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Trees with the Partial Order Topology and Selections

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1. Introduction

In this paper, first we shall describe an interesting example which relates to Michael's result [1, p. 178]: *Suppose that a Hausdorff space (X, T) can be partially ordered as a tree, such that the order topology P is coarser than T, and such that X has only a finite number of branch points. Then X is an S_4 space.*

Next we shall give a necessary and sufficient condition in order that a tree with the partial order topology (X, P) be a Hausdorff space, and consider some conditions under which (X, P) be an S_4 space.

2. Definitions

We now give main definitions which are used in this paper. Definitions which are not given here will be defined later when they come up or may be found in reference [2].

A partially ordered set is a set with a reflexive, anti-symmetric and transitive binary relation $\leqq$. A tree is a partially ordered set X, such that $\{t|t\leqq x\}$ is linearly ordered for all $x\in X$, and such that any two elements of X have a greatest lower bound. Hereafter, greatest lower bound is abbreviated to g.1.b. A branch point of a tree X is a point $x\in X$ which is the g.1.b. of two distinct points of X, both of which are different from x. An end point of a tree X is a point $x\in X$ which has no point $t\in X$ such that $x<t$.

The partial order topology P on a tree X is generated by sub-basic open sets of the following two kinds:

$$\{t\,|\,t<x\},\quad x\in X,$$

$$\{t\,|\,t \text{ is not } \leqq x\},\quad x\in X.$$

In this paper, we denote by (X, P) a tree X with the partial order topology P.

Let Y be a topological space. We denote by 2^Y the collection of all nonempty closed subsets of Y with the Vietoris topology. The topology is generated by the base consisting of all the sets of the form $\langle U_1,\dots, U_k\rangle=\{A\in 2^Y | A\subset\cup_{i=1}^k U_i$ *and* $A\cap U_i\neq\phi$ for $i=1,\dots, k\}$, where $U_1,\dots, U_k$ are open sets of Y.

Let Γ be a subcollection of 2^Y. A function f of Γ in Y is called a selection for Γ provided f is continuous and for each $A\in\Gamma$, $f(A)\in A$.

Following E. Michael [1], we define an S_4 space to be a Hausdorff space Y which admits a selection for each covering of Y by mutually disjoint nonempty compact subsets.

3. Example.

In the plane R^2, let

$X=\{b, e_1, e_2\}\cup(\cup_{n=0}^\infty I_n)\cup(\cup_{n=1}^\infty J_n)$ where

$$b=(-2, 0),\quad e_1=(0, 1),\quad e_2=(0, -1),$$

$$I_n=\left\{(x, (-1)^n)\ \Big|\ -\frac{1}{2^n}<x<-\frac{1}{2^{n+1}}\right\}\quad \text{for each } n=0, 1,\dots,$$

$$J_n=\left\{\left(-\frac{1}{2^n}, y\right)\ \Big|\ -1\leqq y\leqq 1\right\}\quad \text{for each } n=1, 2,\dots.$$

Let us define a partial order relation $\leqq$ on X as follows:

$p\leqq p$ for all $p\in X$,

$b\leqq p$ for all $p\in X$

$p_1<p_2$ for any two distinct points $p_1=(x_1, y_1)$, $p_2=(x_2, y_2)$

of $(\cup_{n=0}^\infty I_n)\cup(\cup_{n=1}^\infty J_n)$ if and only if any one of the following conditions is satisfied:

$x_1<x_2$,

$p_1, p_2\in J_{2n-1}$ and $y_1>y_2$ for some n,

$p_1, p_2\in J_{2n}$ and $y_1<y_2$ for some n.

It is easy to see that the above set X is a tree.

We shall consider the space (X, P) with the partial order topology P and the space (X, T) with the topology T induced from the usual topology on the plane R^2.

Then the following facts hold:

(i) *X has one branch point b and two end points e_1, e_2,*

(ii) *(X, T) is a Hausdorff space, but (X, P) is not,*

(iii) *P is coarser than T,*

(iv) *(X, T) is not an S_4 space.*

For, (i) and the first part of (ii) are evident.

In view of the definition of (X, P), we can take as a base for P the collection of $\{b\}$ and all subsets of the following two forms:

$$\{t \mid p<t<q \text{ and } p,\, q \in (\cup_{n=0}^{\infty} I_n) \cup (\cup_{n=1}^{\infty} J_n)\},$$

$$\{e_i\} \cup \{t \mid p<t \text{ and } p \in (\cup_{n=0}^{\infty} I_n) \cup (\cup_{n=1}^{\infty} J_n)\}.$$

Therefore P is coarser than T, and e_1 and e_2 can not be separated by open sets in (X, P) and hence (X, P) is not a Hausdorff space.

