Some Integral Formulas in Fubini-Study Spaces

By

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§ 1. Introduction

In the previous paper [1], we got some integral formulas in a complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. In this paper, firstly we will extend all formulas in [1] to those in Fubini-study spaces of constant holomorphic sectional curvature $4\lambda$ ($\lambda > 0$). Next we will give the complex version of the formula (14.70) in [2].

Let $C^{n+1} = \{ z = (z^0, \ldots, z^n) \}$ be the complex Euclidean $(n+1)$-space with natural inner product $(z, w) = \sum_{k=0}^{n} z^k \overline{w}^k$, for $z, w \in C^{n+1}$. The Euclidean metric $g$ on $C^{n+1}$ is given by $g(z, w) = \text{Re}(z, w)$. Put $S^{2n+1}(\lambda^{-1/2}) = \{ z \in C^{n+1}, g(z, z) = 1/\lambda \}$. Then it is a principal fibre bundle over the complex projective $n$-space $P^n(C)$ with structure group $S^1$ and projection $\pi$. We may regard $z = (z^0, \ldots, z^n)$ as the homogeneous coordinate system of the point $[z] \in P^n(C)$, where $[z] = \pi(z)$. For $z \in S^{2n+1}(\lambda^{-1/2})$, we may put $T_zS^{2n+1} = \{ w \in C^{n+1}; g(z, w) = 0 \}$. The space given by $T'_z = \{ w \in C^{n+1}; g(z, w = 0), g(iz, w = 0) \}$ is a subspace of $T_zS^{2n+1}$ whose orthogonal complement is $\{ iz \}$. The projection $\pi$ induces a linear isomorphism $\pi_* : T'_z$ onto $T_{[z]}P^n(C)$. The standard Riemannian metric on $S^{2n+1}(\lambda^{-1/2})$ is given by $(1/\lambda)g(W, Z)$, for $W, Z \in T_zS^{2n+1}$. We define the Fubini-Study metric $g$ of constant holomorphic sectional curvature $4\lambda$ by

$$g(X, Y) = \frac{1}{\lambda} g(X', Y'),$$

where $X, Y \in T_{[z]}P^n(C)$ and $X', Y'$ are their respective horizontal lifts at $z$.

§ 2. Total volumes of complex Grassmann manifolds

Let $a_0, \ldots, a_n$ be a unitary frame field on $C^{n+1}$. Put $da_k = \sum_{j=0}^{n} \omega_{jk}a_j$. Let $L^0_r$ be a fixed projective $r$-space of $P^n(C)$. Assume that $L^0_r$ is defined by the points $a_0, \ldots, a_r$. Then we have $\omega_{jk} = 0$ for $0 \leq k \leq r$ and $r + 1 \leq j \leq n$. Thus a density for projective $r$-spaces which is invariant under the unitary group $U(n+1)$ may be given by
(2.1) \[ dL_r = \left( \frac{\sqrt{-1}}{2\lambda} \right)^{(n-r)(r+1)} \bigwedge_{j,k} (\omega_{jk} \wedge \overline{\omega}_{jk}), \quad (0 \leq k \leq r, \ r + 1 \leq j \leq n). \]

For \( r = 0 \), we get the density for points, that is, the volume element of \( P^n(C) \) deduced from the Fubini-Study metric given in §1.

The point \( \lambda^{-1/2}a_0 \) moves on the sphere \( S^{2n+1}(\lambda^{-1/2}) \) centered at the origin. Since \( d(\lambda^{-1/2}a_0) = \sum_{j=0}^{n} (\lambda^{-1/2})\omega_{j0}a_j \). The volume element \( ds^{2n+1} \) for \( S^{2n+1}(\lambda^{-1/2}) \) is given by

(2.2) \[ ds^{2n+1} = (\sqrt{-1})^{n(\lambda)}(\sqrt{-1} \omega_{00}) \wedge (\omega_{i0} \wedge \omega_{i0}), \quad (1 \leq i \leq n). \]

