On Some Structures Defined on Dominant Vector Bundles

By

Yoshihiro ICHIYŌ and Radu MIRON
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§ 0. Introduction.

About ten years ago, Miron, the second author of the present paper, put forward to study the differential geometry of vector bundles ([6], [7])\textsuperscript{1).} He and his collaborators investigated it actively and made clear the structure of the modern Finsler manifolds. Now, in the present paper, we will investigate some structures defined on dominant vector bundles. Here the dominant vector bundle means a vector bundle whose dimension of each fiber is greater than or equals to the dimension of the base manifold. In this case, the theory of $f$-structures and the classical König connections play an important role.

Now, throughout the paper, we settle to use the indices as follows:

$$
\begin{align*}
&\{a, b, c,\ldots, s, t, u\} \text{ run over the range } \{1, 2, 3,\ldots, n\}, \\
&\{x, y \text{ and } z\} \text{ run over the range } \{n + 1, n + 2,\ldots, m\}, \\
&\{\alpha, \beta, \gamma,\ldots, \lambda, \mu, \nu,\ldots\} \text{ run over the range } \{\bar{1}, \bar{2}, \bar{3},\ldots, \bar{m}\}, \\
&\{A, B, C,\ldots, P, Q, R,\ldots\} \text{ run over the range } \{1, 2, 3,\ldots, n; \bar{1}, \bar{2}, \bar{3},\ldots, \bar{m}\}.
\end{align*}
$$

§ 1. Almost dominant tangent structures and non-linear connections.

Let $\xi=(E, \pi, M)$ be a dominant vector bundle ([6], [7]). That is to say, $M$ is an $n$-dimensional manifold, $E$ is a vector bundle over $M$ whose fibres are $m (\geq n)$-dimensional vector spaces and $\pi$ is the natural projection from $E$ to $M$. Let $\{U, x^i\}$ be any local coordinate neighbourhood of $M$ and $(x^a)=(x^i, y^a)$ is a canonical coordinate system of $\pi^{-1}(U)$. Let $\{\bar{U}, \bar{x}^i\}$ be another local coordinate neighbourhood of $M$ and $(\bar{x}^a)=(\bar{x}^i, \bar{y}^a)$ is that of $\pi^{-1}(\bar{U})$. If $U \cap \bar{U}=\emptyset$, then, in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$, we have $\bar{x}^i=\tilde{\bar{x}}^i(x^a)$ and $\bar{y}^a=M_{\bar{y}}^a(x^a)\nu^\beta$, where $M_{\bar{y}}^a$ is the function of $x^i$ alone and det $|M_{\bar{y}}^a|=0$. Let us put $(M_{\bar{y}}^a)^{-1}=(\bar{M}_{\bar{y}}^a)$. The Jacobian matrix is given by

\textsuperscript{1)} Numbers in brackets refer to the references at the end of the paper.
Let $B^i_1 \hat{\partial}_i \otimes dx^i$ be a tensor field on $E$. This is a tensor field on $E$ if and only if $B^i_1$, which is defined in each $\pi^{-1}(U)$, satisfies the relation $B^i_2 \hat{\partial}_i \bar{x}^m = M^i_2 B^i_1$ in $\pi^{-1}(U) \cap \pi^{-1}(U')$. Here, $(B^i_1)$ is called the composite tensor on $M$. In the following, to avoid the confusion of notations, we denote by $E(M)$ the total space $E$ of the dominant vector bundle $\xi$, and call $E(M)$ simply a vector bundle.

In this paper we write the partition of a $(m+n)$-squared matrix as

$$
(m^i_\mu) = 
\begin{pmatrix}
m^i_\mu
m^i_\nu
m^i_\lambda
m^i_\kappa
\end{pmatrix}
$$

or

$$
(m^i_\kappa) = 
\begin{pmatrix}
m^i_\mu
m^i_\nu
m^i_\lambda
m^i_\kappa
\end{pmatrix}.
$$

Now we assume that the manifold $M$ admits a composite tensor field $(B^i_1)$ such that the rank of the matrix $(B^i_1)$ is $n$. If we put $B=(B^i_1)$ and $Q=(0,0,0,0)$ then $Q$ is the matrix of components of the tensor field $B^i_1 \hat{\partial}_i \otimes dx^i$ with respect to the canonical coordinate $(x^i, y^i)$ and $Q$ satisfies $Q^2=0$ and rank $Q=n$.

From the analogy of the case of tangent bundles ([2], [4]), we call $Q$ an almost dominant tangent structure and abbreviate it by a.d.t.str. hereafter. Now $Q$ gives $E(M)$ a $(1,1)$-tensor field. So we may calculate the Nijenhuis tensor $N_Q(U, V)$ of $Q$ defined by

$$
N_Q(U, V) = [Q(U, V) + Q^2(U, V) - Q[Q(U, V)] - Q[U, V] - Q[U, QV],
$$

$U$ and $V$ being arbitrary vector fields in $E(M)$. By using the relations $Q^2=0$, $Q(\hat{\partial}_i, \hat{\partial}_j)=B^j_1 \hat{\partial}_i$ and $Q(\hat{\partial}_i, \hat{\partial}_j)=0$, we have $N_Q(\hat{\partial}_i, \hat{\partial}_j)=(B^j_1 \hat{\partial}_i B^j_1 - B^j_1 \hat{\partial}_j B^j_1) \delta_{\mu} = 0$ and $N_Q(\hat{\partial}_i, \hat{\partial}_j) = 0$. Hence we obtain

**Theorem 1.** Let $Q = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ be an almost dominant tangent structure assigned on a vector bundle $E(M)$. Then, the structure $Q$ is integrable if and only if $B^j_1 \hat{\partial}_i B^j_1 - B^j_1 \hat{\partial}_j B^j_1 = 0$ holds good. If the components of the composite tensor $B^i_1$ are functions of $x^i$ alone, the structure $Q$ is always integrable.

If an a.d.t.str. is integrable, it is called a dominant tangent structure.