Now, to prove (iv), assume, on the contrary, that (X, T) is an S_4 space. We consider the covering of (X, T) by the following mutually disjoint compact sets

$$\{b\},\ \{e_1, e_2\},$$

$$\{p\} \qquad \text{for all} \quad p \in \cup_{n=0}^{\infty} I_n,$$

$$\left\{\left(-\frac{1}{2^{2n-1}},\, y\right),\ \left(-\frac{1}{2^{2n}},\, -y\right)\right\} \qquad \text{for all } n=1, 2,\ldots,\ -1 \leqq y \leqq 1.$$

For simplicity, we denote the covering by $\{K_\alpha | \alpha \in A\}$. Since (X, T) is an S_4 space, we can choose a selection f for $\{K_\alpha | \alpha \in A\}$. We may assume without loss of generality that $f(\{e_1, e_2\})=e_1$. Let V_1, V_2 be open sets such that $e_1 \in V_1$, $e_2 \in V_2$ and $V_1 \cap V_2 = \phi$.

Then, by virtue of the continuity of f, we can find two open sets U_1, U_2 such that

$$e_1 \in U_1 \subset V_1, \quad e_2 \in U_2 \subset V_2,$$

$$f(K_{\alpha'}) \in V_1 \qquad \text{for every} \quad K_{\alpha'} \in \{K_\alpha \mid \alpha \in A\} \quad \text{such that} \quad K_{\alpha'} \in \langle U_1, U_2 \rangle.$$

Since the sequences of points $\left\{\left(-\frac{1}{2^n},\, 1\right) \middle|\, n=1, 2,\ldots\right\}$ and $\left\{\left(-\frac{1}{2^n},\, -1\right) \middle|\, n=1, 2,\ldots\right\}$ converge to e_1 and e_2 respectively, we can take a sufficiently large integer m such that

$$\text{both } \left\{\left(-\frac{1}{2^{2m-1}},\, 1\right),\ \left(-\frac{1}{2^{2m}},\, -1\right)\right\} \text{ and } \left\{\left(-\frac{1}{2^{2m-1}},\, -1\right),\ \left(\frac{1}{2^{2m}},\, 1\right)\right\}$$

belong to $\langle U_1, U_2 \rangle$.

Then we have

$$(1) \qquad f\left(\left\{\left(-\frac{1}{2^{2m-1}},\, 1\right), \left(-\frac{1}{2^{2m}},\, -1\right)\right\}\right) = \left(-\frac{1}{2^{2m-1}},\, 1\right),$$

$$(2) \qquad f\left(\left\{\left(-\frac{1}{2^{2m-1}},\, -1\right),\ \left(-\frac{1}{2^{2m}},\, 1\right)\right\}\right) = \left(-\frac{1}{2^{2m}},\, 1\right).$$

On the other hand, $\left\{\left(-\frac{1}{2^{2m-1}},\, 1\right), \left(-\frac{1}{2^{2m}},\, -1\right)\right\}$ and $\left\{\left(-\frac{1}{2^{2m-1}},\, -1\right), \left(-\frac{1}{2^{2m}},\, 1\right)\right\}$ are the end points of the simple arc in $2^{(X,T)}$

$$\left\{\left\{\left(-\frac{1}{2^{2m-1}}, y\right), \left(-\frac{1}{2^{2m}}, -y\right)\right\} \middle| -1 \leqq y \leqq 1\right\}.$$

Hence, by use of the continuity of f and (1), we have

$$f\left(\left\{\left(-\frac{1}{2^{2m-1}}, -1\right), \left(-\frac{1}{2^{2m}}, 1\right)\right\}\right) = \left(-\frac{1}{2^{2m-1}}, -1\right),$$

which contradicts (2).

Thus (X, T) is not an S_4 space.

4. Theorems

Theorem 1. *Let (X, P) be a tree with the partial order topology. Then, (X, P) is a Hausdorff space if and only if any one of the following three conditions is satisfied:*

(i) *X has no end point,*

(ii) *X has only one end point,*

(iii) *X has only a finite number of end points $e_1,\dots, e_n$, and $X = \bigcup_{i=1}^{n} \{t | t \leqq e_i\}$.*

Proof. To prove the "only if" part, we assume that (X, P) is a Hausdorff space. We shall show that if X has at least two end points, then X satisfies the condition (iii). Let p_1 and p_2 be two distinct end points of X. Every end point of (X, P) has an open neighborhood base consisting of open sets of the form $\bigcap_{i=1}^{m} \{t | t$ is not $\leqq x_i\}$, because any set of the form $\{t | t < x\}$ contains no end point of X.

