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On Serrin's Boundary Point Lemma at a Corner

By

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Let Ω be a domain with smooth boundary in $\boldsymbol{R}^n$ and

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_i u + c(x)u$$

be uniformly elliptic in Ω, that is there exists a positive constant κ such that $a^{ij}(x)\xi_i\xi_j \geqq \kappa|\xi|^2$ for all $x \in \Omega$, $\xi \in \boldsymbol{R}^n$. Throughout this paper, it is assumed that the coefficients $a^{ij}(x)$, $b^i(x)$, $c(x)$ are at least of class $C(\bar{\Omega})$. Under these assumptions, it is well-known that the following boundary point lemma is valid. (cf. [1])

Lemma 1. *Let $u(x) \in C^2(\Omega)$, $x_0 \in \partial\Omega$ be such that*

(i) $Lu \leqq 0$ *in* Ω,

(ii) $u(x)$ *is continuous at* x_0,

(iii) $u(x_0) < u(x)$ *for all* $x \in \Omega$,

(iv) Ω *satisfies the interior sphere condition at* x_0.

Then if $c(x) = 0$ *in* Ω, *the inner derivative of* u *at* x_0, *if it exists, satisfies the strict inequality*

$$\frac{\partial u}{\partial \nu}(x_0) > 0, \quad \nu\text{: inner normal at } x_0.$$

Moreover if $c(x) \leqq 0$ and is bounded from below in Ω, the same conclusion holds provided $u(x) \geqq 0$ in Ω. (On the definition of the interior sphere condition, see [1].)

In the case when there are corner points on $\partial\Omega$, it does not in general follow the same kind of result, even for Laplacian. At a corner point $x_0 \in \partial\Omega$, we must consider the following type of derivative instead of inner derivative.

$\frac{\partial u}{\partial s}$ where s is any direction at x_0 which enters Ω non-tangentially.

Example. (cf. [1]. We adopt here a little modification.) Let $\Omega = \{(x_1, x_2)$: $x_1^2 - x_2^2 > 0,\ x_1 > 0\}$ and $u(x) = x_1^2 - x_2^2$. Then we have $u(x) > u(0) = 0$ in Ω and $\Delta u = 0$. But we see

$$\frac{\partial u}{\partial x_1} = 0 \quad \text{at} \quad x = 0$$

so we can't obtain the positivity of derivative.

At a corner point, Serrin has proved the following boundary point lemma. (cf. [2])

Lemma 2. *Let* D^* *be a domain with* C^2*-boundary and* T *be a plane containing the normal to* ∂D^* *at some point* $x_0 \in \partial D^*$. *Denote by* D *a portion of* D^* *lying on some particular side of* T.

For the coefficients $a^{ij}(x)$, *we assume the validity of the following inequality.*

$$(1) \qquad |a^{ij}(x)\xi_i\eta_j| \leqq C(|\xi\cdot\eta| + |\xi|\cdot d) \quad C = const. > 0$$

where $\xi \in \boldsymbol{R}^n$ *is an arbitrary vector,* η *is the unit normal to the plane* T *and* $d = d(x, T)$ *is the distance between* x *and* T.

Suppose $c(x) = 0$ *and* $u(x)$ *which is of class* $C^2(\bar{D})$ *satisfies*

(i) $Lu \leqq 0$ *in* D,

(ii) $u(x_0) \leqq u(x)$ *for all* $x \in D$.

Then, for s *which is any direction at* x_0 *entering* D *non-tangentially, we have*

$$\text{either } \frac{\partial u}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial s^2} > 0 \quad \text{at} \quad x_0,$$

unless $u(x) \equiv u(x_0)$.

In this paper, we want to prove Lemma 4, which gives the same kind of result as Lemma 2 under simplified condition of (1). In doing so we need the following simple lemma concerning a bilinear form.

Lemma 3. *Let* η *be any fixed unit vector in* $\boldsymbol{R}^n$. *In order that*

$$(2) \qquad |a^{ij}\xi_i\eta_j| \leqq C|\xi\cdot\eta| \quad \text{for every} \quad \xi \in \boldsymbol{R}^n$$

it is necessary and sufficient that η *is an eigen vector of the matrix* $(a^{ij}) = A$.