The restriction of the form \( \sqrt{-1} \omega_{00} \) to each fibre of the fibre bundle \( \pi: S^{2n+1}(\lambda^{-1/2}) \rightarrow P^n(C) \) is regarded as the standard volume element of \( S^1(\lambda^{-1/2}) \). Hence the total volume \( m(P^n(C)) \) is given

(2.3) \[ m(P^n(C)) = \frac{1}{2\pi\sqrt{\lambda}} m(S^{2n+1}(\lambda^{-1/2})) = \frac{\pi^n}{\lambda^{n+1}}. \]

Let \( L_{n-1} \) be the \((n-1)\)-plane in \( P^n(C) \) perpendicular to \( a_0 \). Put \( L_{n-1}^P = L_r \cap L_{n-1} \). Then we have the density for \((r-1)\)-planes \( L_{n-1}^P \) in \( L_{n-1} \) as follows.

(2.4) \[ dL_{n-1} = \left( \frac{\sqrt{-1}}{2\lambda} \right)^{(n-r)r} \bigwedge (\omega_{ki} \wedge \overline{\omega}_{ki}), \quad (1 \leq i \leq r, \ r + 1 \leq h \leq n). \]

If \( ds^{2r+1} \) denotes the volume element of \( S^{2r+1}(\lambda^{-1/2}) \) in \( L_r \), we have

(2.5) \[ dL_r \wedge ds^{2r+1} = dL_{n-1} \wedge ds^{2n+1}. \]

Successive exterior multiplication by \( ds^{2r+1}, ds^{2r-3}, \ldots, ds^3, ds^1 \) gives

(2.6) \[ dL_r \wedge ds^{2r+1} \wedge ds^{2r-1} \wedge \ldots \wedge ds^3 \wedge ds^1 = ds^{2(n-r)+1} \wedge ds^{2(n-r)+3} \wedge \ldots \wedge ds^{2n+1}. \]

Integrating over the spheres \( S^{2n+1}(\lambda^{-1/2}), S^{2n-1}(\lambda^{-1/2}), \ldots, S^3(\lambda^{-1/2}), S^1(\lambda^{-1/2}), \) we get (see [1], [2])

**Proposition 1.** The total volume of the \( r \)-planes in the Fubini-Study space \( P^n(C) \) of constant holomorphic sectional curvature \( 4\lambda \), that is, the total volume of the complex Grassmann manifold \( G_{r+1,n-r} \) of \((r+1)\)-planes in \( C^{n+1} \), is given by

\[ m(G_{r+1,n-r}) = \frac{1!2! \cdots r!}{n!(n-1)! \cdots (n-r)!} \left( \frac{\pi}{\lambda} \right)^{(n-r)(r+1)}. \]

§ 3. **Densities for linear subspaces**

Let \( L_r^q \) be a fixed projective \( r \)-space of \( P^n(C) \). Assume that \( L_r^q \) is defined by the points \( a_0, \ldots, a_r \). Let \( L_q^q \) be a fixed projective \( q \)-subspace contained in \( L_r^q \).
Suppose that \(a_0, \ldots, a_q\) span \(L_q^0\). Then the density for projective \(r\)-spaces containing \(L_q^0\) is
\[
(3.1) \quad dL_{r(q)} = \left(\frac{\sqrt{-1}}{2\pi}\right)^{(r-q)(r+q)} \wedge (\omega_{i\bar{j}} \wedge \bar{\omega}_{i\bar{j}}), \quad (q+1 \leq i \leq r, \ r+1 \leq h \leq n). \]

Let \(L_{n-q-1}\) be the projective \((n-q-1)\)-subspace perpendicular to \(L_q^0\). Each \(L_{r(q)}\) can be defined by the intersection \(L_{r(q)} \cap L_{n-q-1}\), which is a projective \((r-q-1)\)-subspace, and consequently the density of all \(L_{r(q)}\) is equal to the density of all \(L_{r(q)}\) in \(L_{n-q-1}\), that is,
\[
(3.2) \quad dL_{r(q)} = dL_{n-q-1}^{r-q-1}. \]