Putting $Y^*_i = B^*_i \hat{\partial}_i$ in each $\pi^{-1}(U)$, we see that the condition $N_Q=0$ is equivalent to $[Y^*_i, Y^*_j]=0$. In $\pi^{-1}(U) \cap \pi^{-1}(U')$, it is evident that $Y^*_i = \frac{\partial x^a}{\partial x^i} Y^*_a$. Then the distribution spanned by $\{Y^*_i\}$ is a global $n$-dimensional distribution on $E(M)$. Thus we obtain

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1) The notations $\partial_i$ and $\hat{\partial}_i$ stand for $\partial/\partial x^i$ and $\partial/\partial y^i$ respectively.
Corollary. If a vector bundle $E(M)$ admits a dominant tangent structure, the $n$-dimensional distribution spanned by $\{Y_i^*\}$ is integrable.

Next, let there be given a non-linear connection $N$ on our vector bundle $E(M)$ ([6]). That is to say, $E(M)$ admits the quantities $N_i^*(x, y)$ in each $\pi^{-1}(U)$ which satisfy
\[
\partial_a N_i^* = M_a^b N_b^* - \partial_a M_a^* y^* \text{ in } \pi^{-1}(U) \cap \pi^{-1}(\overline{U}).
\]
In this case we can put
\[
X_i = \partial_i - N_i^* \partial_a \quad \text{in each } \pi^{-1}(U),
\]
which satisfy $X_i = \partial_i \tilde{x}^m X_m$ in each $\pi^{-1}(U) \cap \pi^{-1}(\overline{U})$. Hence the distribution spanned by $\{X_i\}$ is a global n-dimensional distribution, which we call the horizontal distribution. If we put $Y_\lambda = \partial_\lambda$ the distribution spanned by $\{Y_\lambda\}$ is an $m$-dimensional distribution and is nothing but a fibre itself and is called the vertical distribution. Of course, $\{X_\lambda, Y_\lambda\}$ becomes a local frame in $\pi^{-1}(U)$, which we call the fundamental frame.

Now let us put
\[
G_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix} \mid a \in GL(n, R), c \in GL(m-n, R), b \in M_{n,m-n} \right\}.
\]
Then $G_1$ is a linear Lie group.

Now, in each $\pi^{-1}(U)$, we can choose $\{Y_1^+, \ldots, Y_n^\ast\}$ out of $\{Y_\lambda\}$ such that $\{Y_1^+, \ldots, Y_n^+, Y_{n+1}^*, \ldots, Y_m^*\}$ forms a local frame of the vertical distribution. Then $\{X_\lambda, Y_1^+\}$ becomes a local frame in $\pi^{-1}(U)$, and satisfies, in $\pi^{-1}(U) \cap \pi^{-1}(\overline{U})$,
\[
\overline{X}_i = \frac{\partial x^m}{\partial \overline{x}^i} X_m, \quad Y_1^+ = \frac{\partial x^m}{\partial \overline{y}^1} Y_m^* \quad \text{and} \quad \overline{Y}_i^* = b_{\lambda} Y_m^* + C_{\lambda} Y_1^*. 
\]
Thus we obtain "A vector bundle which is assigned a composite tensor $B_i^\lambda$ and a non-linear connection $N_i^*$ admits the $G_1$-structure in the sense of the theory of $G$-structures". In the next section we shall consider the converse of this fact.

§ 2. $G$-structures defined on $E(M)$.

On a vector bundle $E(M)$, the natural frame $\{\partial_\lambda, \partial_\lambda\}$ in each local coordinate neighbourhood satisfies
\[
\partial / \partial x^i = \partial_i \overline{x}^m \partial / \partial \overline{x}^m + \partial_i M_a^\lambda y^\sigma \partial / \partial \overline{y}^\lambda, \quad \partial / \partial y^\lambda = M_{\overline{y}^1} \partial / \partial \overline{y}^\lambda
\]
in each $\pi^{-1}(U) \cap \pi^{-1}(\overline{U})$. Therefore, if we put
\[
G_0 = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \mid A \in GL(n, R), B \in GL(m, R), C \in M_{m,n} \right\},
\]
then we see that $G_0$ is a linear Lie group and $E(M)$ admits a $G_0$-structure in the sense of $G$-structures and, moreover, the $G_0$-structure is integrable. So we denote
by \( \mathcal{P}_0 \) this integrable \( G_0 \)-structure and call it the standard \( G_0 \)-structure on \( E(M) \).

Now let us consider, on \( E(M) \), a \( G \)-structure which is a reduction of \( \mathcal{P}_0 \). That is, let \( G \) be a Lie subgroup of \( G_0 \) and let \( E(M) \) admit such the \( G \)-structure that the local frame adapted to the \( G \)-structure is, at the same time, adapted to the structure \( \mathcal{P}_0 \). In the paper [4] we have named these \( G \)-structures \( G \)-structures depending on \( \mathcal{P}_0 \). In order that an adapted frame \( \{ W_A \} \) to the \( G \)-structure under consideration be depending on \( \mathcal{P}_0 \) it is necessary and sufficient that \( W_i = a_i \partial/\partial x^i + b_i \partial/\partial y^k \) and \( W_\xi = C_\xi \partial/\partial y^\rho \) hold good in each \( \pi^{-1}(U) \).

In this paper we treat, for \( G \)-structures on \( E(M) \), only the \( G \)-structures depending on \( \mathcal{P}_0 \).

Now we put

\[
Q_0 = \begin{pmatrix} 0, & 0, & 0 \\ E_n, & 0, & 0 \\ 0, & 0 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} E_n, & 0, & 0 \\ 0, & -E_n, & 0 \\ 0, & 0, & -E_{m-n} \end{pmatrix}.
\]

The straightforward calculation gives us

**Lemma 1.** Let \( P \) be an element of \( GL(n+m, R) \). The conditions \( P Q_0 = Q_0 P \) and \( P P_0 = P_0 P \) are satisfied if and only if \( P \in G_1 \).