The above Lemma 3 seems to be almost self-evident but for the completeness we give a proof.

PROOF. If η is an eigen vector of A, then it is easy to obtain (2). Conversely if (2) holds, then denoting by $[\eta]$ the subspace spanned by η, we have

$$[\eta]^\perp \subset [A\eta]^\perp.$$

By taking orthogonal complement, we have

$$[A\eta] \subset [\eta]$$

and this is the same fact that η is an eigen vector of A.

Using the above Lemma 3, we can prove the following boundary point lemma.

Lemma 4. *Let* D^*, x_0, T *and* D *be the same as in Lemma* 2, *and* η *be the unit* *normal to T. We assume that* $c(x)=0$, $a^{ij}(x)$ *are of class* C^1 *and satisfy the following condition.*

(3) η *is an eigen vector of the coefficient matrix* $(a^{ij}(x))$ *for every* $x \in T$.

Suppose that $u(x)$ *is in* $C^2(\bar{D})$ *and satisfies*

(i) $Lu \leqq 0$ *in* D,

(ii) $u(x_0) \leqq u(x)$ *for all* $x \in D$.

Then, for s which is any direction at x_0 *entering D non-tangentially, we have*

$$\text{either } \frac{\partial u}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial s^2} > 0 \quad \text{at} \quad x_0$$

unless $u(x) \equiv u(x_0)$.

PROOF. We proceed along the same line as that of Serrin. First we introduce the ball K_1 which is internally tangent to ∂D^* at x_0, and which touches ∂D^* only at x_0. This is possible because ∂D^* is of class C^2. We denote the radius of K_1 by r_1. Next we take a ball K_2 with center at x_0 and radius θr_1, where $\theta \leqq 1/2$ is a positive constant to be determined later.

Here we choose coordinates with origin at the center of K_1, with T being the plane $x_1=0$ and D being $x_1>0$. Now we put $K'=K_1 \cap K_2 \cap D$ and define the auxiliary function $z(x)$ in K' as the following manner.

$$z(x)=[\exp(-\alpha(x_1-r_1)^2)-\exp(-\alpha r_1^2)]\,[\exp(-\alpha r^2)-\exp(-\alpha r_1^2)]$$

where α is a positive constant to be determined.

Then it is clear that

$$z(x)>0 \quad \text{in} \quad K',\ z(x)=0 \quad \text{on} \quad \partial K_1 \quad \text{and on} \quad T.$$

On the other hand, by direct computations, we have

$$\begin{aligned} Lz = & \exp(-\alpha r^2)\,[\exp(-\alpha(x_1-r_1)^2)-\exp(-\alpha r_1^2)] \\ & \cdot [4\alpha^2 a^{ij}x_i x_j - 2\alpha(a^{ii}+b^i x_i)] + \\ & + \exp(-\alpha(x_1-r_1)^2)\,[\exp(-\alpha r^2)-\exp(-\alpha r_1^2)] \\ & \cdot [4\alpha^2 a^{11}(x_1-r_1) - 2\alpha(a^{11}+b^1(x_1-r_1))] + \\ & + 8\alpha^2 \exp(-\alpha r^2)\exp(-\alpha(x_1-r_1)^2)(x_1-r_1)a^{1j}x_j. \end{aligned}$$

By the uniform ellipticity, we have

$$a^{ij}x_i x_j \geqq \kappa r^2 \geqq \frac{\kappa}{4} r_1^2 \quad \text{in} \quad K',$$

and also

$$a^{11}(x_1-r_1)^2 \geqq \frac{\kappa}{4} r_1^2 \quad \text{in} \quad K'.$$

Since we have assumed that η, the unit normal of T, is an eigen vector of $(a^{ij}(x))$ when $x \in T$, by Lemma 3 we have

$$|a^{1j}(0, x')x_j| \leqq Cx_1.$$

Here we used notations $x'=(x_2, x_3,\dots, x_n)$ and $x=(x_1, x')$. So we obtain, by the mean value theorem

$$|a^{1j}(x)x_j| \leqq |[a^{1j}(x_1, x')-a^{1j}(0, x')]x_j| + \\ +|a^{1j}(0, x')x_j| \leqq Cx_1.$$

