Geometric Characterization of Singular Points of Nonlinear Equations Involving Parameters

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§ 1. Introduction

We consider a point $\hat{x}, \hat{B} \in \Omega$ satisfying an $n$-dimensional nonlinear equation

$$F(x, B) = 0$$

such that the rank of the Jacobian matrix $F_y(x, B)$ of $F(x, B)$ with respect to $x$ is $n-1$ at $(x, B)=(\hat{x}, \hat{B})$, where $F(x, B)$ is defined in some region $\Omega$ of the $(x, B)$-space and $F(x, B)$ is $(d+2)$ times continuously differentiable with respect to $(x, B)$ in $\Omega$, and $B$ is a parameter and we assume that the dimension of the parameter $B$ is $(d+1)$, that is, $B=(B_1, B_2, \ldots, B_{d+1})^T$ $(d \geq 0)$. Here $(\cdots)^T$ denotes the transposed vector of a vector $(\cdots)$.

We call the point $\hat{x}, \hat{B}$ above a "singular point" of the nonlinear equation (1.1). Especially, in the case $d=0$, that is, the dimension of the parameter $B$ is one, $(\hat{x}, \hat{B})$ is called a "turning point" or "fold point", see [3], [4], [5], [6], [7]. Further, in the case $d=1$, that is, the dimension of the parameter $B$ is two, $(\hat{x}, \hat{B})$ is called a "cusp point", see [1], [6], [7].

We shall show that the $B$-component $\hat{B}$ of such a singular point $(\hat{x}, \hat{B})$ is geometrically characterized as an extremum of some function which expresses a curve in the parameter space, that is, the $B$-space. Here $\hat{B}(=B(\hat{\sigma}))$ is called an "extremum" of a function $B(\sigma)$ if $\frac{dB}{d\sigma}(\hat{\sigma}) = 0$ and $\frac{d^2B}{d\sigma^2}(\hat{\sigma}) \neq 0$, where $\sigma$ (scalar) is a real variable and $B(\sigma)$ is defined in some neighborhood of $\hat{\sigma}$ and is twice continuously differentiable with respect to $\sigma$ in such a neighborhood.

When the dimension of the parameter $B$ is $\leq 2$, that is, $d \leq 1$, H. Kawakami [6] defined a singular point $(\hat{x}, \hat{B})$ as the $B$-component $\hat{B}$ of $(\hat{x}, \hat{B})$, where the component is an extremum of some function. Such a function coincides with the above one in the case $d=0$ but is different from that in the case $d=1$, and he proposed a method for computing it. But he did not give any condition for guaranteeing the isolatedness of a singular point and he did not describe anything about the case $d \geq 2$.

In the case $d \leq 1$, A. Spence and B. Werner [7] also considered the $B$-component $\hat{B}$ of a singular point $(\hat{x}, \hat{B})$ as an extremum of some function. In the case $d=0$,
their characterization of a singular point is similar to ours, but in the case $d=1$, theirs is different from ours. And they also did not describe anything about the case $d \geq 2$.

In our case, on the other hand, we of course study the case $d \geq 2$ and we give a condition for guaranteeing the isolatedness of a singular point.

In this paper, in §2, we give geometric characterization of singular points of nonlinear equations involving parameters and we propose a method for computing them with high accuracy.

§ 2. Geometric Characterization of Singular Points of Nonlinear Equations Involving Parameters

We consider a singular point $(\hat{x}, \hat{B}) \in \Omega$ of the nonlinear equation (1.1). In order to simplify the following argument, for the singular point $(\hat{x}, \hat{B})$, we assume that

\[(2.1) \quad n-1 = \text{rank } F_x(\hat{x}, \hat{B}) = \text{rank } F_0(\hat{x}, \hat{B}),\]

where $F_0(\hat{x}, \hat{B})$ is the $n \times (n-1)$ matrix obtained from $F_x(\hat{x}, \hat{B})$ by deleting the first column vector.