Now, let us assume that \( E(M) \) admits a \( G_1 \)-structure depending on \( \mathcal{P}_0 \), which we denote by \( \mathcal{P}_1 \) hereafter. By the definition, the local frame \( \{ W_A \} \) adapted to \( \mathcal{P}_1 \) is also adapted to \( \mathcal{P}_0 \). So, \( \{ W_A \} \) can be written, in each \( \pi^{-1}(U), \ (x^i, y^k) \), as \( W_A = \Phi^A_0 \partial_B \) where \( \Phi^A_0 \in G_0 \), that is, \( W_a = \gamma^A B \partial_i + \sigma^A \partial_j \) and \( W_\xi = \tau^A_\xi \partial_\rho \). We put \( \Phi = (\Phi^A_0) = \begin{pmatrix} \gamma & 0 \\ \sigma & \tau \end{pmatrix} \) where \( \gamma = (\gamma^A_0), \ \sigma = (\sigma^A_0) \) and \( \tau = (\tau^A_\xi) \). Then we have \( \Phi^{-1} = \begin{pmatrix} \gamma^{-1} & 0 \\ -\tau^{-1} \sigma \gamma^{-1} & \tau^{-1} \end{pmatrix} \). Putting \( E'(m,n) = \begin{pmatrix} E_n \\ 0 \end{pmatrix} \) and \( Q = \Phi Q_0 \Phi^{-1} \), we see \( Q = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \). Since \( E(M) \) admits the \( G_1 \)-structure \( \mathcal{P}_1 \), as is well-known, \( Q \) is independent to the choice of the adapted frame \( \{ W_A \} \) and gives \( E(M) \) a global \( (1, 1) \)-tensor field. Now we put \( \tau E \gamma^{-1} = B = (B^A_\xi) \). The relation \( QJ = JQ \) shows us that \( B^A_\xi \) is a composite tensor field of rank \( n \). Of course, \( Q^2 = 0 \) holds true. Next, let us put \( P = \Phi P_0 \Phi^{-1} \). It is evident that \( P \) is also a global \( (1, 1) \)-tensor field on \( E(M) \) and satisfies \( P^2 = E_{n+m} \), that is, an almost product structure. Moreover it is easy to see that \( P = \begin{pmatrix} E_n & 0 \\ 2 \sigma \gamma^{-1} & -E_m \end{pmatrix} \) with respect to the canonical coordinate \( \{ \pi^{-1}(U), \ (x^i, y^k) \} \). And the relation \( \bar{P}J = JP \) shows us that \( N = -\sigma \gamma^{-1} \) is a non-linear connection. Thus, combining these results with the one stated in the last of \$1 \$, we obtain

**Theorem 2.** In order that a vector bundle \( E(M) \) admits a \( G_1 \)-structure depending on \( \mathcal{P}_0 \), it is necessary and sufficient that \( E(M) \) admits an almost dominant
tangent structure $Q$ (or a composite tensor $B_i^j$) and a non-linear connection $N_i^j$.

It is clear that the Lie algebra $\mathfrak{g}_1$ of the Lie group $G_1$ is given by
\[
\mathfrak{g}_1 = \left\{ \begin{pmatrix} p & 0 & 0 \\ 0 & p & r \\ 0 & 0 & q \end{pmatrix} \middle| p \in \mathfrak{g}(n, R), q \in \mathfrak{g}(m-n, R), r \in M_{n,m-n} \right\}.
\]

Let $\Gamma$ be a $G$-connection relative to the structure $\mathfrak{g}_1$, denote by $\mathcal{F}$ the covariant derivative with respect to the $\Gamma$ and let $\{W_A\}$ be a local frame adapted to the structure $\mathfrak{g}_1$. Then we see that $F_{0}W_A = \Gamma_{AB}^C W_C U^B$ where $\Gamma_{AB}^C U^B \in \mathcal{F}$ for any $U = U^B W_B$. From the last equation, we have $\Gamma_{jA}^j = \Gamma_{jA}^j = 0$, $\Gamma_{jA}^j = 0$, $\Gamma_{jA}^j = 0$ and $\Gamma_{jA}^j = 0$. Moreover the tensor $Q$ and $P$ have such components as $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} E_n & 0 & 0 \\ 0 & -E_m \end{pmatrix}$ with respect to any adapted frame $\{W_A\}$. So, using these relations, we can easily verify that $FP = 0$ and $FP = 0$ hold true. Hence, $\Gamma$ leaves the vertical distribution parallel and the horizontal distribution, too.

Conversely, let $\Gamma$ be a linear connection on $E(M)$ satisfying $FP = 0$ and $FP = 0$. Then, first, $FP = 0$ leads us to $\Gamma_{jA}^j = 0$, $\Gamma_{jA}^j = 0$, and $\Gamma_{jA}^j = \Gamma_{jA}^j$. Moreover the condition $FP = 0$ leads us to $\Gamma_{jA}^j = 0$. Therefore, $\Gamma$ becomes a $G$-connection relative to the structure $\mathfrak{g}_1$. Thus we obtain

**Theorem 3.** Let us assume that a vector bundle $E(M)$ admits a $G_1$-structure depending on $\mathfrak{g}_1$. A linear connection on $E(M)$ is a $G$-connection relative to the $G_1$-structure if and only if $FP = 0$ and $FP = 0$ hold good.

**[Remark]** The theory of $G$-structures teaches us that there always exists a $G$-connection relative to a $G$-structure ([10]). So, if $E(M)$ admits the structure $\mathfrak{g}_1$, $E(M)$ also admits, at least, a linear connection satisfying $FP = 0$ and $FP = 0$.

§ 3. Local expressions.

