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On the Poincaré Series of a Semigraded Module

By

Hiroshi OKUYAMA

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In his paper [10], M. Steurich introduced the notion of the semigraded local ring as a generalized concept of a power series ring over a field and in [9], we studied the homogeneous Golod homomorphism and some change of ring theorems about Poincaré series for semigraded rings. In this paper, we investigate how the properties of inert modules introduced by J. Lescot [7] which are connected with Golod homomorphisms can be transfered to the semigraded case.

Throughout the paper, all rings are commutative and Noetherian, and the symbol $(R, \mathfrak{m}, k)$ stands for R is a local ring with maximal ideal $\mathfrak{m}$ and residue field k.

By definition, a local ring $(R, \mathfrak{m}, k)$ is semigraded if (i) $R=\prod_{i=0}^{\infty} R_i$ as an abelian group, (ii) $R_iR_j \subseteq R_{i+j}$. An R-module M is called semigraded if M satisfies the conditions: (i) $M=\prod_{i=0}^{\infty} M_i$, (ii) $R_iM_j \subseteq M_{i+j}$. For any semigraded (abbreviated by s.g.) R-modules M and N, R-homomorphism $f: M \to N$ is said to be homogeneous of degree $d \geqq 0$ if (i) $f(M_i) \subseteq N_{i+d}$ for all i, (ii) $f(\sum_{i=0}^{\infty} x_i) = \sum_{i=0}^{\infty} f(x_i)$ where $x_i \in M_i$.

Since there exists a minimal free resolution for a finitely generated s.g. R-module M, this extends to a grading on the modules $\mathrm{Tor}_i^R(k, M)$ and we can define the Poincaré series $P_R^M(X, Y)$ of M as the power series in two variables X and Y:

$$P_R^M(X, Y) = \sum_{i,j \geqq 0} \dim_k \mathrm{Tor}_{i,j}^R(k, M) X^i Y^j$$

where $\mathrm{Tor}_{i,j}^R(k, M)$ is the j-th homogeneous component of $\mathrm{Tor}_i^R(k, M)$. For the detail of the definitions and results the reader is referred to [9], [10].

Unless otherwise specified, we shall use the same notations and the same terminology which appeared in [9].

1. Inequalities between the Poincaré series of semigraded module

Let $(R, \mathfrak{m})$, $(S, \mathfrak{n})$ be s.g. local rings and let $f: S \to R$ be a surjective homogenous homomorphism of degree 0. Suppose $f(\mathfrak{n}) \subset \mathfrak{m}$ and f induces an isomorphism of residue fields $k = R/\mathfrak{m} \cong S/\mathfrak{n}$. And, suppose further that R is a finite S-module via f.

Let X (resp. Y) be a minimal homogeneous S (resp. R) algebra resolution of k and let M be a s.g. R-module. We consider the double complex $L=Y\otimes_R M\otimes_R T$ where $T=X\otimes_S R$ is a homogeneous R-algebra which is minimal. Then the filtration $F_pL=\bigoplus_{i\leqq p}(Y_i\otimes M\otimes T)$ determines a spectral sequence of the following form in the same way as the non-homogeneous case [7]:

$$E(f, S, R, M)\colon E^2_{p,q}=\mathrm{Tor}^R_p(k, H_q(M\otimes_R T))\cong \mathrm{Tor}^R_p(k, k)\otimes_k \mathrm{Tor}^S_q(M, k)$$
$$\xRightarrow{p} (\mathrm{Tor}^R_*(M, k)\otimes_k \mathrm{Tor}^S_*(k, k))_{p+q}.$$

If we represent the completion $\hat{R}$ of R as the form $\hat{R}\cong S/I$ where $(S, \mathfrak{n})$ is regular and $I\subseteqq \mathfrak{n}^2$, then the canonical homomorphism s: $(S, \mathfrak{n})\to(\hat{R}, \hat{\mathfrak{m}})$ is a surjective homogeneous homomorphism of degree 0 and if we write $\hat{M}$ for $M\otimes_R\hat{R}$ and $\hat{K}$ for $X\otimes_S\hat{R}$ which is identified with the Koszul complex of $\hat{R}$, then the corresponding spectral sequence $E(s, S, \hat{R}, \hat{M})$ which we denote by $E(\hat{R}, \hat{M})$ is

$$E(\hat{R}, \hat{M})\colon E^2_{p,q}=\mathrm{Tor}^{\hat{R}}_p(k, k)\otimes_k H_q(\hat{M}\otimes_{\hat{R}}\hat{K})\xRightarrow{p} (\mathrm{Tor}^{\hat{R}}_*(\hat{M}, k)\otimes_k(\hat{K}\otimes_{\hat{R}}k))_{p+q}.$$

Since the completion of s.g. modules is an exact functor, we also get a spectral sequence $E(R, M)$ which is isomorphic to $E(\hat{R}, \hat{M})$. That is, for any s.g. local ring R and for any s.g. R-module M, there exists a spectral sequence

$$E(R, M)\colon E^2_{p,q}=\mathrm{Tor}^R_p(k, k)\otimes_k H_q(M\otimes_R K)\xRightarrow{p} (\mathrm{Tor}^R_*(M, k)\otimes_k(K\otimes_R k))_{p+q}$$

where K is the Koszul complex of R.

The existence of spectral sequence $E(f, S, R, M)$ and $E(R, M)$ guarantees the following inequalities between the Poincaré series of s.g. module.

Proposition 1. *Let $f\colon(S, \mathfrak{n})\to(R, \mathfrak{m})$ be a surjective homogeneous homomorphism of s.g. local rings of degree* 0. *Let M be a s.g. R-module and suppose the Poincaré series $P^M_S(X, Y)$ of M considered as S-module is defined, then we have a coefficientwise inequality*:

$$A(M)\colon \quad P^k_R(X, Y)\cdot P^M_S(X, Y)\geqq P^M_R(X, Y)\cdot P^k_S(X, Y).$$

In particular, let $x_1,\dots, x_n$ be a minimal homogeneous generators of $\mathfrak{m}$, $\deg x_i=d_i$ $(i=1,\dots, n)$ and let K be the Koszul complex of R and suppose the Hilbert series of $H(M\otimes_R K)$ which we denote by $H_{H(M\otimes_R K)}(X, Y)$ is defined, then it holds

$$B(M)\colon P^k_R(X, Y)\cdot H_{H(M\otimes_R K)}(X, Y)\geqq P^M_R(X, Y)\cdot \prod_{i=1}^n(1+XY^{d_i}).$$

Proposition 2 ([9], [10]). *Under the same conditions as in Proposition* 1, *we have*

$$(1)\qquad P^k_R(X, Y)\leqq \frac{P^k_S(X, Y)}{1-X[P^R_S(X, Y)-1]}$$

and the equality holds if and only if $f\colon S\to R$ is a homogeneous Golod homomorphism.

(2)
$$P_R^k(X, Y) \leqq \frac{\prod_{i=1}^{n}(1+XY^{d_i})}{1-\sum_{i=1}^{n}\sum_{j=0}^{\infty} c_{ij}X^{i+1}Y^j}$$

where $c_{ij}=\dim_k H_{ij}(K)$. And the equlity holds if and only if R is a homogeneous Golod ring.