Hence, since (X, P) is a Hausdorff space, there exist disjoint open sets $\bigcap_{i=1}^{m_1} \{t | t$ is not $\leqq x_{1i}\}$ and $\bigcap_{j=1}^{m_2} \{t | t$ is not $\leqq x_{2j}\}$ containing p_1 and p_2 respectively. By taking the complements of the two open sets, we get

$$(\bigcup_{i=1}^{m_1} \{t \mid t \leqq x_{1i}\}) \cup (\bigcup_{j=1}^{m_2} \{t \mid t \leqq x_{2j}\}) = X.$$

Therefore, the set E of all end points of X is a subset of the set

$$\{x_{11},\dots, x_{1m_1}, x_{21},\dots, x_{2m_2}\},$$

and so we can put $E = \{e_1,\dots, e_n\}$.

Then we can easily see that $X = \bigcup_{i=1}^{n} \{t | t \leqq e_i\}$ and hence (X, P) satisfies the condition (iii).

Thus the "only if" part is proved.

Next, to prove the "if" part, we assume that (X, P) satisfies any one of the three conditions (i), (ii) and (iii). Let x and y be a given pair of distinct points of X.

Case 1: $x < y$. If there is a point $z \in X$ such that $x < z < y$, then $\{t | t < z\}$ and $\{t | t$ is not $\leqq z\}$ are disjoint open sets containing x and y, respectively. If there is no point z such that $x < z < y$, then $\{t | t < y\}$ and $\{t | t$ is not $\leqq x\}$ are disjoint open sets containing x and y, respectively.

Case 2: neither $x<y$ nor $y<x$ holds, and at least one of x and y is not an end point of X. Assume that x is not an end point, and let x_1 be a point such that $x<x_1$. Then, since $\{t|t\leqq x_1\}$ is a linearly ordered set containing x, y does not belong to $\{t|t\leqq x_1\}$.

Hence $\{t|t<x_1\}$ and $\{t|t$ is not $\leqq x_1\}$ are disjoint open sets containing x and y, respectively.

Case 3: both x and y are end points of X. Then, note that X necessarily satisfies the condition (iii). Hence, without loss of generality, we may assume $x=e_1$ and $y=e_n$, where e_1 and e_n are the same ones as in (iii).

To prove that $\bigcap_{i=2}^{n}\{t|t$ is not $\leqq e_i\}$ and $\bigcap_{i=1}^{n-1}\{t|t$ is not $\leqq e_i\}$ are disjoint open sets containing x and y respectively, it remains only to show that these sets are disjoint.

In fact, $(\bigcap_{i=2}^{n}\{t|t$ is not $\leqq e_i\})\cap(\bigcap_{i=1}^{n-1}\{t|$ is not $\leqq e_i\})=X-[(\bigcup_{i=2}^{n}\{t|t\leqq e_i\})\cup(\bigcup_{i=1}^{n-1}\{t|t\leqq e_i\})]=X-\bigcup_{i=1}^{n}\{t|t\leqq e_i\}=X-X=\phi$.

Therefore, the "if" part is proved.

Thus the proof of Theorem 1 is complete.

In what follows, we assume the axiom of choice.

Theorem 2. *Let (Y, T) be a Hausdorff space. Then (Y, T) is an S_4 space if and only if there exists a covering $\{O_\alpha|\alpha\in A\}$ of (Y, T) by mutually disjoint open sets such that O_α is an S_4 space for every $\alpha\in A$.*

Proof. The "only if" part is evident.

To prove the "if" part, we assume that there exists a covering $\{O_\alpha|\alpha\in A\}$ of (Y, T) satisfying the condition in Theorem 2. Here, by Zermelo theorem, we may consider A as a well order set. Let $\{K_\beta|\beta\in B\}$ be an arbitrary covering of (Y, T) by mutually disjoint compact sets. We note that, for every $K_\beta\in\{K_\beta|\beta\in B\}$,

(1) K_β intersects only finitely many members of $\{O_\alpha|\alpha\in A\}$,

(2) if $K_\beta\cap O_\alpha\neq\phi$, then $K_\beta\cap O_\alpha$ is a compact set, because K_β is a compact subset of a Hausdorff space and O_α is a closed set, which is the complement of an open set $\cup\{O_{\alpha'}|\alpha'\in A-\alpha\}$.

Now, we shall construct a selection for $\{K_\beta|\beta\in B\}$.

First, let us define a function ψ of B in A by, for each $\beta\in B$, $\psi(\beta)$ is the first element of $\{\alpha\in A|K_\beta\cap O_\alpha\neq\phi\}$ with respect to the well order. Put $B_\alpha=\{\beta\in B|\psi(\beta)=\alpha\}$.