First, let us examine the components $\Gamma_{hC}^g$ of a linear connection $\Gamma$ on $E(M)$ with respect to the fundamental frame $\{Z_a\} = \{X_\mu, Y_\nu\}$ in $\pi^{-1}(U)$. Of course, $F_{Z_a}Z_b = \Gamma_{hC}^g Z_a$. In $\pi^{-1}(U)$, also is the relation $F_{Z_a}Z_b = \Gamma_{hC}^g Z_a$. Hence, using the relations
\[
X_j = \partial_j \bar{X}^c X_c \quad \text{and} \quad Y_\lambda = M_\lambda^\gamma Y_\gamma \quad \text{in} \quad \pi^{-1}(U) \cap \pi^{-1}(\bar{U}),
\]
we obtain
\[
\begin{align*}
\Gamma_{hC}^g \partial_j \bar{X}^c \partial_k \bar{X}^c &= -\Gamma_{jk}^m \partial_m \bar{X}^a - \partial_j \partial_k \bar{X}^a, \\
\Gamma_{hC}^g \partial_j \bar{X}^a M_\lambda^\nu &= \Gamma_{jk}^m M_\lambda^\nu - \partial_j M_\lambda^\nu, \\
\Gamma_{hC}^g \partial_j \bar{X}^b \partial_k \bar{X}^c &= \Gamma_{jk}^m \partial_m \bar{X}^b, \\
\Gamma_{hC}^g \partial_j \bar{X}^b M_\lambda^\nu &= \Gamma_{jk}^m M_\lambda^\nu - \partial_j \partial_m \bar{X}^b, \\
\Gamma_{hC}^g \partial_j \bar{X}^b M_\lambda^\nu &= \Gamma_{jk}^m M_\lambda^\nu - \partial_j \partial_m \bar{X}^b, \\
\Gamma_{hC}^g \partial_j \bar{X}^b M_\lambda^\nu &= \Gamma_{jk}^m M_\lambda^\nu - \partial_j \partial_m \bar{X}^b.
\end{align*}
\]
These relations, however, have been already shown by Miron ([6]). From these we see that $\Gamma^i_j(x, y)$ undergoes the transformation law of a linear connection of a base manifold, $\Gamma^\mu_{ij}(x, y)$ undergoes the transformation law of a König connection (or a dominant affine connection or a hyperconnection ([1], [3], [5], [6], [8], [10])) and the other components are the composite tensors.

The a.d.t.str. $Q$ operates as $Q(Y_x) = Q(\hat{\delta}_i) = 0$ and $Q(X_i) = Q(\hat{\delta}_i) - N^i_\mu Q(\hat{\delta}_\mu) = Q(\hat{\delta}_i) = B^i_\mu \hat{\delta}_\mu$. So $Q$ has the components $\begin{pmatrix} 0 & 0 \\ B^i_\mu & 0 \end{pmatrix}$ with respect to the fundamental frame $\{X_i, Y_x\}$. Now, calculating the components of $\nabla Q$, we have

$$
\begin{align*}
F_{xj}Q^b_j = & \Gamma^b_{ij}B^i_j, \\
F_{xj}Q^a_j = & X_j(B^j_i) + \Gamma^b_{ij}B^i_j - \Gamma^b_{ij}B^i_m, \\
F_{y_j}Q^b_i = & \Gamma^b_{ia}B^a_i, \\
F_{y_j}Q^a_i = & Y_x(B^j_i) + \Gamma^b_{ia}B^i_j - \Gamma^b_{ia}B^i_m.
\end{align*}
$$

From the fact that the rank of $(B^j_i)$ is $n$, we obtain

**Theorem 4.** Let us assume that a vector bundle $E(M)$ admits a non-linear connection $N$ and an almost dominant tangent structure $Q$. Let $\Gamma^a_{bc}$ be the components of a linear connection $\Gamma$ on $E(M)$ with respect to the fundamental frame $\{X_i, Y_x\}$. Then $\Gamma$ satisfies $\nabla Q = 0$ if and only if $\Gamma^b_{ij} = 0$, $\Gamma^b_{ia} = 0$, $X(B^j_i) + \Gamma^b_{ij}B^i_j - \Gamma^b_{ij}B^i_m = 0$ and $Y_x(B^j_i) + \Gamma^b_{ia}B^i_j - \Gamma^b_{ia}B^i_m = 0$ hold good.

By virtue of Theorem 2, the assumption in Theorem 4 means that $E(M)$ admits a $G_1$-structure depending on $\mathcal{P}_0$, which we have denoted by $\mathcal{P}_1$. For the intelligibility, however, we call the structure $\mathcal{P}_1$ the $(N, Q)$-structure and the $G$-connection relative to the structure $\mathcal{P}_1$ the $(N, Q)$-connection. Of course, according to Theorem 3, the $(N, Q)$-connection is a linear connection on $E(M)$ satisfying $\nabla Q = 0$ and $\nabla P = 0$.

Here, we examine the condition $\nabla P = 0$ with respect to the frame $\{Z_x\} = \{X_i, Y_x\}$. Since $P(\hat{\delta}_i) = \hat{\delta}_i - 2N^j_\mu \hat{\delta}_\mu$ and $P(\hat{\delta}_i) = -\hat{\delta}_i$, we see $P(Y_x) = -Y_x$ and $P(X_i) = X_i$. Namely, $P$ has the components $\begin{pmatrix} E_n & 0 \\ 0 & -E_m \end{pmatrix}$ with respect to $\{Z_x\} = \{X_i, Y_x\}$. Now calculating the components of $\nabla P$, we obtain $F_A P^b_A = 0$, $F_A P^b_A = -2\Gamma^b_{CA}$, $F_A P^a_A = 2\Gamma^a_C A$ and $F_A P^a_A = 0$ where we put $F_A Z_x = F_A^A$. Therefore the condition $\nabla P = 0$ can be written as $\Gamma^b_{CA} = 0$ and $\Gamma^a_C A = 0$. Hence we obtain

**Theorem 5.** Let $E(M)$ be a vector bundle admitting an $(N, Q)$-structure and let $\Gamma^a_{bc}$ be components of a linear connection $\Gamma$ on $E(M)$ with respect to the fundamental frame $\{X_i, Y_x\}$. The condition for the $\Gamma$ to be an $(N, Q)$-connection is given by
\[
\begin{align*}
\Gamma^\mu_{ij} &= 0, \quad \Gamma^\mu_{ij} = 0, \quad \Gamma^i_{ij} = 0, \quad \Gamma^i_{ij} = 0, \\
X_j(B_i^j) + \Gamma_j^j B_i^j - \Gamma_i^j B_m^j &= 0, \\
Y_i(B_i^j) + \Gamma_i^j B_i^j - \Gamma_i^j B_m^j &= 0.
\end{align*}
\]