Proof. (1) is obtained by $A(\mathfrak{m})$: $P_R^{\mathfrak{m}}(X, Y)\cdot P_S^k(X, Y)\leqq P_R^k(X, Y)\cdot P_S^{\mathfrak{m}}(X, Y)$ combined with the following formulas which are induced from the exact sequence $0\to\mathfrak{m}\to R\to k\to 0$:

$$P_R^{\mathfrak{m}}(X, Y)=\frac{P_R^k(X, Y)-1}{X}$$

and

$$P_S^{\mathfrak{m}}(X, Y)\leqq [P_S^R(X, Y)-1]+\frac{P_S^k(X, Y)-1}{X}.$$

Also, (2) is obtained by $B(\mathfrak{m})$ combined with the inequality

$$H_{H(\mathfrak{m}\otimes_R K)}(X, Y)\leqq H_{H(K)}(X, Y)-1+\frac{\prod_{i=1}^{n}(1+XY^{d_i})-1}{X}.$$

Corollary. *Under the same conditions as above, we have*

(1) *$f\colon S\to R$ is a homogeneous Golod homomorphism if and only if there exists a s.g. R-module M such that the Poincaré series $P_S^M(X, Y)$ of M considered as S-module is defined and*

(*)
$$P_R^M(X, Y)=\frac{P_S^M(X, Y)}{1-X[P_S^R(X, Y)-1]}.$$

(2) *R is a homogeneous Golod ring if and only if there exists a s.g. R-module M such that $H_{H(M\otimes_R K)}(X, Y)$ is defined and*

(**)
$$P_R^M(X, Y)=\frac{H_{H(M\otimes_R K)}(X, Y)}{1-X[H_{H(K)}(X, Y)-1]}.$$

Proof. (1) This condition is necessary. Conversely, let M be a s.g. R-module such that (*) holds. Then, by using the inequality $A(M)$, we have

$$P_R^k(X, Y)\geqq\frac{P_S^k(X, Y)}{1-X[P_S^R(X, Y)-1]}.$$

From Proposition 2 (1), it is easily seen that f is a Golod homomorphism,

(2) is proved in the same way by using the inequality $B(M)$.

2. Semigraded inert module

Under the same notations as Proposition 1, we define the inert module due to Lescot [7] in semigraded case.

Definition. *Let $f: (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a surjective homogeneous homomorphism of s.g. local rings of degree 0, then the s.g. R-module M is said to be inert by f if one of the following equivalent conditions are satisfied:*

(1) *The spectral sequence $E(f, S, R, M)$ is degenerate.*

(2) $P_R^k(X, Y) \cdot P_S^M(X, Y) = P_R^M(X, Y) \cdot P_S^k(X, Y)$.

Especially, a s.g. R-module M is said to be inert R-module when one of the following equivalent conditions are satisfied:

(1′) *The spectral sequence $E(R, M)$ is degenerate.*

(2′) $P_R^k(X, Y) \cdot H_{H(M \otimes_R K)}(X, Y) = P_R^M(X, Y) \cdot \prod_{i=1}^{n} (1 + XY^{d_i})$.

Remark 1. Let M be a s.g. R-module and $\hat{M}$ be its completion, then the observation before Proposition 1 shows that the following conditions are equivalent:

a) M is an inert R-module.

b) $\hat{M}$ is an inert $\hat{R}$-module.

c) $\hat{M}$ is inert by $s: (S, \mathfrak{n}) \to (\hat{R}, \hat{\mathfrak{m}})$ where $\hat{R} \cong S/I$ is a quotient of regular local ring $(S, \mathfrak{n})$ by a homogeneous ideal $I \subset \mathfrak{n}^2$.

Remark 2. When R is a s.g. regular local ring, then every s.g. R-module is an inert module.

Proposition 3. *Let $f: (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a surjective homogeneous homomorphism of s.g. local rings of degree 0 and let M be a s.g. R-module such that $\mathfrak{m}M = 0$, then M is an inert R-module and M is also inert by f.*

PROOF. It is enough to prove when M is a finitely generated module, since, in general case, by considering M as an inductive limit of finitely generated s.g. submodules M_i, the spectral sequence $E(R, M)$ (resp. $E(f, S, R, M)$) is degenerate if each corresponding spectral sequences $E(R, M_i)$ (resp. $E(f, S, R, M_i)$) are degenerate.

Let M be a finitely generated s.g. R-module and let T be a minimal homogeneous generating system of M and we write $\chi_M(Y)$ for $\sum_{t \in T} Y^{\deg t}$. Then we have

$$P_R^k(X, Y) \cdot P_S^M(X, Y) = P_R^k(X, Y) \cdot \chi_M(Y) \cdot P_S^k(X, Y) = P_R^M(X, Y) \cdot P_S^k(X, Y)$$

and

$$P_R^k(X, Y)\cdot H_{H(M\otimes_R K)}(X, Y)=P_R^k(X, Y)\cdot\chi_M(Y)\cdot\prod_{i=1}^{n}(1+XY^{d_i})$$
$$=P_R^M(X, Y)\cdot\prod_{i=1}^{n}(1+XY^{d_i})$$

which shows that M is an inert R-module and M is also inert by f.

Proposition 4. *Let $f: S\to R$ be a surjective homogeneous homomorphism of s.g. local rings of degree* 0 *and let M, N are s.g. R-modules, then $M\oplus N$ is inert by f if and only if M and N are inert by f. Also, $M\oplus N$ is an inert R-module if and only if M and N are inert R-modules.*

The proof follows from Proposition 1 by using the additive property of torsion and homology functor.

Proposition 5. *Let $f: S\to R$ be a surjective homogeneous homomorphism of degree* 0. *Let M, M' are s.g. R-modules and suppose that there exists a homogeneous R-homomorphism h: $M\to M'$ of degree* 0, *then*

(I) 1) *If M is inert by f and h_*: $\mathrm{Tor}_{ij}^S(M, k)\to\mathrm{Tor}_{ij}^S(M', k)$ is surjective for each i, j, then M' is inert by f.*

2) *If M' is inert by f and h_*: $\mathrm{Tor}_{ij}^S(M, k)\to\mathrm{Tor}_{ij}^S(M', k)$ is injective for each i, j, then M is inert by f.*

(II) *Let K be the Koszul complex of R.*

1′) *If M is an inert R-module and h_*: $H_{ij}(M\otimes_R K)\to H_{ij}(M'\otimes_R K)$ is surjective for each i, j, then M' is an inert R-module.*

2′) *If M' is an inert R-module and h_*: $H_{ij}(M\otimes_R K)\to H_{ij}(M'\otimes_R K)$ is injective for each i, j, then M is an inert R-module.*

The proof follows by applying Proposition 2.11 A) and Proposition 4.1 of Lescot [7] to the semigraded case.

3. Small, large and Golod homomorphism of semigraded local rings

Lescot investigated in [7] the homomorphisms of local rings such as small, large and Golod homomorphism and he characterized these homomorphisms in connection with the inert property of modules in the non semigraded case and he also proved that the inert property of modules is preserved under the composition of homomorphisms. Most of these results are realized in the semigraded case.