Then, for each $\alpha\in A$, the collection $\{K_\beta\cap O_\alpha|\beta\in B_\alpha\}\cup\{\{y\}|y\in O_\alpha-\bigcup_{\beta\in B_\alpha}K_\beta\}$ is a covering of O_α by mutually disjoint compact sets. Since O_α is an S_4 space, we can choose a selection for the above covering of O_α, and denote it by f_α.

Finally, we define a function f of $\{K_\beta|\beta\in B\}$ in Y as follows:

$$f(K_\beta)=f_{\psi(\beta)}(K_\beta \cap O_{\psi(\beta)}) \qquad \text{for each} \quad K_\beta \in \{K_\beta | \beta \in B\}.$$

Then, it is obvious that $f(K_\beta) \in K_\beta$.

Further, to show that f is continuous, let $K_{\beta_0} \in \{K_\beta | \beta \in B\}$ and let V be an open neighborhood of $f(K_{\beta_0})$. By (1), we can put $\{O_\alpha | K_{\beta_0} \cap O_\alpha \neq \phi\} = \{O_{\alpha_1}, O_{\alpha_2}, \dots, O_{\alpha_m}\}$ where $\psi(\beta_0)=\alpha_1 < \alpha_2 < \cdots < \alpha_m$.

In view of the continuity of $f_{\psi(\beta_0)}$, we can choose open sets $U_1, \dots, U_n$ of $O_{\psi(\beta_0)}$ such that

$$K_{\beta_0} \cap O_{\psi(\beta_0)} \in \langle U_1, \dots, U_n \rangle,$$

$$\text{if} \quad \beta \in B_{\psi(\beta_0)} \quad \text{and} \quad K_\beta \cap O_{\psi(\beta_0)} \in \langle U_1, \dots, U_n \rangle, \quad \text{then} \quad f_{\psi(\beta_0)}(K_\beta \cap O_{\psi(\beta_0)}) \in V.$$

Then, it is easy to see that

$$K_{\beta_0} \in \langle U_1, \dots, U_n, O_{\alpha_2}, \dots, O_{\alpha_m} \rangle,$$

$$\text{if} \quad K_\beta \in \langle U_1, \dots, U_n, O_{\alpha_2}, \dots, O_{\alpha_m} \rangle, \quad \text{then}$$

$$f(K_\beta)=f_{\psi(\beta)}(K_\beta \cap O_{\psi(\beta)})=f_{\psi(\beta_0)}(K_\beta \cap O_{\psi(\beta_0)}) \in V.$$

Therefore, f is continuous, and hence f is a selection for $\{K_\beta | \beta \in B\}$.

Thus the proof of Theorem 2 is complete.

We define a chain to be a subset of a partially ordered set X which is linear with respect to the partial order. A maximal chain of X is a chain which is properly contained in no other chain of X.

We first prove some preliminary lemmas which will be used in the proof of Theorem 3. It is well-known that Zorn's lemma assures Lemma 1.

Lemma 1. *Let L be a chain of a tree X. Then there exists a maximal chain of X which contains L.*

Lemma 2. *Let M be a maximal chain of (X, P). Then we can take as an open sub-base of M, the collection of all intervals of the form*

$$\{t \mid t<x\}, \quad \{t \mid t \leqq b\} \quad \text{or} \quad M-\{t \mid t \leqq x\},$$

where $x \in M$ and b is a branch point of X such that $b \in M$.

Proof. First, note that for each $x \in M$, $\{t | t \leqq x\}$ is a subset of M, because M is a maximal chain. By virtue of the definition of (X, P), the collection of all sets of the form $M \cap \{t | t<x\}$ or $M \cap \{t | t \text{ is not } \leqq x\}$, where $x \in X$, is an open sub-base of M.

Case 1: $x \in M$. Then we obtain

$$M \cap \{t \mid t<x\}=\{t \mid t<x\},$$

$$M \cap \{t \mid t \text{ is not } \leqq x\}=M-\{t \mid t \leqq x\}.$$

Case 2: $x \in X - M$. Let b be the g.l.b. of x and a point of $M - \{t | t \leqq x\}$, then b is a branch point of X such that $b \in M$. Then

$$M \cap \{t \mid t < x\} = \{t \mid t \leqq b\},$$

$$M \cap \{t \mid t \text{ is not } \leqq x\} = M - M \cap \{t \mid t \leqq x\} = M - \{t \mid t \leqq b\}.$$

Thus Lemma 2 is proved.

Remark. Let L be a chain of (X, P) and M a maximal chain of (X, P) containing L. Then we can take as an open sub-base of L, the collection of all intersections of L and the members of the open sub-base of M described in Lemma 2.