Now we define $n \times n$ matrices $X^{(m+1)}$ $(0 \leq m \leq d)$ and $n$-dimensional vectors $l_m$ $(1 \leq m \leq d+1)$ by

\[(2.2) \quad X^{(m+1)} = \sum_{i=0}^{m} C_i X^{(i)} h_{m+1-i} \quad (0 \leq m \leq d)\]

and

\[(2.3) \quad l_m = \sum_{i=1}^{m} C_i X^{(i)} h_{m+1-i} \quad (1 \leq m \leq d+1)\]

respectively, where $X^{(0)} = F_x(x, B)$, and $X^{(i)}$ $(i=0, 1, \ldots, d)$ are the derivatives of $X^{(i)}$ $(i=0, 1, \ldots, d)$ with respect to $x$, respectively, and $h_j$ $(j=1, 2, \ldots, d+1)$ are $n$-dimensional vectors. Moreover, we define $n$-dimensional vectors $\mu_m (1 \leq m \leq d+1)$ by

\[(2.4) \quad \mu_m = \sum_{j=0}^{m-1} C_j X^{(j+1)} h_{m-j} \quad (1 \leq m \leq d+1).\]

Now we consider the following equation

\[(2.5) \quad G(x, B) = \begin{pmatrix} F(x, B) \\ X^{(0)} h_1 \\ \vdots \\ X^{(0)} h_2 + X^{(1)} h_1 \\ \vdots \\ \sum_{j=0}^{d-1} s^{-1} C_j X^{(j)} h_{d-j} \\ \psi(x, B) \end{pmatrix} = \begin{pmatrix} F(x, B) \\ X^{(0)} h_1 \\ \vdots \\ X^{(0)} h_2 + l_1 \\ \vdots \\ X^{(0)} h_d + l_{d-1} \\ \psi(x, B) \end{pmatrix} = 0,

\]
where \( x = (x_1, x_2, \ldots, x_n)^T, x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \) \((j = 1, 2, \ldots, d)\), \(B = (B_1, B_2, \ldots, B_{d+1})^T\), and \(\psi(x, B) = (h_1 - 1, h_2, \ldots, h_{d+1})^T\). Then the function \(G(x, B)\) defined by the equality (2.5) is a \((d + 1)(n + 1) - 1\)-dimensional vector and is twice continuously differentiable with respect to \((x, B)\) due to the assumption on \(F(x, B)\).

In particular, in the case \(d = 0\), the equation (2.5) becomes

\[
(2.6) \quad G(x, B) = F(x, B) = 0
\]

since \(x = x\). Moreover, in the case \(d = 1\), the equation (2.5) becomes

\[
(2.7) \quad G(x, B) = \begin{pmatrix} F(x, B) \\ F_x(x, B)h_1 \\ h_1^T - 1 \end{pmatrix} = 0 \]

because \(x = (x, h_1)^T\) and \(\psi(x, B) = h_1^T - 1\).

Assume that there exists a vector \((\hat{x}, \hat{B}) = (\hat{x}, \hat{h}_1, \ldots, \hat{h}_d, \hat{B})^T\) such that the conditions

\[
(2.8) \quad (\hat{x}, \hat{B}) \in \Omega \text{ satisfies the equation (1.1) and the condition (2.1)}
\]

\[
(2.9) \quad (\hat{x}, \hat{B}) \text{ satisfies the equation (2.5)}
\]

\[
(2.10) \quad \text{rank } (F_0(\hat{x}, \hat{B}), \lambda_d) = n - 1
\]

\[
(2.11) \quad \text{rank } (G_x(\hat{x}, \hat{B}), G_B(\hat{x}, \hat{B}), \ldots, G_{B_{d+1}}(\hat{x}, \hat{B})) = (d + 1)(n + 1) - 1
\]

are satisfied, where \(\lambda_d\) denotes the value of \(\lambda_q\) at \((x, B) = (\hat{x}, \hat{B})\), and \(G_x(x, B)\) denotes the derivative of \(G(x, B)\) with respect to \(x\), that is,

\[
(2.12) \quad G_x(x, B) = \begin{pmatrix}
0 & C_0 \dot{X}^{(0)} \\
\vdots & \vdots \\
0 & C_{d-1} \dot{X}^{(d-1)} \\
1 & C_0 \dot{X}^{(1)} \\
\vdots & \vdots \\
1 & C_{d} \dot{X}^{(d)}
\end{pmatrix}
\]

and \(G_B(x, B)\) \((i = 1, 2, \ldots, d + 1)\) denote the partial derivatives of \(G(x, B)\) with respect to \(B_i\), respectively, that is,
\begin{equation}
G_B(x, B) = \begin{pmatrix}
F_B(x, B) \\
X_B^{(0)}h_1 \\
X_B^{(1)}h_2 + X_B^{(1)}h_1 \\
\vdots \\
\sum_{j=0}^{d-1} C_j X_B^{(j)}h_{d-j} \\
0
\end{pmatrix} 
(i = 1, 2, \ldots, d + 1).
\end{equation}

Here \( F_B(x, B) \) and \( X_B^{(q)} (i = 1, 2, \ldots, d + 1; q = 0, 1, \ldots, d - 1) \) denote the partial derivatives of \( F(x, B) \) and \( X^{(q)} (q = 0, 1, \ldots, d - 1) \) with respect to \( B_i \), respectively, and \( 0 \) denotes the \( d \)-dimensional zero vector.