The following definition is due to Avramov [1] and Levin [6].

Definition. *A surjective homogeneous homomorphism $f: S\to R$ of s.g. local rings is called small (resp. large) homomorphism if the induced map f_*: $\mathrm{Tor}_{ij}^S(k, k)\to\mathrm{Tor}_{ij}^R(k, k)$ is injective (resp. surjective) for each i, j.*

The following theorem is the semigraded version of the result of Lescot [7].

Theorem 1. *Let $f: S\to R$ be a surjective homogeneous homomorphism of s.g. local rings of degree 0 and let M be a s.g. R-module.*

(I) a) *If the induced map $(M,f)_*: \operatorname{Tor}^S_{ij}(M, k)\to\operatorname{Tor}^R_{ij}(M, k)$ is injective for each i, j, then M is inert by f. Moreover if $\operatorname{Tor}^R_{ij}(M, k)\neq 0$, then f is a small homomorphism.*

b) *If f is small, then M is inert by f if and only if $(M,f)_*: \operatorname{Tor}^S_{ij}(M, k)\to \operatorname{Tor}^R_{ij}(M, k)$ is injective for each i, j.*

(II) *Let K be the Koszul complex of R, then M is an inert R-module if and only if the canonical map $H_{ij}(M\otimes_R K)\to\operatorname{Tor}^R_{ij}(M, k)$ is injective for each i, j.*

Theorem 2. *Let $f: S\to R$ be a surjective homogeneous homomorphism of s.g. local rings of degree 0, then the following conditions are equivalent:*

1) *f is a large homomorphism.*

2) *Any s.g. R-module is inert by f.*

3) *Semigraded R-module R is inert by f.*

4) *The map $p_*: \operatorname{Tor}^S_{ij}(R, k)\to\operatorname{Tor}^S_{ij}(k, k)$ induced by the projection p: $R\to k$ is injective for each i, j.*

5) *For any s.g. R-module M, the map $(M,f)_*: \operatorname{Tor}^S_{ij}(M, k)\to\operatorname{Tor}^R_{ij}(M, k)$ is surjective for each i, j.*

6) $P^k_S(X, Y)=P^k_R(X, Y)\cdot P^R_S(X, Y)$.

7) $P^M_S(X, Y)=P^M_R(X, Y)\cdot P^R_S(X, Y)$ *for any s.g. R-module M.*

PROOF. The equivalences from 1) to 5) are easily seen by the similar argument in Theorem 4.8 [7].

2)$\to$7). Since M is inert by f, we have $P^k_R(X, Y)\cdot P^M_S(X, Y)=P^M_R(X, Y)\cdot P^k_S(X, Y)$. Especially, for $M=R$, it holds $P^k_R(X, Y)\cdot P^R_S(X, Y)=P^k_S(X, Y)$. Therefore we have $P^M_S(X, Y)=P^M_R(X, Y)\cdot P^R_S(X, Y)$.

7)$\to$2). Suppose $P^M_S(X, Y)=P^M_R(X, Y)\cdot P^R_S(X, Y)$ for any s.g. R-module M, then especially for $M=k$ we have $P^k_S(X, Y)=P^k_R(X, Y)\cdot P^R_S(X, Y)$. From this we get $P^k_R(X, Y)\cdot P^M_S(X, Y)=P^M_R(X, Y)\cdot P^k_S(X, Y)$. This implies that M is inert by f.

7)$\to$6). Obvious.

6)$\to$7). Let X be a minimal homogeneous S-algebra resolution of k and let Y be a minimal homogeneous R-algebra resolution of M. We consider the double complex $C_{p,q}=Y_p\otimes_S X_q$, then the filtration $F_pC=\bigoplus_{i\leqq p} Y_i\otimes_S X$ gives rise to the change of rings spectral sequence $E^r_{p,q}$ with

$$E^2_{p,q}\cong\operatorname{Tor}^R_p(M, k)\otimes\operatorname{Tor}^S_q(R, k)\overset{p}{\Longrightarrow}\operatorname{Tor}^S_{p+q}(M, k).$$

From this it follows that

$$P^M_R(X, Y)\cdot P^R_S(X, Y)\geqq P^M_S(X, Y).$$

On the other hand, by using the inequality $A(M)$, we have

$$P_S^M(X, Y) \geqq P_R^M(X, Y) \cdot P_S^R(X, Y)$$

which proves our assertion.

Corollary 1 ([4]). *Let $g: R \to S$ be a homogeneous algebra retract of degree 0, that is, there exists a homogeneous homomorphism $f: S \to R$ of degree 0 such that $f \circ g = id.\ R$. Let M be a s.g. R-module, then by considering M as a s.g. S-module via f, we have*

$$P_S^M(X, Y) = P_R^M(X, Y) \cdot P_S^R(X, Y).$$

The proof follows by Theorem 2, since the map $f: S \to R$ is a large homomorphism.

As an application of Corollary 1, we obtain a generalized form of Theorem 6 [9].

Corollary 2. *Let M and N are s.g. modules over a s.g. local ring R, and let $R(M)$ be a trivial extension of R by M, then we have*

$$P_{R(M)}^N(X, Y) = \frac{P_R^N(X, Y)}{1 - X\, P_R^M(X, Y)}.$$

PROOF. The canonical map $g: R \to R(M)$ is a homogeneous algebra retract such that $f: R(M) \to R$ is a large homomorphism. Hence we get from Corollary 1

$$P_{R(M)}^N(X, Y) = P_R^N(X, Y) \cdot P_{R(M)}^R(X, Y)$$

and

$$P_{R(M)}^M(X, Y) = P_R^M(X, Y) \cdot P_{R(M)}^R(X, Y).$$

On the other hand, since the exact sequence of s.g. R-modules $0 \to M \to R(M) \to R \to 0$ is regarded as a sequence of s.g. $R(M)$-modules, we have

$$P_{R(M)}^R(X, Y) = 1 + X P_{R(M)}^M(X, Y).$$

From these relations we get our result.

The following theorems are also semigraded version of the result of Lescot.

Theorem 3. (I) *Let $f: (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a surjective homogeneous homomorphism of s.g. local rings of degree 0. Then the following conditions are equivalent:*

1) *$f: S \to R$ is a homogeneous Golod homomorphism.*

2) *$f: S \to R$ is a small homomorphism and $\mathfrak{m}$ is inert by f.*

3) *The canonical homomorphism $(m, f)_*: \mathrm{Tor}_{ij}^S(\mathfrak{m}, k) \to \mathrm{Tor}_{ij}^R(\mathfrak{m}, k)$ is injective for each i, j.*

4) *$\mathfrak{m}$ is inert by f and the map $\mathrm{Tor}_{ij}^S(\mathfrak{m}, k) \to \mathrm{Tor}_{ij}^S(R, k)$ induced by the injection $\mathfrak{m} \to R$ is surjective for each $i \geqq 1, j \geqq 0$.*

5) *$\mathfrak{m}$ is inert by f and the map $\mathrm{Tor}_{ij}^S(\mathfrak{n}, k) \to \mathrm{Tor}_{ij}^S(\mathfrak{m}, k)$ induced by the*

surjection $\mathfrak{n}\to\mathfrak{m}$ *is injective for each* i, j.