Concerning composite tensors, for example \(T^h_{ia}\), we use the following notations:

\[
T^h_{ij} = X_j(T^h_{ij}) + \Gamma^h_{ij} T^h_{ij} + \Gamma^h_{ij} T^h_{ij} - \Gamma^h_{ij} T^h_{ij} - \Gamma^h_{ij} T^h_{ij},
\]

\[
T^h_{ia} = Y_i(T^h_{ia}) + \Gamma^h_{ia} T^h_{ia} + \Gamma^h_{ia} T^h_{ia} - \Gamma^h_{ia} T^h_{ia} - \Gamma^h_{ia} T^h_{ia}.
\]

\section*{§ 4. An f-structure.}

In this section we use the notations defined in §2. First, we put

\[
J_0 = \begin{pmatrix}
0 & -E_n \\
E_n & 0
\end{pmatrix}, \quad G_2 = \begin{pmatrix}
a & 0 & 0 \\
0 & a & 0 \\
0 & 0 & b
\end{pmatrix}, \quad a \in \text{GL}(n, R), \quad b \in \text{GL}(m-n, R).
\]

The direct calculation and Lemma 1 lead us to

**Lemma 2.** Let \( P \) be an element of \( \text{GL}(n+m, R) \). The conditions \( PQ_0 = Q_0P, PP_0 = P_0P \) and \( PJ_0 = J_0P \) are satisfied if and only if \( P \in G_2 \).

Let us suppose that a vector bundle \( E(M) \) admits a \( G_2 \)-structure depending on \( \mathcal{B}_0 \) and \( \{W_A\} \) is a local frame adapted to the \( G_2 \)-structure, and put \( W_A = \Phi_0^a \delta_B \). Then \( F = \Phi J_0 \Phi^{-1} \) is independent to the choice of the local frame \( \{W_A\} \) adapted to the \( G_2 \)-structure, and \( F \) becomes a global, \( (1, 1) \)-tensor field on \( E(M) \). Moreover we can easily see that \( F \) is a tensor of rank 2 with respect to the fundamental frame \( \{X_i, Y_i\} \), as \( F = \Psi_0 \Psi^{-1} \). And we see \( F = \begin{pmatrix} 0 & -t^E \gamma^{-1} \\ \gamma^{-1} E^E \gamma^{-1} & 0 \end{pmatrix} \) with respect to \( \{X_i, Y_i\} \). In §2, we have put \( \tau E^{-1} = B = (B_i^j) \) and we have seen \( B_i^j \) is a composite tensor. Here let us put \( \gamma E^{-1} = C = (C_i^j) \). Then rank \( C = n \). Since \( F \) is a tensor field on \( E(M) \) and the Jacobian matrix from \( \{X_i, Y_i\} \) to \( \{X_i, Y_i\} \) is given by \( J = \begin{pmatrix} \partial_i \xi^i & 0 \\ 0 & M_{i}^{j} \end{pmatrix} \), so relation \( \tilde{F}J = JF \) leads us to \( \tilde{C}^j_i M_i^{j} = \xi_i \xi^i C_i^j \), that is, \( C_i^j \) is a composite tensor. From the definition, it follows that \( CB = \gamma E^{-1} \tau E^{-1} = E_n \), that is, \( C_i^j B_j^i = \delta_i^j \).

Conversely, let there be given a non-linear connection \( N_i^j \) and composite tensors \( B_i^j \) and \( C_i^j \) satisfying \( C_i^j B_j^i = \delta_i^j \) and rank \( (B_i^j) = \text{rank}(C_i^j) = n \). We define a (1, 1)-
tensor $F$ by $F = (F^i_j) = \begin{pmatrix} 0 & -C^n_i \\ B^j_i & 0 \end{pmatrix}$ with respect to the frame $\{X_i, Y_i\}$ in each $\pi^{-1}(U)$. Then $F$ becomes a global tensor field on $E(M)$ and operates as $F(X_i) = B^j_i Y_j = Y^*_i$, $F(Y_j) = -C^n_j X_m$, from which it follows that $F^3 + F = 0$. Thus $F$ gives $E(M)$ an $f$-structure of rank $2n$. Now consider the following simultaneous linear equation $C^i_j p^j = 0$. Since the rank of $(C^i_j)$ is $n$, there exist $(m-n)$ linearly independent solutions $p^i_j$. On putting $Y^*_i = p^i_j Y_j$, we can show that $\{Y^*_i, Y^*_j\}$ is a local frame of each vertical distribution. Then $\{X_i, Y^*_i, Y^*_k\}$ is a local frame in $\{\pi^{-1}(U), (x^i, y^a)\}$ by $\{X_i, Y^*_i, Y^*_k\}$, we see, in $\pi^{-1}(U) \cap \pi^{-1}(\overline{U})$, that $X_i = \partial_i \bar{x}^m X_m$, $Y^*_i = \partial_i \bar{x}^m Y^*_m$. Moreover, from $C^i_j p^j = 0$, we see $C^i_j (M^j_k p^k) = 0$. Then we have $M^i_j p^j = k^i_j p^k_j$ for certain $k^i_j$. That is, $p^i_j = M^i_j p^k_j k^i_j$. Then we see

$$Y^*_i = \bar{p}^i_j Y_j = M^i_j p^k_j k^i_j Y_j = p^k_j k^i_j Y_j = k^i_j Y^*_j.$$ 

Namely, $X_i = \partial_i \bar{x}^m X_m$, $Y^*_i = \partial_i \bar{x}^m Y^*_m$, $Y^*_y = k^i_j Y^*_j$ and $X_i = \partial_i N^i_j \partial_j$, $Y^*_i = B^i_j \partial_j$, $Y^*_y = p^i_j \partial_j$ hold. Therefore $E(M)$ admits a $G_2$-structure depending of $\mathcal{P}_0$. Thus we obtain

**Theorem 6.** In order that a vector bundle $E(M)$ admits a $G_2$-structure depending on $\mathcal{P}_0$, it is necessary and sufficient that $E(M)$ admits a non-linear connection $\nabla^1$ and composite tensors $B^1_i$ and $C^1_i$ satisfying $C^1_i B^1_j = \delta^1_i j$ and rank $(B^1 j) = \text{rank}(C^1 j) = n$. In this case, $E(M)$ admits an $f$-structure of rank $2n$.