By (2.8), (2.9) and (2.10), \( \dot{\xi} = (\dot{x}, \dot{h}_{d+1}, \dot{B})^T \) (where \( \dot{h}_{d+1} \) is a solution of the equation \( \dot{X}^{(0)}h_{d+1} + \dot{l}_d = 0, h_{d+1} = 0 \)) is certainly a solution of the system

\begin{equation}
S(z) = \begin{pmatrix}
F(x, B) \\
X^{(0)}h_1 \\
X^{(0)}h_2 + X^{(1)}h_1 \\
\vdots \\
\sum_{j=0}^{d} C_j X^{(j)}h_{d+1-j} \\
\psi_{d+1}(z)
\end{pmatrix} = \begin{pmatrix}
F(x, B) \\
X^{(0)}h_1 \\
X^{(0)}h_2 + l_1 \\
\vdots \\
X^{(0)}h_{d+1} + l_d \\
\psi_{d+1}(z)
\end{pmatrix} = 0,
\end{equation}

where \( \dot{X}^{(0)} = F_x (\dot{x}, \dot{B}) \) and \( \dot{l}_d \) denotes the value of \( l_d \) at \( (x, B) = (\dot{x}, \dot{B}) \), and \( z = (x, h_{d+1}, B)^T, x = (x, h_1, \ldots, h_d)^T, x = (x_1, \ldots, x_d)^T, h_j = (h_{j1}, h_{j2}, \ldots, h_{j1})^T \) \((j = 1, 2, \ldots, d + 1)\) and \( \psi_{d+1}(z) = (\psi(x, B), h_{d+1})^T = (h_{d+1} - 1, h_{d+1}, h_{d+1}, h_{d+1})^T \). For the solution \( \dot{\xi} \), we have the following theorem.

**Theorem.**

The matrix \( S'(\dot{\xi}) \) is non-singular if and only if

\begin{equation}
\text{rank } (F_0(\dot{x}, \dot{B}), \dot{l}_{d+1}) = n,
\end{equation}

where \( S'(z) \) denotes the Jacobian matrix of \( S(z) \) with respect to \( z \) and \( \dot{l}_{d+1} \) denotes the value of \( l_{d+1} \) at \( z = \dot{\xi} \).

**Proof.** Since \( F(x, B) \) is \((d + 2)\) times continuously differentiable with respect to \((x, B)\) in \( \Omega \), \( S(z) \) defined by the equality (2.14) is continuously differentiable with respect to \( z \). Then we have
where $S_{B_i}(z) (i=1, 2, \ldots, d+1)$ are $(d+2)n+d+1$-dimensional vectors defined by

\begin{align}
(2.17) 
S_{B_i}(z) &= \begin{pmatrix}
F_{B_i}(x, B) \\
X_{B_i}^{(0)}h_1 \\
X_{B_i}^{(0)}h_2 + X_{B_i}^{(1)}h_1 \\
\vdots \\
\sum_{j=0}^{d} C_j X_{B_i}^{(j)}h_{d+1-j} \\
\theta
\end{pmatrix} 
\quad (i=1, 2, \ldots, d+1),
\end{align}

where $X_{B_i}^{(j)} (i=1, 2, \ldots, d+1)$ denote the partial derivatives of $X^{(d)}$ with respect to $B_i$, respectively, and $\theta$ denotes the $(d+1)$-dimensional zero vector. From (2.16) it follows that for the solution $\hat{x}$, we have

\begin{align}
(2.18) 
\det S'(\hat{x}) \equiv 0 \quad \text{is equivalent to} \quad (2.15),
\end{align}

because

\begin{align*}
\hat{X}^{(0)} \hat{h}_{m+1} + \hat{l}_m = \hat{X}^{(0)} \hat{h}_{m+1} + \sum_{i=1}^{m} C_i \hat{X}^{(i)} \hat{h}_{m+1-i} = \sum_{i=0}^{m} C_i \hat{X}^{(i)} \hat{h}_{m+1-i} = 0 \quad (1 \leq m \leq d),
\end{align*}

where $\hat{X}^{(i)} (i=0, 1, \ldots, d)$ and $\hat{l}_j (j=1, 2, \ldots, d)$ denote the values of $X^{(i)} (i=0, 1, \ldots, d)$ and $l_j (j=1, 2, \ldots, d)$ at $(x, B)=(\hat{x}, \hat{B})$, respectively. This completes the proof.