Lemma 3. *Let (X, P) be a tree with the partial order topology. Assume that (X, P) is a Hausdorff space. Let K, L and M be a compact set, a compact chain and a maximal chain of (X, P), respectively. Let b be a branch point of X such that $b \in M$. Then*

(i) *L has the first point,*

(ii) *if $M \cap K \neq \phi$, then $M \cap K$ has the first point,*

(iii) *if $(M - \{t | t \leqq b\}) \cap K \neq \phi$, then $(M - \{t | t \leqq b\}) \cap K$ has the first point.*

Proof. To prove (i), suppose, on the contrary, that L does not have the first point. Let us assign to each $x \in L$ a point $x' \in L$ such that $x' < x$, and define $U(x) = L - \{t | t \leqq x'\}$. Then, since the collection $\{U(x) | x \in L\}$ is an open covering of the compact set L, we can find a finite set $\{x_1, \dots, x_n\}$ such that $L = U(x_1) \cup \cdots \cup U(x_n)$. Let x'_k be the first point of the set $\{x'_1, \dots, x'_n\}$ with respect to the order on L. Then, in view of the definition of $U(x)$, we have $L = U(x_1) \cup \cdots \cup U(x_n) = U(x_k)$, and hence x'_k is not a point of L. This contradiction proves (i).

Next, to prove (ii), let x_0 be a point of $M \cap K$. Since M is a maximal chain, $\{t | t \leqq x_0\} \cap K$ is a subset of M and a compact chain.

Therefore, by (i) shown in the above, there exists the first point of $\{t | t \leqq x_0\} \cap K$. It is obvious that the point is also the first point of $M \cap K$. Thus (ii) is proved.

Finally, the proof of (iii) is quite analogous to that of (ii), if we replace M in the proof of (ii) with $M - \{t | t \leqq b\}$.

Lemma 4. *Let (X, P) be a tree with the partial order topology. Assume that (X, P) is a Hausdorff space. Then every chain L of (X, P) is an S_4 space.*

Proof. Let $\{K_\alpha | \alpha \in A\}$ be a covering of L by mutually disjoint compact subsets of L. Then, by (i) of Lemma 3, every $K_\alpha \in \{K_\alpha | \alpha \in A\}$ has the first point, and let denote it by x_α.

Now, we define a function f of $\{K_\alpha | \alpha \in A\}$ in L by

$$f(K_\alpha) = x_\alpha \qquad \text{for each} \quad K_\alpha \in \{K_\alpha | \alpha \in A\}.$$

To prove the continuity of f, let K_α be a given element of $\{K_\alpha|\alpha\in A\}$ and V an open neighborhood of x_α in L. In view of Remark, we can find an open interval I of L such that $x_\alpha\in I\subset V$.

In case $K_\alpha\neq\{x_\alpha\}$ (resp. $K_\alpha=\{x_\alpha\}$), the open neighborhood $\langle I, L-\{t|t\leqq x_\alpha\}\rangle$ (resp. $\langle I\rangle$) of K_α in 2^L satisfies the following condition:

$$f(K_{\alpha'})=x_{\alpha'}\in I\subset V \qquad \text{for every} \quad K_{\alpha'}\in\{K_\alpha|\alpha\in A\} \text{ such that}$$
$$K_{\alpha'}\in\langle I, L-\{t|t\leqq x_\alpha\}\rangle \quad (\text{resp. } \langle I\rangle).$$

Therefore, f is continuous, and hence f is a selection for $\{K_\alpha|\alpha\in A\}$. Thus Lemma 4 is proved.

Theorem 3. *Let (X, P) be a tree with the partial order topology. Assume that (X, P) is a Hausdorff space, and that the set B of all branch points of X satisfies the following conditions:*

(i) *B is at most countable,*

(ii) *for every point $x\in X-\bigcup_{b\in B}\{t|t\leqq b\}$, $B\cap\{t|t\leqq x\}$ has the last point relative to the order.*

Then (X, P) is an S_4 space.

Proof. By (i), we can put $B=\{b_i|i=1, 2,\ldots\}$. Since (X, P) is Hausdorff, the proof will be carried out by considering three cases each of which satisfies any one of the three conditions of Theorem 1.

Case I: X has no end point.

Put $b_{i_1}=b_1$. Let $M_{b_{i_1}}$ be a maximal chain of (X, P) containing b_{i_1}. To show that $M_{b_{i_1}}$ is an open set, let x be any point of $M_{b_{i_1}}$. Since $M_{b_{i_1}}$ is a maximal chain of (X, P) in case I, there is a point $x'\in M_{b_{i_1}}$ such that $x<x'$. Then $\{t|t<x'\}$ is an open set which satisfies $x\in\{t|t\in x'\}\subset M_{b_{i_1}}$. Therefore $M_{b_{i_1}}$ is an open set.

We put $O_{\mathrm{I}1}=M_{b_{i_1}}$.