(II) *Let* $(R, \mathfrak{m})$ *be a s.g. local ring and let* K *be the Koszul complex of* R, *then the following conditions are equivalent*:

1) R *is a homogeneous Golod ring.*

2) $\mathfrak{m}$ *is an inert* R*-module.*

3) *The canonical homomorphism* $H_{ij}(\mathfrak{m}\otimes_R K)\to \mathrm{Tor}_{ij}^R(\mathfrak{m}, k)$ *is injective for each* i, j.

Theorem 4. 1) *Let* $f:(S, \mathfrak{n})\to(R, \mathfrak{m})$ *and* $g:(R, \mathfrak{m})\to(R', \mathfrak{m}')$ *are surjective homogeneous homomorphisms of s.g. local rings of degree* 0. *Put* $h=g\circ f$ *and let* M *be a s.g.* R'*-module such that the Poincaré series* $P_{R'}^M(X, Y)$ *is defined. Then* M *is inert by* h *if and only if* M *is inert by* g *and is inert by* f *considered as a s.g.* R*-module via* g.

2) *Let* $g:(R, \mathfrak{m})\to(R', \mathfrak{m}')$ *be a surjective homogeneous homomorphism of s.g. local rings of degree* 0. *Let* M *be a s.g.* R'*-module such that the Poincaré series* $P_{R'}^M(X, Y)$ *is defined. Then* M *is an inert* R'*-module if and only if* M *is inert by* g *and is an inert* R*-module considered as a s.g.* R*-module via* g.

Corollary. *Let* $f:(S, \mathfrak{n})\to(R, \mathfrak{m})$ *and* $g:(R, \mathfrak{m})\to(R', \mathfrak{m}')$ *are two surjective homogeneous homomorphisms of s.g. local rings of degree* 0. *Suppose the composite map* $g\circ f$ *is homogeneous Golod homomorphism, then the following three conditions in* (A) *and* (B) *are respectively equivalent*:

(A) 1) g *is a homogeneous Golod homomorphism.*

2) g *is a small homomorphism.*

3) *The induced homomorphism* $\mathrm{Tor}_{ij}^R(\mathfrak{m}, k)\to\mathrm{Tor}_{ij}^R(\mathfrak{m}', k)$ *is injective for each* i, j.

(B) 1) f *is a homogeneous Golod homomorphism.*

2) $\mathfrak{m}$ *is inert by* f.

3) *The induced map* $\mathrm{Tor}_{ij}^S(\mathfrak{m}, k)\to\mathrm{Tor}_{ij}^R(\mathfrak{m}, k)$ *is injective for each* i, j.

PROOF. (A) Since the composite map $g\circ f$ is homogeneous Golod, the s.g. R'-module $\mathfrak{m}'$ is inert by $g\circ f$ from Theorem 3, (I). Therefore $\mathfrak{m}'$ is also inert by g by Theorem 4. Now the equivalence of 1), 2) and 3) follows from Theorem 3, (I).

(B) Since $g\circ f$ is homogeneous Golod, it is a small homomorphism by Theorem 3, (I). Therefore f is also a small homomorphism [1, Lemma 3.8]. Now the equivalence of 1), 2) and 3) in (B) follows from the above theorem.

Theorem 5. *Let* $f:(S, \mathfrak{n})\to(R, \mathfrak{m})$ *and* $g:(R, \mathfrak{m})\to(R', \mathfrak{m}')$ *are two surjective homogeneous homomorphisms of s.g. local rings of degree* 0. *Then the following conditions are equivalent*:

1) f, g *and* $g\circ f$ *are homogeneous Golod homomorphisms.*

2) g *and* $g\circ f$ *are homogeneous Golod homomorphisms and the induced homo-*

morphism $\mathrm{Tor}_{ij}^{S}(\mathfrak{m}, k)\to\mathrm{Tor}_{ij}^{S}(\mathfrak{m}', k)$ *is injective for each* i, j.

3) f, g *are homogeneous Golod homomorphisms and* $\mathfrak{m}'$ *considered as a s.g.* R*-module is inert by* f.

4) f, g *are homogeneous Golod homomorphisms and the ideal* $J=\mathrm{Ker}\, g$ *is inert by* f.

PROOF. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Tor}_{ij}^{S}(\mathfrak{m}, k) & \longrightarrow & \mathrm{Tor}_{ij}^{S}(\mathfrak{m}', k) \\ \big\downarrow{\scriptstyle (\mathfrak{m}, f)_*} & & \big\downarrow{\scriptstyle (\mathfrak{m}', f)_*} \\ \mathrm{Tor}_{ij}^{R}(\mathfrak{m}, k) & \longrightarrow & \mathrm{Tor}_{ij}^{R}(\mathfrak{m}', k) \end{array}$$

1)$\to$2). Since $g\circ f$ and g are homogeneous Golod, the induced map $\mathrm{Tor}_{ij}^{R}(\mathfrak{m}, k)\to\mathrm{Tor}_{ij}^{R}(\mathfrak{m}', k)$ is injective by the Corollary to Theorem 4 and since f is Golod, the homomorphism $(\mathfrak{m}, f)_*$ is also injective by the same Corollary. Hence the homomorphism $\mathrm{Tor}_{ij}^{S}(\mathfrak{m}, k)\to\mathrm{Tor}_{ij}^{S}(\mathfrak{m}', k)$ is injective for each i, j.

2)$\to$1). Since $g\circ f$ is homogeneous Golod, the homomorphism $(\mathfrak{m}', g\circ f)_*$: $\mathrm{Tor}_{ij}^{S}(\mathfrak{m}', k)\to\mathrm{Tor}_{ij}^{R'}(\mathfrak{m}', k)$ is injective. Hence the homomorphism $(\mathfrak{m}', f)_*$ is also injective. Since the map $\mathrm{Tor}_{ij}^{S}(\mathfrak{m}, k)\to\mathrm{Tor}_{ij}^{S}(\mathfrak{m}', k)$ is injective, it follows that $(\mathfrak{m}, f)_*$ is also injective. Therefore f is Golod by the Corollary to Theorem 4.

1)$\to$3). Since s.g. R'-module $\mathfrak{m}'$ is inert by $g\circ f$ from Theorem 3, it is also inert by f considered as a s.g. R-module in view of Theorem 4.

3)$\to$1). By the similar argument in [1, Lemma 3.8], $g\circ f$ is a small homomorphism. Since $\mathfrak{m}'$ is inert by g and by f, it is also inert by $g\circ f$ from Theorem 4. This implies that $g\circ f$ is homogeneous Golod by Theorem 3, (I).