Q.E.D.

Thus, if the condition (2.15) is satisfied, we can get an approximation to the
solution \( \hat{\xi} \) of (2.14) as accurately as we desire by applying the Newton method to the system (2.14). Hence we can also obtain a desired approximation to the singular point \((\hat{\xi}, \hat{B})\) of the equation (1.1). We call this singular point \((\hat{\xi}, \hat{B})\) satisfying \( \det S'(\xi) \neq 0 \) an "isolated singular point".

From the conditions (2.9) and (2.11), due to the theorem on implicit function, we have the following results: The equation (2.5) defines a curve in some neighbourhood of \((\hat{\xi}, \hat{B})\) in the \((x, B)\)-space. We denote such a curve by \( \Gamma \). Then, taking some parameter \( \sigma \), we can write the curve \( \Gamma \) in the form

\[
(2.19) \quad x = x(\sigma) = (x(\sigma), h_1(\sigma), \ldots, h_d(\sigma))^T \quad \text{and} \quad B = B(\sigma) = (B_1(\sigma), \ldots, B_{d+1}(\sigma))^T
\]

and we have

\[
(2.20) \quad G(x(\sigma), B(\sigma)) = 0
\]

for \((x(\sigma), B(\sigma))\). Since \((\hat{\xi}, \hat{B})\) is of course a point on the curve \( \Gamma \), there exists one and only one \( \hat{\sigma} \) corresponding to \((\hat{\xi}, \hat{B})\), and we have

\[
(2.21) \quad \hat{x} = x(\hat{\sigma}) \quad \text{and} \quad \hat{B} = B(\hat{\sigma}).
\]

Further, \( x(\sigma) \) and \( B(\sigma) \) defined by (2.19) are twice continuously differentiable with respect to \( \sigma \) because \( G(x, B) \) is twice continuously differentiable with respect to \((x, B)\).

Then, differentiating the both sides of (2.20) with respect to \( \sigma \), we have

\[
(2.22) \quad G_x \cdot \frac{dx}{d\sigma} + \sum_{i=1}^{d+1} G_{B_i} \cdot \frac{dB_i}{d\sigma} = 0,
\]

where \( G_x \) and \( G_{B_i} \) \((i = 1, 2, \ldots, d+1)\) denote \( G_x(x(\sigma), B(\sigma)) \) and \( G_{B_i}(x(\sigma), B(\sigma)) \) \((i = 1, 2, \ldots, d+1)\), respectively, and \( \frac{dx}{d\sigma} \) and \( \frac{dB_i}{d\sigma} \) \((i = 1, 2, \ldots, d+1)\) are the derivaties of \( x(\sigma) \) and \( B_i(\sigma) \) \((i = 1, 2, \ldots, d+1)\) with respect to \( \sigma \), respectively, that is,

\[
\frac{dx}{d\sigma} = \left( \frac{dx}{d\sigma}(\sigma), \frac{dh_1}{d\sigma}(\sigma), \ldots, \frac{dh_d}{d\sigma}(\sigma) \right)^T \quad \text{and} \quad \frac{dB_i}{d\sigma} = \frac{dB_i}{d\sigma}(\sigma)
\]

\((1 \leq i \leq d+1)\).

Differentiating the both sides of (2.22) with respect to \( \sigma \), we have

\[
(2.23) \quad \left\{ \begin{array}{l}
G_x \cdot \frac{dx}{d\sigma} + \sum_{i=1}^{d+1} G_{B_1} \cdot \frac{dB_i}{d\sigma} \cdot \frac{dx}{d\sigma} + \sum_{i=1}^{d+1} \left( \frac{d}{d\sigma} G_{B_i} \right) \frac{dB_i}{d\sigma} \\
+ \sum_{i=1}^{d+1} G_{B_i} \cdot \frac{d^2 B_i}{d\sigma^2} = 0,
\end{array} \right.
\]

where

\( G_{xx} \) denotes the second derivative of \( G(x, B) \) with respect to \( x \);
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\[ G_{xB_i}(i=1, 2, \ldots, d+1) \] denote the partial derivatives of \( G_x(x, B) \) with respect to \( B_i \);
\[ \frac{d^2 x}{d\sigma^2} \text{ and } \frac{d^2 B_i}{d\sigma^2}(i=1, 2, \ldots, d+1) \] denote the second derivatives of \( x(\sigma) \) and \( B(\sigma) \) \((i=1, 2, \ldots, d+1)\) with respect to \( \sigma \), respectively, that is,