Now, assume that $\{M_{b_{i_n}}|n=1,\ldots,j-1\}$ and $\{O_{\mathrm{I}n}|n=1,\ldots,j-1\}$ have been defined. Let b_{i_j} be the point with the least index of the branch points of X which belong to $X-\bigcup_{n=1}^{j-1}M_{b_{i_n}}$, and let $M_{b_{i_j}}$ be a maximal chain of (X, P) containing b_{i_j}. For each $n=1,\ldots,j-1$, let denote by $b_{(i_j,i_n)}$ the g.l.b. of b_{i_j} and a point of $M_{b_{i_n}}-M_{b_{i_j}}$.

We put $O_{\mathrm{I}j}=M_{b_{i_j}}-\bigcup_{n=1}^{j-1}\{t|t\leqq b_{(i_j,i_n)}\}$.

Thus, by the indutive process, we get a collection of chains $\{O_{\mathrm{I}n}|n=1, 2,\ldots\}$. About $\{O_{\mathrm{I}n}|n=1, 2,\ldots\}$, we note that

(1) the members of $\{O_{\mathrm{I}n}|n=1, 2,\ldots\}$ are mutually disjoint open sets,

(2) $B\subset\bigcup_i\{t|t\leqq b_i\}\subset\bigcup_n O_{\mathrm{I}n}$.

Next, let us denote by $\{O_{\mathrm{II}\lambda}|\lambda\in\Lambda\}$ the collection of all maximal chains of $X-\bigcup_n O_{\mathrm{I}n}$. We shall show that the members of $\{O_{\mathrm{II}\lambda}|\lambda\in\Lambda\}$ are also mutually disjoint open sets. Suppose, on the contrary, that there exists a point $c\in O_{\mathrm{II}\lambda_1}\cap$

$O_{\mathrm{II}\lambda_2}$ for some $\lambda_1 \neq \lambda_2$. Let $x_1 \in O_{\mathrm{II}\lambda_1} - O_{\mathrm{II}\lambda_2}$, $x_2 \in O_{\mathrm{II}\lambda_2} - O_{\mathrm{II}\lambda_1}$ and let b be the g.l.b. of x_1 and x_2. Then b is a branch point of X such that $c \leqq b$. Hence $c \in \bigcup_n O_{\mathrm{I}n}$. This contradicts the fact that $c \in O_{\mathrm{II}\lambda_1} \subset X - \bigcup_n O_{\mathrm{I}n}$.

Thus $O_{\mathrm{II}\lambda_1} \cap O_{\mathrm{II}\lambda_2} = \phi$ for any $\lambda_1 \neq \lambda_2$.

For any fixed $O_{\mathrm{II}\lambda}$, let x_0 be a point of $O_{\mathrm{II}\lambda}$ and M_{x_0} a maximal chain of X containing x_0. Since $x_0 \in O_{\mathrm{II}\lambda} \subset X - \bigcup_n O_{\mathrm{I}n} \subset X - \bigcup_i \{t \mid t \leqq b_i\}$, we can choose the last point of $B \cap \{t \mid t \leqq x_0\}$ by the condition (ii), and denote it by $b_{i(x_0)}$. Then, to prove that $O_{\mathrm{II}\lambda}$ is an open set, it suffices to show that $O_{\mathrm{II}\lambda} = M_{x_0} - \{t \mid t \leqq b_{i(x_0)}\}$. To do this, suppose that there exists a point $x \in M_{x_0} - \{t \mid t \leqq b_{i(x_0)}\}$ such that $x \in O_{\mathrm{I}j}$ for some j. Let $M_{b_{i_j}}$ be the member of $\{M_{b_{i_n}} \mid n = 1, 2, \dots\}$ containing the set $O_{\mathrm{I}j}$.

In case $x_0 \leqq x$, then $x_0 \in \{t \mid t \leqq x\} \subset M_{b_{i_j}} \subset \bigcup_n O_{\mathrm{I}n}$, contradicting $x_0 \in O_{\mathrm{II}\lambda} \subset X - \bigcup_n O_{\mathrm{I}n}$. In case $b_{i(x_0)} < x < x_0$, we denote by b the g.l.b. of x_0 and a point of $M_{b_{i_j}} - M_{x_0}$. Then we have $x \leqq b < x_0$, contradicting the definition of $b_{i(x_0)}$. Therefore $M_{x_0} - \{t \mid t \leqq b_{i(x_0)}\} \subset X - \bigcup_n O_{\mathrm{I}n}$. Thus, by the definition of $O_{\mathrm{II}\lambda}$, we have $M_{x_0} - \{t \mid t \leqq b_{i(x_0)}\} \subset O_{\mathrm{II}\lambda}$.