Finally we have the following inequality by the mean value theorem.

$$\begin{aligned} &\exp(-\alpha(x_1-r_1)^2)-\exp(-\alpha r_1^2) \\ &\geqq 2\alpha(1-\theta)r_1 \exp(-\alpha r_1^2)x_1 \\ &\geqq \alpha x_1 r_1 \exp(-2\alpha\theta r_1^2-\alpha(x_1-r_1)^2) \end{aligned}$$

Inserting these inequalities into the earlier expression for Lz and using the fact that the terms $(a^{ii}+b^i x_i)$ and $(a^{11}+b^1(x_1-r_1))$ are bounded, we have, for large α,

$$Lz \geqq \alpha^2 x_1 r_1 \exp(-\alpha(r^2+(x_1-r_1)^2))[(\alpha\kappa r_1^2-B)\exp(-2\alpha\theta r_1^2)-C]+ \\ +\alpha\exp(-\alpha(x_1-r_1)^2)[\exp(-\alpha r^2)-\exp(-\alpha r_1^2)](\alpha\kappa r_1^2-B)$$

where B and C are appropriate constants.

By choosing $\theta=1/\alpha$ and taking α sufficiently large, we can make the quantities $[(\alpha\kappa r_1^2-B)\exp(-2\alpha\theta r_1^2)-C]$ and $(\alpha\kappa r_1^2-B)$ positive. We have then $Lz>0$ in K'.

Next, we consider the portion of $\partial K'$ lying on ∂K_2, and suppose $u(x)$ is not identically $u(x_0)$ in D. Without loss of generality, we may assume that $u(x_0)=0$. Then by the strong maximum principle, we have $u(x)>0$ in D. Noting that $\partial K' \cap \partial K_2$ intersects ∂D only on the plane T, and moreover the intersection set lies at a finite distance from the corners of D, it is easy to see that there exists a constant $\varepsilon>0$ such that

$$u(x) \geqq \varepsilon x_1 \quad \text{on} \quad \partial K' \cap \partial K_2$$

by virtue of Lemma 1. Moreover, since we have already seen that $u(x)>0$ in D, we have

$$u(x) \geqq 0 \quad \text{on} \quad \partial K' \cap \partial K_1 \quad \text{and on} \quad \partial K' \cap T.$$

On the other hand, it is clear that

$$z(x) \leqq \exp(-\alpha(x_1-r_1)^2)-\exp(-\alpha r_1^2) \leqq 2\alpha r_1 x_1 \quad \text{on} \quad \partial K' \cap \partial K_2$$

by the mean value theorem. Consequently the function

$$u(x)-\frac{\varepsilon}{2\alpha r_1}z(x)$$

is non-negative on $\partial K'$, and is zero at x_0. Also we have

$$L\left(u-\frac{\varepsilon}{2\alpha r_1}z\right)=Lu-\frac{\varepsilon}{2\alpha r_1}Lz<0 \quad \text{in} \quad K'.$$

Hence we have

$$\text{either } \frac{\partial\left(u-\frac{\varepsilon}{2\alpha r_1}z\right)}{\partial s}>0 \quad \text{or} \quad \frac{\partial^2\left(u-\frac{\varepsilon}{2\alpha r_1}z\right)}{\partial s^2}\geqq 0 \quad \text{at} \quad x_0.$$

On the other hand, we have by a direct calculation

$$\frac{\partial z}{\partial s}=0, \quad \frac{\partial^2 z}{\partial s^2}>0 \quad \text{at} \quad x_0.$$

Thus the proof is complete. Q. E. D.

Remark. Compared with the condition (1), the condition (3) is only on T, and it is not difficult to see that from the condition (1) we obtain the condition (3) by Lemma 3. Moreover the converse may be valid, but the condition (3) seems to be easier to apply.

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References

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