\[
\frac{d^2 x}{d\sigma^2} = \left( \frac{d^2 x}{d\sigma^2}(\sigma), \frac{d^2 h_1}{d\sigma^2}(\sigma), \ldots, \frac{d^2 h_d}{d\sigma^2}(\sigma) \right)^T \quad \text{and} \quad \frac{d^2 B_i}{d\sigma^2} = \frac{d^2 B_i}{d\sigma^2}(\sigma)
\]

\((1 \leq i \leq d+1)\);

\[ \frac{d}{d\sigma} G_{B_i} \] \((i=1, 2, \ldots, d+1)\) denote \( \frac{d}{d\sigma} \{G_{B_i}(x(\sigma), B(\sigma))\} \) \((i=1, 2, \ldots, d+1)\).

We shall show that

\[
\frac{d B}{d\sigma} (\dot{\theta}) = \left( \frac{d B_1}{d\sigma}(\dot{\theta}), \frac{d B_2}{d\sigma}(\dot{\theta}), \ldots, \frac{d B_{d+1}}{d\sigma}(\dot{\theta}) \right)^T = 0
\]

and

\[
\frac{d^2 B}{d\sigma^2} (\ddot{\theta}) = \left( \frac{d^2 B_1}{d\sigma^2}(\ddot{\theta}), \frac{d^2 B_2}{d\sigma^2}(\ddot{\theta}), \ldots, \frac{d^2 B_{d+1}}{d\sigma^2}(\ddot{\theta}) \right)^T = 0.
\]

This implies that \( \dot{\theta} = B(\dot{\theta}) \) is an extremum of the function \( B(\sigma) \) which expresses a curve projected from the curve \( \Gamma \) into the parameter space, that is, the \( B \)-space. Hence it is sufficient to show that (2.24) and (2.25) hold.

From (2.8), (2.9) and (2.10), it follows that

\[
\text{rank } G_x(\dot{x}, \dot{B}) = (d+1)n - 1.
\]

By (2.11) and (2.26), we see that

\[
\frac{d B}{d\sigma} (\dot{\theta}) = \left( \frac{d B_1}{d\sigma}(\dot{\theta}), \frac{d B_2}{d\sigma}(\dot{\theta}), \ldots, \frac{d B_{d+1}}{d\sigma}(\dot{\theta}) \right)^T = 0.
\]

This shows that the equality (2.24) holds. Next we will also show that (2.25) holds. From (2.27), at \((x, B)=(\dot{x}, \dot{B})\) or at \(\sigma = \dot{\theta}\), it follows that both the equations (2.22) and (2.23) become

\[
G_x(\dot{x}, \dot{B}) \frac{d x}{d\sigma}(\dot{\theta}) = 0
\]

and

\[
\left\{ G_{xx}(\dot{x}, \dot{B}) \frac{d x}{d\sigma}(\dot{\theta}) \right\} \frac{d x}{d\sigma}(\dot{\theta}) + G_x(\dot{x}, \dot{B}) \frac{d^2 x}{d\sigma^2}(\dot{\theta})
\]

\[
+ \sum_{i=1}^{d+1} G_{B_i}(\dot{x}, \dot{B}) \frac{d^2 B_i}{d\sigma^2}(\dot{\theta}) = 0.
\]
respectively. We consider the vector \( \{ G_{xx}(x, \dot{B}) \frac{dx}{d\sigma}(\dot{\theta}) \} \frac{dx}{d\sigma}(\dot{\theta}) \). When we put

\[
(2.30) \quad k_1(\sigma) = \frac{dx}{d\sigma}(\sigma), \quad k_2(\sigma) = \frac{dh_1}{d\sigma}(\sigma), \quad k_3(\sigma) = \frac{dh_2}{d\sigma}(\sigma), \ldots, \quad k_{d+1}(\sigma) = \frac{dh_d}{d\sigma}(\sigma),
\]

from (2.12), we have

\[
(2.31) \quad G_{xx}(x, B) \frac{dx}{d\sigma}(\sigma) = \begin{pmatrix}
0 & C_0 X^{(0)}k_1(\sigma) \\
1 C_0 X^{(0)}k_2(\sigma) + 1 C_1 X^{(1)}k_1(\sigma) \\
\vdots \\
\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} 2 C_i X^{(i)}k_{d-i}(\sigma) \\
\phi(k_1(\sigma), k_2(\sigma), \ldots, k_{d+1}(\sigma))
\end{pmatrix},
\]

where \( k_i(\sigma) = (k_i^1(\sigma), k_i^2(\sigma), \ldots, k_i^d(\sigma))^T \) \((i = 1, 2, \ldots, d+1)\), and \( \phi(k_1(\sigma), k_2(\sigma), \ldots, k_{d+1}(\sigma)) = (k_{d+1}^1(\sigma), k_{d+1}^2(\sigma), \ldots, k_{d+1}^d(\sigma))^T \).