3)$\leftrightarrow$4). Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}_{ij}^{S}(\mathfrak{m}, k) & \longrightarrow & \mathrm{Tor}_{ij}^{S}(\mathfrak{m}', k) & \longrightarrow & \mathrm{Tor}_{i-1,j}^{S}(J, k) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle (\mathfrak{m}, f)_*} & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Tor}_{ij}^{R}(\mathfrak{m}, k) & \longrightarrow & \mathrm{Tor}_{ij}^{R}(\mathfrak{m}', k) & \longrightarrow & \mathrm{Tor}_{i-1,j}^{R}(J, k) & \longrightarrow & 0 \end{array}$$

Since g is homogeneous Golod, the homomorphism $\mathrm{Tor}_{ij}^{R}(\mathfrak{m}, k)\to\mathrm{Tor}_{ij}^{R}(\mathfrak{m}', k)$ is injective by Theorem 3, (I) 5). Therefore the lower sequence is exact and the injectivity $(\mathfrak{m}, f)_*$ implies that the upper sequence is also exact. From this sequence, we have

$$P_S^{\mathfrak{m}'}(X, Y)-P_S^{\mathfrak{m}}(X, Y)=X\, P_S^J(X, Y)$$

and

$$P_R^{\mathfrak{m}'}(X, Y)-P_R^{\mathfrak{m}}(X, Y)=X\, P_R^J(X, Y).$$

Since f is Golod, $\mathfrak{m}$ is inert by f and it holds

$$P_R^k(X, Y)\cdot P_S^{\mathfrak{m}}(X, Y)=P_R^{\mathfrak{m}}(X, Y)\cdot P_S^k(X, Y).$$

From these relations we have

$$P_R^k(X, Y)\cdot P_S^J(X, Y)=P_R^J(X, Y)\cdot P_S^k(X, Y)$$

if and only if

$$P_R^k(X, Y)\cdot P_S^{\mathfrak{m}'}(X, Y)=P_R^{\mathfrak{m}'}(X, Y)\cdot P_S^k(X, Y).$$

This completes the proof.

Corollary. *Let (R, m) be a s.g. local ring and let x be a homogeneous non zero divisor in $\mathfrak{m}$ and J be a proper homogeneous ideal of R. Then*

1) *Any s.g. R-module M such that $JM=0$ is inert by $g\colon R\to R/xJ$ considered M as a s.g. R/xJ-module.*

2) *If R is a homogeneous Golod ring and J is an inert R-module, then R/xJ is a homogeneous Golod ring.*

3) *Let $f\colon S\to R$ be a surjective homogeneous Golod homomorphism. If J is inert by f, then the composite map $g\circ f\colon S\to R/xJ$ is a homogeneous Golod homomorphism.*

Proof. 1) It is well known that the homomorphism $g\colon R\to R/xJ$ is homogeneous Golod [9, Theorem 3]. Therefore we get

$$P_{R/xJ}^k(X, Y)=\frac{P_R^k(X, Y)}{1-X[P_R^{R/xJ}(X, Y)-1]}.$$

Since $JM=0$, we have by the similar argument as in [2, Theorem 5]

$$P_{R/xJ}^M(X, Y)=\frac{P_R^M(X, Y)}{1-X[P_R^{R/xJ}(X, Y)-1]}.$$

Hence it holds $P_{R/xJ}^k(X, Y)\cdot P_R^M(X, Y)=P_{R/xJ}^M(X, Y)\cdot P_R^k(X, Y)$.

2) is a special case of 3) and 3) follows by Theorem 5 since $g\colon R\to R/xJ$ is a homogeneous Golod homomorphism.

Proposition 6. *Let $(R, \mathfrak{m})$ be a homogeneous Golod ring and M be a finitely generated s.g. inert R-module. Then $R(M)$ is a homogeneous Golod ring.*

Proof. Let K be the Koszul complex of $R(M)$. From the exact sequence of $R(M)$-modules $0\to M\to(\mathfrak{m}, M)\to\mathfrak{m}\to 0$ we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & H_{ij}(M\otimes_{R(M)}K) & \longrightarrow & H_{ij}((\mathfrak{m}, M)\otimes_{R(M)}K) & \longrightarrow & H_{ij}(\mathfrak{m}\otimes_{R(M)}K) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}_{ij}^{R(M)}(M, k) & \longrightarrow & \mathrm{Tor}_{ij}^{R(M)}((\mathfrak{m}, M), k) & \longrightarrow & \mathrm{Tor}_{ij}^{R(M)}(\mathfrak{m}, k) & \longrightarrow & 0
\end{array}$$

for each $i \geqq 0$, $j \geqq 0$.

The lower row is exact, since $\mathrm{Tor}_{ij}^{R(M)}((\mathfrak{m}, M), k) \cong \mathrm{Tor}_{i+1,j}^{R(M)}(k, k)$ for $i \geqq 0$, $j \geqq 0$ and the map $\mathrm{Tor}_{i+1,j}^{R(M)}(k, k) \to \mathrm{Tor}_{ij}^{R(M)}(\mathfrak{m}, k)$ induced by the exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$ is surjective for each i, j from Theorem 2, 4).

Since M and $\mathfrak{m}$ are inert R-modules, they are also inert considered as $R(M)$-modules by Theorem 4, 2). Hence the left vertical map $H_{ij}(M \otimes_{R(M)} K) \to \mathrm{Tor}_{ij}^{R(M)}(M, k)$ is injective for each i, j by Theorem 1, (II). So we have the exact sequence

$$0 \longrightarrow H_{ij}(M \otimes_{R(M)} K) \longrightarrow H_{ij}((\mathfrak{m}, M) \otimes_{R(M)} K) \longrightarrow H_{ij}(\mathfrak{m} \otimes_{R(M)} K) \longrightarrow 0$$

for each i, j, from which we get the following additive property for trivial extension ring.

$$P_{R(M)}^{(\mathfrak{m}, M)}(X, Y) = P_{R(M)}^{M}(X, Y) + P_{R(M)}^{\mathfrak{m}}(X, Y)$$

and

$$H_{H((\mathfrak{m}, M) \otimes_{R(M)} K)}(X, Y) = H_{H(M \otimes_{R(M)} K)}(X, Y) + H_{H(\mathfrak{m} \otimes_{R(M)} K)}(X, Y).$$

By calculating the Poincaré series, it is easily seen that the module $(\mathfrak{m}, M)$ is an inert $R(M)$-module and therefore $R(M)$ is a homogeneous Golod ring in view of Theorem 3, (II).

4. Applications

As applications of the preceding sections, we treat of three cases. The first is the reduction to the Artin case and the second is the inert property of syzygy modules and the third is the fibre product of semigraded local rings.

A) Reduction to the Artin case.

In this part, we fix a s.g. local ring $(R, \mathfrak{m})$ of embedding dimension n and let $x_1, \ldots, x_n$ be a minimal homogeneous generators of $\mathfrak{m}$, deg $x_i = d_i$ $(i = 1, \ldots, n)$ and K be the Koszul complex of R. All s.g. R-modules considered are assumed to be finitely generated and all unspecified tensor products are over R. For any finitely generated s.g. R-module M we denote by $H(M)(X, Y)$ the Hilbert series of M associated with a $\mathfrak{m}$-adic filtration: $H(M)(X, Y) = \sum_{i,j \geqq 0} c_{ij} X^i Y^j$ where $c_{ij} = \dim_k(\mathfrak{m}^i M / \mathfrak{m}^{i+1} M)_j$.