Let \(L_r\) and \(L_q\) be a moving projective \(r\)-subspace and a fixed projective \(q\)-subspace respectively in \(P^n(C)\). Assume that \(r+q > n\). Denote by \(L_{r+q-n}\) the intersection \(L_q \cap L_r\). Take a unitary frame field \([a_0, \ldots, a_r]\) such that \(a_0, \ldots, a_{r+q-n}\) span \(L_{r+q-n}\) and \(a_{r+q-n+1}, \ldots, a_r\) lie on \(L_r\). Moreover take points \(b_{r+q-n+1}, \ldots, b_n\) such that \(a_0, \ldots, a_{r+q-n}, b_{r+q-n+1}, \ldots, b_n\) form a unitary frame field and \(a_0, \ldots, a_{r+q-n}, b_{r+q-n+1}, \ldots, b_r\) span \(L_q\). As similarly as (3.5) in [1], we get
\[
(3.3) \quad dL_r = |A|^{[2(r+q-n+1)]} dL_{r+q-n}^{q-r-q-n} \wedge dL_{r+q-n}, \]
where \(A = det(a_h, b_q), \quad (r+1 \leq k, h \leq n)\). By putting \(N = 2n - r - q - 1, \quad v = r + q - n + 1, \quad p = n - q - 1\), we get (see Proposition 2 in [1])
\[
(3.4) \quad \int G_{N-\rho, -\rho+1} |A|^{2r} dL_N = \frac{m(G_{-\rho, -\rho+1})}{m(G_{-\rho, -\rho+1})}. \]

Let \(F(L_r)\) be an integrable function that depends only on \(L_{r+q-n} = L_q \cap L_r\). From (3.3), it follows that
\[
\int f(L_r) \ dL_r = \int |A|^{2(r+q-n+1)} dL_{r+q-n} \int F(L_{r+q-n}) dL_{r+q-n}^{q-r-q-n}. \]
Applying (3.2) and (3.4), we obtain

**Proposition 2.** Let \(F(L_r)\) be an integrable function that depends only on \(L_{r+q-n} = L_q \cap L_r\). Then
\[
\int F(L_r) dL_r = \frac{m(G_{r+1, n-r})}{m(G_{r+q-n+1, n-r})} \int F(L_{r+q-n}) dL_{r+q-n}^{q-r-q-n}. \]

**§ 4. Intersections of projective subspaces and submanifolds**

Let \(L_r\) be a moving projective \(r\)-space and \(M^q\) be a Kaehlerian submanifold in \(P^n(C)\). Let \([a_0, \ldots, a_r]\) be a unitary frame field such that \(a_0, \ldots, a_r\) span \(L_r\). For any submanifold \(X\) of \(P^n(C)\), denote by \(PT_x X\) the projective tangent space of \(X\) at
\( x \in X \), that is, \( \pi((d\pi)^{-1}(T_xX)) \), where \( \pi: C^{n+1} \rightarrow P^q(C) \). We may assume that \( x = a_r \in L_r \cap M^q \) and \( a_0, \ldots, a_{r+q-n} \) span \( PT_x(L_r \cap M^q) \). Let \( b_{r+1}, \ldots, b_n \) be a set of unitary vectors such that \( PT_xM \) is spanned by \( a_1, \ldots, a_{r+q-n}, b_{r+1}, \ldots, b_n \). We can put, up to a constant factor,

\[
(4.1) \quad dL_r = ^{\wedge}_h (\omega_{hr} \wedge \overline{\omega}_{hr}) \wedge (\omega_{kj} \wedge \overline{\omega}_{kj}), \quad (r+1 \leq h \leq n, 1 \leq j \leq r-1, r+1 \leq k \leq n).
\]

Let \( L'_{n-r} \) be the linear \((n-r)\)-space in \( C^{n+1} \) orthogonal to \( L_r \) and \( L_{r-1|x} \) be the projective \((r-1)\)-space in \( L_r \) orthogonal to \( x = a_r \). The volume element of \( L'_{n-r} \) is given by, up to a constant factor,

\[
(4.2) \quad d\sigma_{n-r} = ^{\wedge}_h (\omega_{hr} \wedge \overline{\omega}_{hr}), \quad (r+1 \leq h \leq n).
\]