We set

\[
(2.32) \quad \begin{cases}
Y^{(0)}(\sigma) = F_{xx}(x(\sigma), B(\sigma)), \\
Y^{(m+1)}(\sigma) = \sum_{i=0}^{d} m C_i X^{(i)}(\sigma)k_{m+1-i}(\sigma) \quad (0 \leq m \leq d),
\end{cases}
\]

where \( k_1(\sigma) = \frac{dx}{d\sigma}(\sigma), k_j(\sigma) = \frac{dh_{j-1}}{d\sigma}(\sigma) \quad (j = 2, 3, \ldots, d+1) \), and \( X^{(i)}(\sigma) \quad (i = 0, 1, \ldots, d) \) denote the values of \( X^{(i)} \) at \( (x, B) = (x(\sigma), B(\sigma)) \). Here \( X^{(i)}(\sigma) \quad (i = 0, 1, \ldots, d) \) are previously mentioned in (2.2). Then, the \( (d+1)n + d \times (d+1)n \) matrix \( G_{xx}(x(\sigma), B(\sigma)) \frac{dx}{d\sigma}(\sigma) \) can be written in the form

\[
(2.33) \quad G_{xx}(x(\sigma), B(\sigma)) \frac{dx}{d\sigma}(\sigma) = \begin{pmatrix}
0 & 1 C_0 Y^{(1)}(\sigma) \\
1 C_0 Y^{(2)}(\sigma) & 2 C_1 Y^{(1)}(\sigma) \\
\vdots & \vdots \\
\sum_{i=0}^{d} \sum_{j=0}^{d} 2 C_i Y^{(i)}(\sigma) & \sum_{i=0}^{d} 2 C_i Y^{(i)}(\sigma) \\
\phi(k_1(\sigma), k_2(\sigma), \ldots, k_{d+1}(\sigma)) & \phi(k_1(\sigma), k_2(\sigma), \ldots, k_{d+1}(\sigma))
\end{pmatrix}.
\]
from which it follows that

\[
\begin{pmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
2C_2 Y^{(1)}(\sigma) & \ldots & 0 \\
\vdots & \ddots & \vdots \\
dC_d Y^{(d-1)}(\sigma) & \ldots & dC_d Y^{(1)}(\sigma) \\
0 \\
\end{pmatrix}
\]

(\ref{2.34})

\[\left\{G_{xx}(x(\sigma), B(\sigma)) \frac{dx}{d\sigma}(\sigma) \right\} \frac{dx}{d\sigma}(\sigma) = \begin{pmatrix}
0C_0 Y^{(1)}(\sigma)k_1(\sigma) \\
1C_0 Y^{(1)}(\sigma)k_2(\sigma) + 1C_1 Y^{(2)}(\sigma)k_1(\sigma) \\
\sum_{i=0}^{2} 2C_i Y^{(i+1)}(\sigma)k_{i+1}(\sigma) \\
\vdots \\
\sum_{i=0}^{d} dC_i Y^{(i+1)}(\sigma)k_{d+i-1}(\sigma) \\
0 \\
\end{pmatrix},\]

where \(0\) denotes the \(d\)-dimensional zero vector.

Assume that the parameter \(\sigma\) of the curve \(\Gamma\) is chosen so that \(\frac{dx_1}{d\sigma}(\sigma) = 1\).

For example, we can take \(\sigma = x_1\), where \(x_1\) is the first component of \(x = (x_1, \ldots, x_n)^T\).