For the integrability, we call the structure stated in Theorem 6 an $(N, Q, F)$-structure hereafter.

For the $f$-structure $F$, as is well-known ([9], [11]), $\mathcal{L} = -F^2$ and $\mathcal{M} = E_n + m - \mathcal{L}$ satisfy $\mathcal{L}^2 = \mathcal{L}$, $\mathcal{M}^2 = \mathcal{M}$, $\mathcal{L} \mathcal{M} = 0$, $\mathcal{M} \mathcal{L} = 0$ and $\mathcal{L} + \mathcal{M} = E_{n+m}$. That is, $\mathcal{L}$ and $\mathcal{M}$ are projection tensors. We denote by $D_1$ and $D_m$ the distributions defined by $\mathcal{L}$ and $\mathcal{M}$ respectively. Of course, dim $D_1 = 2n$ and dim $D_m = m - n$ and $T_p(E(M)) = (D_1)_p \oplus (D_m)_p$ for any point $p$ of $E(M)$.

With respect to $\{X_i, Y_j\}$, $F$ is represented by $F = \begin{pmatrix} 0 & -C^n_i \\ B^j_i & 0 \end{pmatrix}$. So, $\mathcal{L} = \begin{pmatrix} E & 0 \\ B & 0 \end{pmatrix}$ and $\mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & E - BC \end{pmatrix}$. Now we put $\mathcal{U} = \frac{1}{2} \begin{pmatrix} E & C \\ B & BC \end{pmatrix}$ and $\mathcal{V} = \frac{1}{2} \begin{pmatrix} E & -C \\ -B & BC \end{pmatrix}$. Then, these satisfy $\mathcal{U}^2 = \mathcal{U}$, $\mathcal{V}^2 = \mathcal{V}$, $\mathcal{U} \mathcal{V} = 0$, $\mathcal{V} \mathcal{U} = 0$, $\mathcal{U} + \mathcal{V} = \mathcal{L}$. Thus $\mathcal{U}$ and $\mathcal{V}$ are also projection tensors. Denoting by $D_a$ and $D_b$ the distributions defined by $\mathcal{U}$ and $\mathcal{V}$ respectively, we have $D_a \oplus D_b = D_1$ and $T_p(E(M)) = (D_a)_p \oplus (D_b)_p \oplus (D_m)_p$.

§ 5. $(N, Q, F)$-connections.

We consider successively the $(N, Q, F)$-structure in this section. The $f$-structure $F$ defined in the preceding section has the following components with respect to $\{X_i, Y_j\}$:

$$F = \begin{pmatrix} F^h_i & F^h_j \\ F^h_i & F^h_j \end{pmatrix} = \begin{pmatrix} 0 & -C^n_i \\ B^j_i & 0 \end{pmatrix}.$$
Now, let $\Gamma$ be an $(N, Q)$-connection on $E(M)$. Calculating $\overline{P}F$ with respect to this $\Gamma$, we obtain

$$\begin{align*}
F_{XJ}F^h_\mu = 0, & \quad F_{XJ}F^j_\mu = -C^h_{\mu l j}, & \quad F_{XJ}F^j_\mu = B^h_{\mu l j} = 0, & \quad F_{XJ}F^j_\mu = 0, \\
F_{Yx}F^h_\mu = 0, & \quad F_{Yx}F^j_\mu = -C^h_{\mu |h x}, & \quad F_{Yx}F^j_\mu = B^h_{\mu |h x} = 0, & \quad F_{Yx}F^j_\mu = 0.
\end{align*}$$

Therefore, on a vector bundle $E(M)$ admitting an $(N, Q, F)$-structure, a necessary and sufficient condition for the $(N, Q)$-connection to satisfy $\overline{P}F=0$ is that $C^h_{\mu l j} = 0$ and $C^h_{\mu |h x} = 0$ hold.

Here we call a linear connection on $E(M)$ satisfying $\overline{P}P=0, \overline{P}Q=0, \overline{P}F=0$ as an $(N, Q, F)$-connection. Then, according to Theorem 5 and the above result, we obtain

**Theorem 7.** Let $E(M)$ be a vector bundle admitting an $(N, Q, F)$-structure, and let $\Gamma^h_{\mu i}$ be components of a linear connection $\Gamma$ on $E(M)$ with respect to the fundamental frame $\{X_i, Y_i\}$. The condition for $\Gamma$ to be an $(N, Q, F)$-connection is given by

$$\begin{align*}
\Gamma^h_{\mu i} = 0, \quad \Gamma^h_{\mu \lambda} = 0, \quad \Gamma^i_{\mu j} = 0, \quad \Gamma^i_{\mu \lambda} = 0, \\
B^h_{\mu i} = 0, \quad B^i_{\lambda i} = 0, \quad C^h_{\mu j} = 0, \quad C^i_{\mu | i} = 0.
\end{align*}$$

[Remark] From the last 4 conditions in Theorem 7, $\Gamma^h_{\mu j}$ and $\Gamma^h_{\mu i}$ are determined as follows:

$$\begin{align*}
\Gamma^h_{\mu j} &= C^h_{\mu j}(X_j(B^i_\mu) + \Gamma^i_{\mu j}B^i_\mu), \\
\Gamma^h_{\mu i} &= C^i_{\mu i}(Y_i(B^i_\mu) + \Gamma^i_{\mu i}B^i_\mu).
\end{align*}$$


