harmonic Gauss map

Put

\[
\phi_j^{i*} = \phi_j^{i*} \delta_k^{i*} - \delta_j^{i*} \phi_k^{i*}.
\]

Then the structure equations in the Kaehler manifold \( F_n \) (resp. \( F_{n+1} \)) are given by

\[
d\phi_j^{i*} = -\sum \phi_j^{i*} \wedge \phi_k^{i*} \quad \text{(resp. } d\phi_0^{i*} = -\sum \phi_0^{i*} \wedge \phi_k^{i*})
\]

On the other hand, the structure equations in \( F_{n,n+1} \) are given by

\[
d\phi_0 = -\sum \phi_0 \wedge \phi_k^{i*}
\]
\[
d\phi_0^0 = -\sum \phi_0^0 \wedge \phi_k^{i*}
\]
\[
d\phi_0^i = -\sum \phi_0^i \wedge \phi_k^{i*} - \sum \phi_0 \wedge \phi_0^{i*} + \phi_0 \wedge \phi_0^0.
\]

Thus, by the very same calculation as in §4 of [2], we can show that the Gauss map \( g \) is \( \pi_1 \)-horizontal and

**Theorem 3.** Let \( f : M \to H^N \) be a holomorphic and isometric immersion of a Kaehler \( n \)-manifold \( M \). Then the corresponding Gauss maps \( g : M \to F_{n,n+1} \) and \( g_1 : M \to F_n \) are harmonic.

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### References