Then \(x(\theta), \frac{dx}{d\sigma}(\theta), B(\theta)^T = (x(\theta), \frac{dx}{d\sigma}(\theta), \frac{dh_1}{d\sigma}(\theta), \ldots, \frac{dh_d}{d\sigma}(\theta), B(\theta))^T\) is a solution of the system (\ref{2.14}). In fact, in this case, we have

(\ref{2.35}) \(\dot{x} = x(\theta), \dot{h}_1 = \frac{dx}{d\sigma}(\theta), \dot{h}_2 = \frac{dh_1}{d\sigma}(\theta), \ldots, \dot{h}_{d+1} = \frac{dh_d}{d\sigma}(\theta), \dot{B} = B(\theta)\),

where \(\xi = (\dot{x}, \dot{h}_1, \dot{h}_2, \ldots, \dot{h}_{d+1}, \dot{B})^T\) is the previously mentioned solution of (\ref{2.14}) which is referred to in Theorem. Therefore we have

(\ref{2.36}) \(Y^{(0)}(\theta) = \tilde{X}^{(0)}(\theta), Y^{(1)}(\theta) = \tilde{X}^{(1)}(\theta), \ldots, Y^{(d+1)}(\theta) = \tilde{X}^{(d+1)}(\theta)\),

where \(\tilde{X}^{(i)}(i = 0, 1, 2, \ldots, d+1)\) denote the values of \(X^{(i)}(i = 0, 1, 2, \ldots, d+1)\) at \(z = \xi\). Then, by (\ref{2.4}) and (\ref{2.36}), we have the equalities

(\ref{2.37}) \(\bar{\mu}_m = \sum_{i=0}^{m-1} C_i Y^{(i+1)}(\theta) \bar{k}_{m-i} \quad (1 \leq m \leq d+1)\),

where \(\bar{\mu}_m (m = 1, 2, \ldots, d+1)\) denote the values of \(\mu_m (m = 1, 2, \ldots, d+1)\) at \(z = \xi\),

and \(\bar{k}_1 = k_1(\theta) = \frac{dx}{d\sigma}(\theta), \bar{k}_2 = k_2(\theta) = \frac{dh_1}{d\sigma}(\theta), \ldots, \bar{k}_{d+1} = k_{d+1}(\theta) = \frac{dh_d}{d\sigma}(\theta)\). Thus, the vector \(\{G_{xx}(x(\theta), B(\theta)) \frac{dx}{d\sigma}(\theta)\} \frac{dx}{d\sigma}(\theta)\) is of the form
(2.38) \[ \{G_{x\sigma}(x(\delta), B(\delta)) \frac{dx}{d\sigma}(\delta), \frac{dx}{d\sigma}(\delta)\} = (\mu_1, \mu_2, \ldots, \mu_{d+1}, 0)^T, \]

where \( \theta \) is the \( d \)-dimensional zero vector. When we put

\[ \delta = (\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_{d+1}, 0)^T, \]

by (2.38), the equation (2.29) can be written in the form

(2.39) \[ \delta + G_* (\bar{x}, \bar{B}) \frac{d^2 x}{d\sigma^2}(\delta) + \sum_{i=1}^{d+1} G_{n_i}(\bar{x}, \bar{B}) \frac{d^2 B_i}{d\sigma^2}(\delta) = 0. \]

From (2.11), (2.26) and (2.39) it follows that

\[ \left\{ \frac{d^2 B}{d\sigma^2}(\delta) = \left( \frac{d^2 B_1}{d\sigma^2}(\delta), \frac{d^2 B_2}{d\sigma^2}(\delta), \ldots, \frac{d^2 B_{d+1}}{d\sigma^2}(\delta) \right)^T \right\} = 0 \]

is equivalent to rank \((\delta, G_* (\bar{x}, \bar{B})) = (d + 1)n.\)

Since

(2.41) \[ \bar{\mu}_{i+1} + \sum_{j=0}^{i} jC_j \bar{x}^{(i-j)} \bar{h}_{2+j} = 0 \quad (0 \leq i \leq d - 1) \]

and

(2.42) \[ \bar{\mu}_{d+1} + \sum_{j=0}^{d-1} dC_j \bar{x}^{(d-j)} \bar{h}_{2+j} = \bar{\lambda}_{d+1}, \]

we have

(2.43) \[ \text{rank} (\delta, G_* (\bar{x}, \bar{B})) = \text{rank} (\bar{\zeta}, G_*(\bar{x}, \bar{B})), \]

where \( \bar{\zeta} = (0, 0, \ldots, 0, \bar{\lambda}_{d+1}, 0)^T. \) Here 0 is the \( n \)-dimensional zero vector and \( \theta \) is the \( d \)-dimensional zero vector. From (2.26), we have

(2.44) \[ \text{rank} (\bar{\zeta}, G_* (\bar{x}, \bar{B})) = (d + 1)n \]

is equivalent to (2.15).