If an $(N, Q, F)$-structure satisfies, in each $\pi^{-1}(U)$,

$$B^i_\mu = B^i_\mu(x), \quad C^i_\mu = C^i_\mu(x) \quad \text{and} \quad N^i_\mu = \Gamma^i_{\mu i}(x)y^\sigma,$$

then the structure is said to be a König structure. That is to say, as for the König structure, $B^i_\mu, C^i_\mu$ and $\Gamma^i_{\mu i}$ are functions of $x^i$ alone and the non-linear connection is given by $N^i_\mu = \Gamma^i_{\mu i}(x)y^\sigma$. The last condition is globally well-defined. Because, taking account of $\Gamma^i_{\mu \alpha} \partial_{\alpha} \tilde{x}^\sigma M^\sigma_\alpha = \Gamma^i_{\mu \alpha} M^\sigma_\alpha - \partial_{\alpha} M^\sigma_\alpha$, we have $(\Gamma^i_{\sigma \alpha} y^\sigma)(\partial_{\alpha} \tilde{x}^\sigma = (\Gamma^i_{\sigma \alpha} y^\sigma) M^\sigma_\alpha - \partial_{\alpha} M^\sigma_\alpha)$, that is, $\Gamma^i_{\sigma \alpha} y^\sigma$ is a globally defined non-linear connection. In this case, $\Gamma^i_{\mu i}$ is a so-called König connection ([1], [3], [8], [10]) (or a dominant affine connection). Due to Theorem 1, Corollary and Theorem 6, it follows directly that

**Theorem 8.** In a vector bundle admitting a König structure, the almost dominant tangent structure $Q$ is integrable and the distribution spanned by $\{Y^*_i\}$ is also integrable.

From now on, we call this f-structure the f-structure derived from the König structure. Now let us examine the integrability conditions of this f-structure $F$. The Nijenhuis tensor $N_F$ of $F$ is given by

$$
$$

$U$ and $V$ being arbitrary vector fields in $E(M)$.

First, the condition for the distribution $D_m$ to be integrable is given by $N_F(\mathcal{M} U, \mathcal{M} V) = 0$ ([8], [10]). On the other hand, $F, \mathcal{M} F(E + F^2) = 0$, $\mathcal{M} X_i = (E + F^2)X_i = 0$ and $\mathcal{M} Y_j = (E + F^2)Y_j = Y_j - B^n X^n Y_j$. So it follows that $N_F(\mathcal{M} X_i, \mathcal{M} X_j) = 0$, $N_F(\mathcal{M} X_i, \mathcal{M} Y_j) = 0$ and $N_F(\mathcal{M} Y_j, \mathcal{M} Y_j) = 0$. Hence we obtain

Theorem 10. The distribution $D_m$ defined by the f-structure derived from a König structure is always integrable.

Next, the condition for the distribution $D_i$ to be integrable is given by $\mathcal{M} N_F(U, V) = 0$ ([9], [11]). By means of $\mathcal{M} F = 0$, we can rewrite this condition as $\mathcal{M} [F_U, F_V] = 0$. On the other hand, we see $[X_i, X_j] = R_{sij}^\beta y^s Y_\beta$ where

$$
R_{sij}^\beta = \partial_j \Gamma_{ai}^\beta - \partial_i \Gamma_{aj}^\beta + \Gamma_{aj}^\gamma \Gamma_{ai}^\gamma - \Gamma_{ai}^\gamma \Gamma_{aj}^\gamma.
$$

This $R_{sij}^\beta$ is called the curvature tensor of the König connection $\Gamma_{ai}^\beta(x)$. Now, direct calculation shows us

\[
\mathcal{M}[FX_i, FX_j] = \mathcal{M}[B_i^s X_s, B_j^s Y_p] = 0,
\]

\[
\mathcal{M}[FX_i, FY_j] = \mathcal{M}[B_i^s Y_s - C^s_m X_m] = (\Gamma_{cm}^s B_i^m B_j^p + \partial_m B_i^s C^p_m) \mathcal{M}(Y_p)
\]

\[
= C^s_m (\partial_m B_i^s + B_i^s \Gamma_{cm}) (\Gamma_{cm}^p - B_i^p C^p_m) Y_p,
\]

\[
\mathcal{M}[FY_i, FY_p] = \mathcal{M}[C^m_m X_m, C^p_i X_i] = C^s_m C^p_i \mathcal{M}[X_m, X_i]
\]

\[
= C^s_m C^p_i R_{cml}^s (\delta_m^i - B_i^h C^h_m) Y_p.
\]

Thus we obtain

Theorem 11. The distribution $D_i$ defined by the f-structure derived from a König structure is integrable if and only if

$$
\begin{align*}
(\delta_m^i - B_m^s C^m_s) (\partial_i B_p^m + \Gamma_{ij}^m B_j^m) &= 0, \\
(\delta_m^i - B_m^s C^m_s) R_{sij}^\beta &= 0,
\end{align*}
$$

where $R_{sij}^\beta$ is the curvature tensor of the König connection $\Gamma_{ai}^\beta(x)$. 
When the distribution $D_t$ is integrable and the induced almost complex structure $f'$ is complex analytic on each integral manifold of $D_t$, we say that the $f$-structure $F$ to be partially integrable. And the condition for the $f$-structure $F$ to be partially integrable is given by $N_F(\mathcal{L} U, \mathcal{L} V) = 0$ ([9], [11]). Since $F, \mathcal{L} = F$, we get


Then, we see

$$N_F(\mathcal{L} X_1, \mathcal{L} X_j) = \{X_1(B_j) + \Gamma^z_\alpha B_j^\alpha - X_j(B_1) - \Gamma^z_\beta B_1^\beta\} C^a_\alpha X_a - R_{\alpha j}^\beta B_m^\beta C^a_m Y_a,$$

$$N_F(\mathcal{L} X_1, \mathcal{L} Y_a) = C^m_\alpha \{X_m(B_1) + \Gamma^z_\alpha B_1^\alpha - B_1^\gamma C^1_\gamma (X_1(B_1) + \Gamma^z_\beta B_1^\beta) + R_{\alpha m}^\gamma Y_a\} Y_a,$$

$$N_F(\mathcal{L} Y_1, \mathcal{L} Y_j) = C^a_\alpha C^\beta_\mu \{X_1(B_m) + \Gamma^z_\alpha B_m^\alpha - X_m(B_1) - \Gamma^z_\beta B_1^\beta\} X_a + C^m_\alpha C^\beta_\mu R_{\alpha ml}^\gamma X_a.$$