Proposition 7. *For any s.g. R-module M, there is an integer s_0 such that for all $s \geqq s_0$*

$$H_{H(\mathfrak{m}^s M \otimes K)}(X, Y) = H(\mathfrak{m}^s M)(-X, Y) \cdot \prod_{i=1}^{n} (1 + X Y^{d_i}).$$

PROOF. The argument similar to [5, Lemma 2.6] shows that there is an integer s_0 such that for any $s \geqq s_0$ the induced homomorphism $H(\mathfrak{m}^{s+1}M \otimes K) \to H(\mathfrak{m}^s M \otimes K)$ is zero.

Therefore, the exact sequence $0 \to \mathfrak{m}^{s+1}M \to \mathfrak{m}^s M \to \mathfrak{m}^s M/\mathfrak{m}^{s+1}M \to 0$ yeilds the exact sequence

$$(*) \qquad 0 \longrightarrow H_{p,q}(\mathfrak{m}^s M \otimes K) \longrightarrow H_{p,q}(\mathfrak{m}^s M/\mathfrak{m}^{s+1}M \otimes K) \longrightarrow H_{p-1,q}(\mathfrak{m}^{s+1}M \otimes K) \longrightarrow 0$$

for $p \geqq 1$, $q \geqq 0$ and $s \geqq s_0$. Put

$$B_{p,q}(i) = \dim_k H_{p,q}(\mathfrak{m}^i M \otimes K), \quad b_{p,q} = \dim_k H_{p,q}(K \otimes k) = \dim_k (K_p \otimes k)_q.$$

Then, $(*)$ implies that

$$B_{p,q}(s) = \sum_{k+h=q} c_{s,k} b_{p,h} - B_{p-1,q}(s+1).$$

Upon iterating, we have

$$B_{p,q}(s) = \sum_{i=0}^{p-1} \sum_{k+h=q} (-1)^i c_{s+i,k} b_{p-i,h} + (-1)^p B_{0,q}(s+p).$$

Since $B_{0,q}(s+p) = \dim_k H_{0,q}(\mathfrak{m}^{s+p}M \otimes K) = \dim_k H_{0,q}(\mathfrak{m}^{s+p}M/\mathfrak{m}^{s+p+1}M \otimes K) = \sum_{k+h=q} c_{s+p,k} b_{o,h}$, we have

$$B_{p,q}(s) = \sum_{i=0}^{p} \sum_{k+h=q} (-1)^i c_{s+i,k} b_{p-i,h}.$$

Consequently,

$$\begin{aligned} H_{H(\mathfrak{m}^s M \otimes K)}(X, Y) &= \sum_{p,q \geqq 0} B_{p,q}(s) X^p Y^q \\ &= \sum_{p,q \geqq 0} \{ \sum_{i=0}^{p} \sum_{k+h=q} (-1)^i c_{s+i,k} b_{p-i,h} \} X^p Y^q \\ &= (\sum_{i,j \geqq 0} (-1)^i c_{s+i,j} X^i Y^j)(\sum_{s,t \geqq 0} b_{s,t} X^s Y^t) \\ &= H(\mathfrak{m}^s M)(-X, Y) \cdot \prod_{i=1}^{n} (1 + XY^{d_i}), \end{aligned}$$

which finish our proof.

Theorem 6. *Let M be a s.g. R-module. Then, there is an integer s_0 such that for any $s \geqq s_0$*

1) *$\mathfrak{m}^s M$ is an inert R-module.*

2) *The canonical map* $\mathrm{Tor}^R_{p,q}(\mathfrak{m}^s M, k) \to \mathrm{Tor}^R_{p,q}(\mathfrak{m}^s M/\mathfrak{m}^{s+1}M, k)$ is injective for each p, q.

3) a) $P_R^{\mathfrak{m}^s M}(X, Y)=H(\mathfrak{m}^s M)(-X, Y)\cdot P_R^k(X, Y)$.

b) $P_R^{M/\mathfrak{m}^s M}(X, Y)=X\cdot H(\mathfrak{m}^s M)(-X, Y)\cdot P_R^k(X, Y)+P_R^M(X, Y)$ for $s\geqq s_0+1$.

PROOF. 1) and 2) follows by the similar argument in [7, Theorem 5.3] since the canonical map $H_{p,q}(\mathfrak{m}^s M\otimes K)\to H_{p,q}(\mathfrak{m}^s M/\mathfrak{m}^{s+1}M\otimes K)$ is injective for $s\geqq s_0$.

3) a). Since $\mathfrak{m}^s M$ is an inert R-module, we have

$$P_R^k(X, Y)\cdot H_{H(\mathfrak{m}^s M\otimes K)}(X, Y)=P_R^{\mathfrak{m}^s M}(X, Y)\cdot\prod_{i=1}^n(1+XY^{d_i}).$$

By applying Proposition 7, we get the desired formula.

b). By using 2), we can easily see that the map $\operatorname{Tor}_{p,q}^R(\mathfrak{m}^s M, k)\to\operatorname{Tor}_{p,q}^R(\mathfrak{m}^{s-1}M, k)$ is zero. Therefore, for $s\geqq s_0+1$, the map $\operatorname{Tor}_{p,q}^R(\mathfrak{m}^s M, k)\to\operatorname{Tor}_{p,q}^R(M, k)$ is also zero, since the map $\mathfrak{m}^s M\to M$ is decomposed into the composite maps $\mathfrak{m}^s M\to\mathfrak{m}^{s-1}M\to M$ for $s-1\geqq s_0$.

Hence, the exact sequence $0\to\mathfrak{m}^s M\to M\to M/\mathfrak{m}^s M\to 0$ yields the exact sequence

$$(**)\quad 0\longrightarrow\operatorname{Tor}_{p,q}^R(M, k)\longrightarrow\operatorname{Tor}_{p,q}^R(M/\mathfrak{m}^s M, k)\longrightarrow\operatorname{Tor}_{p-1,q}^R(\mathfrak{m}^s M, k)\longrightarrow 0$$

for $p\geqq 1$, $q\geqq 0$ and $s\geqq s_0+1$. Put

$$b_{p,q}(\mathfrak{m}^s M)=\dim_k\operatorname{Tor}_{p,q}^R(\mathfrak{m}^s M, k),\quad b_{p,q}(M/\mathfrak{m}^s M)=\dim_k\operatorname{Tor}_{p,q}^R(M/\mathfrak{m}^s M, k).$$

Then, $(**)$ implies that

$$b_{p,q}(M/\mathfrak{m}^s M)=b_{p,q}(M)+b_{p-1,q}(\mathfrak{m}^s M)\quad\text{and}\quad b_{0,q}(M/\mathfrak{m}^s M)=b_{0,q}(M)$$

for $p\geqq 1$, $q\geqq 0$ and $s\geqq s_0+1$. Therefore, we have

$$\begin{aligned}
P_R^{M/\mathfrak{m}^s M}(X, Y)&=\sum_{p,q\geqq 0}b_{p,q}(M/\mathfrak{m}^s M)X^pY^q\\
&=\sum_{q\geqq 0}b_{0,q}(M/\mathfrak{m}^s M)Y^q+\sum_{p\geqq 1}\sum_{q\geqq 0}b_{p,q}(M/\mathfrak{m}^s M)X^pY^q\\
&=\sum_{p,q\geqq 0}b_{p,q}(M)X^pY^q+X\cdot\sum_{p,q\geqq 0}b_{p,q}(\mathfrak{m}^s M)X^pY^q\\
&=P_R^M(X, Y)+X\cdot P_R^{\mathfrak{m}^s M}(X, Y)
\end{aligned}$$

for $s\geqq s_0+1$, which proves our assertion.