Conversely, let y be any point of $O_{\mathrm{II}\lambda}$. Since M_{x_0} is a maximal chain, there exists a point $z \in M_{x_0} - \{t \mid t \leqq b_{i(x_0)}\}$ such that $y < z$. Then $y \in \{t \mid t \leqq z\} \subset M_{x_0}$. On the other hand,

$$y \in O_{\mathrm{II}\lambda} \subset X - \bigcup_n O_{\mathrm{I}n} \subset X - \bigcup_i \{t \mid t \leqq b_i\},$$

and hence $y \notin \{t \mid t \leqq b_{i(x_0)}\}$. Therefore $y \in M_{x_0} - \{t \mid t \leqq b_{i(x_0)}\}$.

Thus $O_{\mathrm{II}\lambda} = M_{x_0} - \{t \mid t \leqq b_{i(x_0)}\}$ holds.

Now we consider the collection $\{O_{\mathrm{I}n} \mid n = 1, 2, \dots\} \cup \{O_{\mathrm{II}\lambda} \mid \lambda \in \Lambda\}$. Then this collection is a covering of (X, P) whose members are mutually disjoint, open and linearly ordered subsapce. Therefore, by Lemma 4 and Theorem 2, (X, P) is an S_4 space.

Thus, in case I, Theorem 3 is proved.

Case II: X has only one end point.

We denote this end point by e and put $M_e = \{t \mid t \leqq e\}$. Note that $M_e - e$ is an open set but in general, M_e is not so.

Let b_{i_1} be the point with the least index of the branch points of X which belong to M_e, and put $M_{b_{i_1}} = M_e$. By the same process as in case I, we construct a collection $\{O_{\mathrm{I}n} \mid n = 2, 3, \dots\}$ of mutually disjoint, open and linearly ordered sets.

Furthermore, we consider the collection $\{O_{\mathrm{II}\lambda} \mid \lambda \in \Lambda\}$ of all maximal chains of $X - (M_e \cup (\cup \{O_{\mathrm{I}n} \mid n = 2, 3, \dots\}))$. Note that the members of $\{O_{\mathrm{II}\lambda} \mid \lambda \in \Lambda\}$ are also mutually disjoint open sets.

To prove that (X, P) is an S_4 space, let $\{K_\alpha \mid \alpha \in A\}$ be a given covering of (X, P) by mutually disjoint compact subsets. Let $\alpha(e)$ be the element of A such that $e \in K_{\alpha(e)}$.

Case (II.1): $K_{\alpha(e)} = \{e\}$. Then the collection whose members are $M_e - e$ and all members of $\{O_{\mathrm{I}n} \mid n = 2, 3, \dots\} \cup \{O_{\mathrm{II}\lambda} \mid \lambda \in \Lambda\}$ is a covering of $X - e$, and $\{K_\alpha \mid \alpha \in A - \alpha(e)\}$ is also a covering of $X - e$. Therefore, by Lemma 4 and Theorem 2, $X - e$

is an S_4 space, and hence we can get a selection f for $\{K_\alpha|\alpha\in A-\alpha(e)\}$.

Now, we define a function g of $\{K_\alpha|\alpha\in A\}$ in X by

$$g(K_{\alpha(e)})=e,$$

$$g(K_\alpha)=f(K_\alpha)\quad\text{for each}\quad K_\alpha\in\{K_\alpha|\alpha\in A-\alpha(e)\}.$$

To show that the function g is continuous, let K_α be a given element of $\{K_\alpha|\alpha\in A\}$ and V be any open neighborhood of $g(K_\alpha)$.

If $K_\alpha=\{e\}$, then the following clearly holds:

$K_\alpha\in\langle V\rangle$,

$$g(K_{\alpha'})\in V\qquad\text{for every}\quad K_{\alpha'}\in\{K_{\alpha'}|\alpha\in A\}\quad\text{such that}\quad K_{\alpha'}\in\langle V\rangle.$$

If $K_\alpha\neq\{e\}$, then, since f is a selection for $\{K_\alpha|\alpha\in A-\alpha(e)\}$, we can choose an open set $\langle U_1,\dots,U_k\rangle$ of $2^{(X,P)}$ such that

$$K_\alpha\in\langle U_1,\dots,U_k\rangle,$$

$$g(K_{\alpha'})=f(K_{\alpha'})\in V-e\subset V\qquad\text{for every}\quad K_{\alpha'}\in\{K_\alpha|\alpha\in A\}$$
$$\text{such that}\quad K_{\alpha'}\in\langle U_1,\dots,U_k\rangle.$$

Therefore g is continuous, and hence g is a selection for $\{K_\alpha|\alpha\in A\}$.