Hence we get, up to a constant factor,

\[
(4.3) \quad dL_r = d\sigma_{n-r} \wedge dL_{r-1|x}.
\]

As \( x \in M^q \), it holds \( dx = \sum_{i=0}^{q-r-n} \lambda_i a_i + \sum_{k=r+1}^n \beta_k b_k \), where \( \lambda_i \) and \( \beta_k \) are differential 1-forms. We have

\[
\omega_{hr} = -(dx, a_h) = -\sum_{k=r+1}^n \beta_k (b_k, a_h).
\]

If follows from (4.2), up to a constant factor,

\[
(4.4) \quad d\sigma_{n-r} = |\Delta|^2 \wedge_k (\beta_k \wedge \overline{\beta}_k), \quad (r+1 \leq k \leq n),
\]

where \( \Delta = \det(b_k, a_h) \). If we denote by \( \sigma_{r+q-n}(x) \) and \( \sigma_q(x) \) the volume elements of \( L_r \cap M^q \) and \( M^q \) respectively. We have, up to a constant factor,

\[
(4.5) \quad d\sigma_q(x) = ^{\wedge}_k (\beta_k \wedge \overline{\beta}_k), \quad (r+1 \leq k \leq n).
\]

From (4.3), (4.4) and (4.5), it follows, up to a constant factor,

\[
(4.6) \quad d\sigma_{r+q-n}(x) \wedge dL_r = |\Delta|^2 d\sigma_q(x) \wedge dL_{r-1|x}.
\]

Since \( \Delta \) is independent of \( x \), integrating the both sides of (4.6) over all projective \( r \)-space intersecting \( M^q \), we obtain

\[
(4.7) \quad \int_{L_r \cap M^q \neq \emptyset} m(M^q \cap L_r) dL_r = C m(M^q),
\]

where \( C \) is a constant. We need to determine it. For this purpose, we calculate the both side of (4.7) for \( M^q = L_q = P^q(C) \). Using (2.3) and Proposition 1, we get

\[
C = \frac{m(G_{r+1, n-r}) \times m(P^q \cap C)}{m(P^q(C))} = \frac{1 \cdot \ldots \cdot r \cdot q!}{n! \cdot (n-r)! \cdot (q+r-n)!} \left( \frac{\pi}{\lambda} \right)^{(n-r)r}.
\]
Thus we obtain

**Proposition 3.** Let $M^q$ be a $q$-dimensional compact Kaehler submanifold of a Fubini-Study space $P^n(C)$ of constant holomorphic sectional curvature $4\lambda$. Let $L_r$ a projective $r$-space in $P^n(C)$. Denote by $m(M^q)$ and $m(M^q \cap L_r)$ the volumes of $M^q$ and $M^q \cap L_r$, respectively. Then we have

$$
\int_{M^q \cap L_r \neq \emptyset} m(M^q \cap L_r) dL_r = \frac{1!2!\ldots\ldots r!q!}{n!(n-1)!\ldots(n-r)!(q+r-n)!} \left( \frac{\pi}{\chi} \right)^{(n-r)r} m(M^q),
$$

where $dL_r$ is the density for projective $r$-spaces in $P^n(C)$.

When $q+r=n$, the intersection of a $q$-dimensional submanifold $M^q$ and a projective $r$-space $L_r$ is, in general, a finite set of points in $P^n(C)$. In this case, the above formula becomes

**Corollary.** Let denote by $\#(M^{n-r} \cap L_r)$ be the number of the points in $M^{n-r} \cap L_r$. Then it holds

$$
\int_{M^{n-r} \cap L_r \neq \emptyset} \#(M^{n-r} \cap L_r) dL_r = \frac{1!2!\ldots\ldots r!}{n!(n-1)\ldots(n+r+1)!} \left( \frac{\pi}{\chi} \right)^{(n-r)r} m(M^{n-r}).
$$

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**References**