As a corollary of Theorem 6, we can easily deduce Theorem 4 [9] by using the fact that the map $f\colon R\to R/\mathfrak{m}^s$ is a homogeneous Golod homomorphism for sufficiently large s.

Corollary 1. *Let $(R, \mathfrak{m})$ be a s.g. local ring, then there is an integer s_0 such that for any $s\geqq s_0$, we have*

$$P_{R/\mathfrak{m}^s}^k(X, Y)=\frac{P_R^k(X, Y)}{1-X^2\cdot H(\mathfrak{m}^s)(-X, Y)\cdot P_R^k(X, Y)}.$$

Corollary 2. *Let M be a s.g. R-module. Then, M is an inert R-module if and only if $M/\mathfrak{m}^s M$ is inert for sufficiently large s.*

PROOF. Since the induced homomorphism $H(\mathfrak{m}^s M\otimes K)\to H(\mathfrak{m}^{s-1}M\otimes K)$ is zero for sufficiently large s, the exact sequence $0\to\mathfrak{m}^s M\to M\to M/\mathfrak{m}^s M\to 0$ yields the exact sequence

$$0\longrightarrow H_{p,q}(M\otimes K)\longrightarrow H_{p,q}(M/\mathfrak{m}^s M\otimes K)\longrightarrow H_{p-1,q}(\mathfrak{m}^s M\otimes K)\longrightarrow 0$$

for $p\geqq 1$, $q\geqq 0$.

Therefore, by using Proposition 7 and Theorem 6, it holds for sufficiently large s

$$\begin{aligned} H_{H(M\otimes K)}(X, Y) &= -X\cdot H_{H(\mathfrak{m}^s M\otimes K)}(X, Y)+H_{H(M/\mathfrak{m}^s M\otimes K)}(X, Y)\\ &= -X\cdot P_R^{\mathfrak{m}^s M}(X, Y)\cdot \prod_{i=1}^{n}(1+XY^{d_i})/P_R^k(X, Y)+H_{H(M/\mathfrak{m}^s M\otimes K)}(X, Y). \end{aligned}$$

By using Theorem 6 again, we can easily see that

$$P_R^k(X, Y)\cdot H_{H(M\otimes K)}(X, Y)=P_R^M(X, Y)\cdot \prod_{i=1}^{n}(1+XY^{d_i})$$

if and only if

$$P_R^k(X, Y)\cdot H_{H(M/\mathfrak{m}^s M\otimes K)}(X, Y)=P_R^{M/\mathfrak{m}^s M}(X, Y)\cdot \prod_{i=1}^{n}(1+XY^{d_i})$$

for sufficiently large s. This completes the proof.

B) Inert property of syzygy modules.

Let R be a s.g. local ring and M be a finitely generated s.g. R-module. Let Y be a minimal R-free resolution of M and we denote the i-th syzygy module of M by M_i; $M_i=\mathrm{Im}\,(Y_i\to Y_{i-1})$, $(M_0=M)$.

Proposition 8. 1) *Let R be a homogeneous Golod ring and M be a finitely generated s.g. inert R-module. Then the syzygy modules M_i of M are also inert.*

2) *Let $f: S\to R$ be a surjective homogeneous Golod homomorphism and let M be a finitely generated s.g. R-module which is inert by f. Then the syzygy modules M_i of M are also inert by f.*

PROOF. Since 1) is a special case of 2), we prove only the case 2).

Let $x_1,\dots, x_b$ be a set of homogeneous generators of M, deg $x_i=d_i$ $(i=1,\dots, b)$. Let $0\to N\to F\to M\to 0$ be an exact sequence of s.g. R-modules and homogeneous homomorphisms of degree 0 where F is a finitely generated s.g. free R-module and $N\subseteq\mathfrak{m}F$. Then it is enough to prove that N is inert by f if M is inert by f.

From the above exact sequence we get

$$P_R^M(X, Y)=\chi_M(Y)+X\cdot P_R^N(X, Y).$$

Since f is small, the map $(M, f)_*$: $\mathrm{Tor}^S_{p,q}(M, k) \to \mathrm{Tor}^R_{p,q}(M, k)$ is injective by Theorem 1. Hence in the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}^S_{p+1,q}(M, k) \arrow[r] \arrow[d, "\delta"] & \mathrm{Tor}^R_{p+1,q}(M, k) \arrow[d, "\wr"] \\ & \mathrm{Tor}^S_{p,q}(N, k) \arrow[r] & \mathrm{Tor}^R_{p,q}(N, k) \end{tikzcd}$$

the connected homomorphism δ is injective and therefore we have the following exact sequence of s.g. S-modules:

$$0 \longrightarrow \mathrm{Tor}^S_{p+1,q}(M, k) \longrightarrow \mathrm{Tor}^S_{p,q}(N, k)$$
$$\longrightarrow \mathrm{Tor}^S_{p,q-d_1}(R, k) \oplus \cdots \oplus \mathrm{Tor}^S_{p,q-d_b}(R, k) \longrightarrow 0.$$

By calculating the Poincaré series we have

$$P^N_S(X, Y) = \frac{P^M_S(X, Y) - \chi_M(Y)}{X} + \chi_M(Y) \cdot [P^R_S(X, Y) - 1].$$

On the other hand, from the assumption we have

$$P^k_R(X, Y) = \frac{P^k_S(X, Y)}{1 - X[P^R_S(X, Y) - 1]}$$

and

$$P^k_R(X, Y) \cdot P^M_S(X, Y) = P^M_R(X, Y) \cdot P^k_S(X, Y).$$

From these relations we get

$$P^k_R(X, Y) \cdot P^N_S(X, Y) = P^N_R(X, Y) \cdot P^k_S(X, Y)$$

which proves our result.

The following theorem is a slight generalization of Theorem 3.1 [3].

Theorem 7. ([3], [7]). *Let $(R, \mathfrak{m})$ be a s.g. local ring with homogeneous ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \mathfrak{m}$ and let $x_1, \ldots, x_n$ are homogeneous elements of $\mathfrak{m}$. Put $R_0 = R$ and $R_i = R/x_1J_1 + \cdots + x_iJ_i$ for $i > 0$. If x_i is regular on R_{i-1} for each i, then the natural map f_i: $R \to R_i$ is a homogeneous Golod homomorphism.*

Moreover, if M is a finitely generated s.g. R-module such that $J_nM = 0$, then M is inert by f_i and

$$P^M_{R_i}(X, Y) = \frac{P^M_R(X, Y)}{1 - X[P^{R_i}_R(X, Y) - 1]} \qquad (i = 0, 1, \ldots, n).$$

PROOF. The Golod homomorphism of f_i is easily seen by the similar argument as in [7, Theorem 6.6] in view of Corollary of Theorem 5 and Proposition 7.

In the next place, since M is considered as a s.g. R_{i-1}-module such that $J_iM = 0$,

M is inert by g_i: $R_{i-1}\to R_i$ from Corollary of Theorem 5. Hence M is inert by f_i by Theorem 4.