Hence, by (2.40), (2.43) and (2.44), we have

\[ \left\{ \frac{d^2 B}{d\sigma^2}(\delta) = \left( \frac{d^2 B_1}{d\sigma^2}(\delta), \frac{d^2 B_2}{d\sigma^2}(\delta), \ldots, \frac{d^2 B_{d+1}}{d\sigma^2}(\delta) \right)^T \right\} = 0 \]

is equivalent to (2.15).

Thus, when the singular point \((\bar{x}, \bar{B})\) of the equation (1.1) satisfies the condition (2.15), the \( B \)-component \( \bar{B} \) of the singular point \((\bar{x}, \bar{B})\) can be characterized as an extremum of the function \( B(\sigma) \) which expresses a curve projected from the curve \( \Gamma \) into the \( B \)-space. From (2.45), due to Theorem, we also have
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\[
\det S'(\hat{z}) \neq 0 \text{ is equivalent to }
\left( \frac{d^2 B}{d\sigma^2}(\hat{\sigma}), \frac{d^2 B_1}{d\sigma^2}(\hat{\sigma}), \ldots, \frac{d^2 B_{d+1}}{d\sigma^2}(\hat{\sigma}) \right)^T \neq 0
\]

for the solution \( \hat{z} \) of (2.14), where \( S'(z) \) is the Jacobian matrix of \( S(z) \) defined by the equality (2.14) which is referred to in Theorem.

Especially, in the case \( d=0 \), both the equations (2.28) and (2.29) become

\[
F_\chi(\hat{x}, \hat{B}) \frac{dx}{d\sigma}(\hat{\sigma}) = 0
\]

and

\[
\left\{ F_{xx}(\hat{x}, \hat{B}) \frac{dx}{d\sigma}(\hat{\sigma}) \right\} \frac{dx}{d\sigma}(\hat{\sigma}) + F_\chi(\hat{x}, \hat{B}) \frac{d^2 x}{d\sigma^2}(\hat{\sigma}) + \frac{d^2 B}{d\sigma^2}(\hat{\sigma}) F_\beta(\hat{x}, \hat{B}) = 0
\]

respectively, because \( G(x, B) = F(x, B) \) from (2.6). As has been mentioned in the above argument, if the parameter \( \sigma \) of the curve \( \Gamma \) is chosen so that \( \frac{dx}{d\sigma}(\sigma) = 1 \), then the equation (2.48) can be rewritten in the form

\[
\hat{l}_1 + F_\chi(\hat{x}, \hat{B}) \frac{d^2 x}{d\sigma^2}(\hat{\sigma}) + \frac{d^2 B}{d\sigma^2}(\hat{\sigma}) F_\beta(\hat{x}, \hat{B}) = 0,
\]

since \( \hat{\mu}_1 = \hat{l}_1 \). Therefore, if the condition \( \text{rank}(F_\chi(\hat{x}, \hat{B}), F_\beta(\hat{x}, \hat{B})) = n \) is satisfied, then we have

\[
\frac{d^2 B}{d\sigma^2}(\hat{\sigma}) \neq 0 \text{ is equivalent to } \text{rank}(F_\chi(\hat{x}, \hat{B}), \hat{l}_1) = n.
\]

In this case, the system (2.14) is of the form

\[
\left( \begin{array}{c}
F(x, B) \\
F_\chi(x, B)h \\
h_1 - 1
\end{array} \right) = 0,
\]

where \( z = (x, h, B)^T, x = (x_1, \ldots, x_n)^T, h = (h_1, \ldots, h_n)^T \). Since \( \frac{dx}{d\sigma}(\hat{\sigma}) \) is a solution of the equation

\[
\left\{ \begin{array}{c}
F_\chi(\hat{x}, \hat{B})h = 0, \\
h_1 - 1 = 0
\end{array} \right.
\]

from (2.47), the system (2.51) certainly has a solution \( \hat{z} = (\hat{x}, \hat{h}, \hat{B})^T \), where \( \hat{h} = \frac{dx}{d\sigma}(\hat{\sigma}) \). Due to Theorem, for this solution \( \hat{z} \), we have

\[
\det S(\hat{z}) \neq 0 \text{ is equivalent to } \text{rank}(F_\chi(\hat{x}, \hat{B}), \hat{l}_1) = n
\]
and also

(2.54) \[ \det S'(\xi) = 0 \text{ is equivalent to } \frac{d^2B}{d\sigma^2}(\theta) = 0. \]

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