Case (II.2): $K_{\alpha(e)}\neq\{e\}$. We establish a well order for the collection $\{M_e-e\}\cup\{O_{\mathrm{I}n}|n=2,3,\dots\}\cup\{O_{\mathrm{II}\lambda}|\lambda\in\Lambda\}$ such that the first member in the well order meets the set $K_{\alpha(e)}-e$, and, for simplicity, we denote this well ordered collection by $\{O_\gamma|\gamma\in\Gamma\}$. For every $K_\alpha\in\{K_\alpha|\alpha\in A\}$, let denote by $O_{\gamma(\alpha)_1}$ the first member of $\{O_\gamma|\gamma\in\Gamma\}$ such that $K_\alpha\cap O_\gamma\neq\phi$. By Lemma 3, there exists the first point x_α of $O_{\gamma(\alpha)_1}\cap K_\alpha$.

We define a function h of $\{K_\alpha|\alpha\in A\}$ in X by

$$h(K_\alpha)=x_\alpha\qquad\text{for each}\quad K_\alpha\in\{K|\alpha\in A\}.$$

To show that the function h is continuous, let K_α be a given element of $\{K_\alpha|\alpha\in A\}$ and V any open neighborhood of x_α.

Then, by Remark to Lemma 2, we can choose an open interval I of $O_{\gamma(\alpha)_1}$ such that $x_\alpha\in I\subset V\cap O_{\gamma(\alpha)_1}$.

If $K_\alpha=K_{\alpha(e)}$, then

$$K_\alpha\in\langle I,\ X-(O_{\gamma(\alpha)_1}\cap\{t|t\leqq x_\alpha\})\rangle,$$

$$h(K_{\alpha'})=x_{\alpha'}\in V\qquad\text{for every}\quad K_{\alpha'}\in\{K_\alpha|\alpha\in A\}\quad\text{such that}$$
$$K_{\alpha'}\in\langle I,\ X-(O_{\gamma(\alpha)_1}\cap\{t|t\leqq x_\alpha\}).$$

If $K_\alpha\neq K_{\alpha(e)}$, K_α intersects only finitely many members of $\{O_\gamma|\gamma\in\Gamma\}$. Let denote $\{O_{\gamma(\alpha)_1}, O_{\gamma(\alpha)_2},\dots, O_{\gamma(\alpha)_k}\}$, where $\gamma(\alpha)_1<\gamma(\alpha)_2<\cdots<\gamma(\alpha)_k$, the above finite members. Then we have

$K_\alpha \in \langle I, O_{\gamma(\alpha)_1} - \{t | t \leqq x_\alpha\}, \quad O_{\gamma(\alpha)_2}, \dots, O_{\gamma(\alpha)_k} \rangle$.

$h(K_{\alpha'}) = x_{\alpha'} \in I \subset V$ for every $K_{\alpha'} \in \{K_\alpha | \alpha \in A\}$

such that $K_{\alpha'} \in \langle I, O_{\gamma(\alpha)_1} - \{t \mid t \leqq x_\alpha\}, \quad O_{\gamma(\alpha)_2}, \dots, O_{\gamma(\alpha)_k} \rangle$.

Therefore g is continuous, and hence g is a selection for $\{K_\alpha \mid \alpha \in A\}$. Thus, in case II, Theorem 3 is proved.

Case III; X has only a finite number of end points $\{e_i | i = 1, \dots, n\}$ and $X = \bigcup_{i=1}^n \{t | t \leqq e_i\}$. Note that $e_i \notin \{t | t \leqq e_j\}$ for any $i \neq j$.

We put $M_{e_i} = \{t | t \leqq e_i\}$, then M_{e_i} is obviously a closed set. Moreover, since M_{e_i} is the union of two open sets $\{t | t < e_i\}$ and $X - \bigcup_j \{M_{e_j} | j = 1, 2, \dots, n$ and $j \neq i\}$, M_{e_i} is an open set.

Put $O_1 = M_{e_1}$.

Next, for any i, j with $1 \leqq i < j \leqq n$, let denote by $b_{(e_j, e_i)}$ the g.l.b. of e_j and e_i. We put

$$O_j = M_{e_j} - \bigcup_{i=1}^{j-1} \{t \mid t \leqq b_{(e_j, e_i)}\}.$$

In this way, we obtain the collection $\{O_j | j = 1, \dots, n\}$. Then the collection is a covering of (X, P) by mutually disjoint, open and linearly ordered sets.

Therefore, by Lemma 4 and Theorem 2, (X, P) is an S_4 space.

Thus the proof of Theorem 3 is complete.

Corollary. *Let (X, P) be a tree with the partial order topology. Assume that (X, P) is a Hausdorff space, and that X has only a finite number of branch points. Then (X, P) is an S_4 space.*

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