The formula for the Poincaré series $P_{R_i}^M(X, Y)$ now follows from the fact that f_i: $R\to R_i$ is a Golod homomorphism and M is inert by f_i.

C) Fibre product of semigraded rings.

Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ are two s.g. local rings with isomorphic residue fields $k\cong R_1/\mathfrak{m}_1\cong R_2/\mathfrak{m}_2$ and let $(R, \mathfrak{m})$ be a fibre product of R_1 and R_2 over k which is a subring of $R_1\oplus R_2$ defined by $R=\{(x, y)|s_1(x)=s_2(y)\}$, where s_i $(i=1, 2)$ are canonical surjective maps from R_i onto k. Put $n=\dim_k \mathfrak{m}/\mathfrak{m}^2$, $n_1=\dim_k \mathfrak{m}_1/\mathfrak{m}_1^2$, $n_2=\dim_k \mathfrak{m}_2/\mathfrak{m}_2^2$ and let let f_i: $R\to R_i$ are canonical projective maps from R to R_i $(i=1, 2)$. Since $n=n_1+n_2$, we can choose a minimal homogeneous generators $x_1,\dots, x_n$ of $\mathfrak{m}$, $\deg x_i=d_i$ $(i=1,\dots, n)$, such that $f_1(x_1),\dots,f_1(x_{n_1})$ is a minimal homogeneous generators of $\mathfrak{m}_1$ and $f_2(x_{n_1+1}),\dots,f_2(x_n)$ is a minimal homogeneous generators of $\mathfrak{m}_2$ and that $f_1(x_i)=0$ for $i>n_1$ and $f_2(x_i)=0$ for $i\leqq n_1$. Let $K=R\langle T_1,\dots, T_n;\ dT_i=x_i\rangle$, $K_1=R_1\langle T'_1,\dots, T'_{n_1};\ dT'_i=f_1(x_i)\rangle$ and $K_2=R_2\langle T'_{n_1+1},\dots, T'_{n_1+n_2};\ dT'_j=f_2(x_j)\rangle$ are Koszul complexes of R, R_1 and R_2 constructed by each minimal homogeneous generators of maximal ideals.

Under these notations we have the following theorem which is a semigraded version of the result of Lescot [8].

Theorem 8. *The fibre product R of s.g. local rings R_1 and R_2 over k is a homogenerous Golod ring if and only if R_1 and R_2 are homogeneous Golod rings. In this case, we have*

$$\frac{1}{P_R^k(X, Y)}=\frac{1-X[H_{H(K_1)}(X, Y)-1]}{\prod_{i=1}^{n_1}(1+XY^{d_i})}+\frac{1-X[H_{H(K_2)}(X, Y)-1]}{\prod_{i=n_1+1}^{n}(1+XY^{d_i})}-1.$$

PROOF. Put $K'_r=R\langle T_1,\dots, T_r;\ dT_i=x_i\rangle$ for $1\leqq r\leqq n$, then the complexes of s.g. R-modules $K'_{n_1}\otimes_R\mathfrak{m}_1$ and $K_1\otimes_{R_1}\mathfrak{m}_1$ are isomorphic and in particular we have $H_{p,q}(K'_{n_1}\otimes_R\mathfrak{m}_1)\cong H_{p,q}(K_1\otimes_{R_1}\mathfrak{m}_1)$ for each p, $q\geqq 0$.

Since $x_i\in \operatorname{Ann}_R\mathfrak{m}_1$ for $i>n_1$, we get the following exact sequence of s.g. R-modules:

$$\begin{aligned}0\longrightarrow H_{p,q}(K'_r\otimes_R\mathfrak{m}_1)\longrightarrow H_{p,q}(K'_{r+1}\otimes_R\mathfrak{m}_1)\longrightarrow H_{p-1,q-d_{r+1}}(K'_r\otimes_R\mathfrak{m}_1)\\ \longrightarrow 0\qquad \text{for } n_1\leqq r<r+1\leqq n,\ p\geqq 0,\ q\geqq 0.\end{aligned}$$

If we write $K^{\mathfrak{m}_i}$ for $K\otimes_R\mathfrak{m}_i$ and $K_i^{\mathfrak{m}_i}=K_i\otimes_{R_i}\mathfrak{m}_i$ $(i=1, 2)$, then we have from the above sequence that

$$H_{H(K^{\mathfrak{m}_1})}(X, Y)=H_{H(K_1^{\mathfrak{m}_1})}(X, Y)\cdot \prod_{i=n_1+1}^{n}(1+XY^{d_i}).$$

By similar arguments we see that

$$H_{H(K^{\mathfrak{m}_2})}(X,\ Y)=H_{H(K_2^{\mathfrak{m}_2})}(X,\ Y)\cdot\prod_{i=1}^{n_1}(1+XY^{d_i}).$$

Consequently, we get from $\mathfrak{m}\cong\mathfrak{m}_1\oplus\mathfrak{m}_2$

$$\begin{aligned}H_{H(K^{\mathfrak{m}})}(X,\ Y)&=H_{H(K^{\mathfrak{m}_1})}(X,\ Y)+H_{H(K^{\mathfrak{m}_2})}(X,\ Y)\\&=H_{H(K_1^{\mathfrak{m}_1})}(X,\ Y)\cdot\prod_{i=n_1+1}^{n}(1+XY^{d_i})+H_{H(K_2^{\mathfrak{m}_2})}(X,\ Y)\cdot\prod_{i=1}^{n_1}(1+XY^{d_i}).\end{aligned}$$

On the other hand, from Proposition 1 and from the proof of Satz 3.3 [10], we have the following two inequalities:

(a) $P_R^{\mathfrak{m}_1}(X,\ Y)\cdot\prod_{i=1}^{n}(1+XY^{d_i})\leqq P_R^k(X,\ Y)\cdot H_{H(K_1^{\mathfrak{m}_1})}(X,\ Y)\cdot\prod_{i=n_1+1}^{n}(1+XY^{d_i})$

(b) $P_R^{\mathfrak{m}_2}(X,\ Y)\cdot\prod_{i=1}^{n}(1+XY^{d_i})\leqq P_R^k(X,\ Y)\cdot H_{H(K_2^{\mathfrak{m}_2})}(X,\ Y)\cdot\prod_{i=1}^{n_1}(1+XY^{d_i})$

with equalities if and only if R_1 and R_2 are homogeneous Golod rings in (a) and (b) respectively by virtue of Theorem 3 (II).

Since $P_R^{\mathfrak{m}}(X,\ Y)=P_R^{\mathfrak{m}_1}(X,\ Y)+P_R^{\mathfrak{m}_2}(X,\ Y)$, it holds the equality

$$P_R^{\mathfrak{m}}(X,\ Y)\cdot\prod_{i=1}^{n}(1+XY^{d_i})=P_R^k(X,\ Y)\cdot H_{H(K^{\mathfrak{m}})}(X,\ Y)$$

if and only if the equality holds in (a) and (b) respectively. This establishes the first part of the Theorem.

The formula for the Poincaré series is easily deduced from Satz 3.3 of [10] by using Proposition 2.

Department of Mathematics
Faculty of Education
Tokushima University

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