Thus the condition under consideration can be written as

$$\begin{align*}
R_{\alpha j}^\beta &= 0, \\
C^h_\gamma \{(\partial_j B_1^\gamma + \Gamma^z_\alpha B_j^\alpha) - (\partial_j B_1^\gamma + \Gamma^z_\beta B_1^\beta)\} &= 0, \\
(\partial_j B_1^\gamma + \Gamma^z_\alpha B_j^\alpha) - B_m^\gamma C^1_\gamma (\partial_j B_1^\gamma + \Gamma^z_\beta B_1^\beta) &= 0.
\end{align*}$$

Substituting (2) into (3), we get

$$(\partial_j B_1^\gamma + \Gamma^z_\alpha B_j^\alpha) - B_m^\gamma C^1_\gamma (\partial_j B_1^\gamma + \Gamma^z_\beta B_1^\beta) = 0.$$  

Namely, we get

$$(\delta_j^\gamma - B_m^\gamma C^1_\gamma) (\partial_j B_1^\gamma + \Gamma^z_\beta B_1^\beta).$$

Conversely, the condition (2) and (4) lead us to (3). Hence we obtain

**Theorem 12.** The $f$-structure derived from a König structure is partially integrable if and only if

$$\begin{align*}
R_{\alpha j}^\beta &= 0, \\
(\delta_j^\gamma - B_m^\gamma C^1_\gamma) (\partial_j B_1^\gamma + \Gamma^z_\beta B_1^\beta) &= 0, \\
C^h_\gamma \{(\delta_j^\gamma B_1^\gamma + \Gamma^z_\alpha B_j^\alpha) - (\partial_j B_1^\gamma + \Gamma^z_\beta B_1^\beta)\} &= 0.
\end{align*}$$

Finally we look for the condition for $F$ to be integrable. An $f$-structure $F$ is said to be integrable when $N_F = 0$. Calculating the components of $N_F$, we get

$$N_F(X_1, X_j) = -R_{\alpha j}^\beta B_m^\beta Y_a$$

$$-C^h_\gamma \{(\partial_j B_1^\gamma + B^\gamma B_1^\gamma) - (\partial_j B_1^\gamma + \Gamma^z_\beta B_1^\beta)\} X_a,$$

$$N_F(X_1, Y_\mu) = \{C^m_\mu (\delta_m B_1^\gamma + \Gamma^z_\alpha B_j^\alpha) + B_m^\gamma (\partial_j C^1_\gamma - \Gamma^z_\beta C^1_\gamma)\} Y_a$$

$$-C^m_\mu R_{\alpha m}^\gamma C^1_\gamma X_a.$$
\[ N_F(Y_\lambda, Y_\mu) = \{ C^\alpha_\mu \left( \partial_m C_\mu^h - \Gamma_{\mu m}^\alpha C_\sigma^h \right) - C^\mu_\mu \left( \partial_m C_\mu^h - \Gamma_{\mu m}^\alpha C_\sigma^h \right) \} X_h + C^h_\mu R^\mu_{\tau ab} \gamma^\tau Y_\sigma. \]

Thus, the condition under consideration is given by

\[
\begin{align*}
R^\lambda_{\sigma i j} & \equiv 0, \\
C^m_\mu \left( \partial_m C_\mu^h - \Gamma_{\mu m}^\alpha C_\sigma^h \right) - C^m_\mu \left( \partial_m C_\mu^h - \Gamma_{\mu m}^\alpha C_\sigma^h \right) & = 0, \\
C^m_\mu \left( \partial_m B_\mu^h + \Gamma_{\mu m}^\alpha B_\mu^h \right) + B^a_\mu \left( \partial_a C_\mu^h - \Gamma_{\mu a}^\alpha C_\sigma^h \right) & = 0, \\
C^h_\lambda \left( \partial_j B_\mu^h + \Gamma_{\alpha j}^\lambda B_\mu^h \right) - \left( \partial_j B_\mu^h + \Gamma_{\alpha j}^\lambda B_\mu^h \right) & = 0.
\end{align*}
\]

Multiplying the second equation by \( B^i_j B^j_k \) and contracting with \( \lambda \) and \( \mu \), we get

\[ B^i_j \left( \partial_j C_\lambda^h - \Gamma_{\lambda j}^\alpha C_\sigma^h \right) - B^i_j \left( \partial_j C_\lambda^h - \Gamma_{\lambda j}^\alpha C_\sigma^h \right) = 0. \]

Taking account of \( C^i_\lambda B^j_\lambda = \delta^i_j \), we have

\[ C^h_\mu \left( \partial_j B^i_j + \Gamma_{\alpha j}^\lambda B^i_j \right) - \left( \partial_j B^i_j + \Gamma_{\alpha j}^\lambda B^i_j \right) = 0, \]

that is, the second condition includes the fourth condition. Therefore we obtain

**Theorem 13.** The \( f \)-structure derived from a König structure is integrable if and only if

\[
\begin{align*}
R^\lambda_{\sigma i j} & \equiv 0, \\
\left( \partial_m C_\mu^h - \Gamma_{\mu m}^\alpha C_\sigma^h \right) C^m_\mu - \left( \partial_m C_\mu^h - \Gamma_{\mu m}^\alpha C_\sigma^h \right) C^m_\mu & = 0, \\
\left( \partial_m B_\mu^h + \Gamma_{\beta m}^\lambda B_\mu^h \right) C^m_\beta + B^a_\mu \left( \partial_a C_\mu^h - \Gamma_{\mu a}^\alpha C_\sigma^h \right) & = 0.
\end{align*}
\]

§ 7. The standard connection.

Let there be given a König structure on a vector bundle \( E(M) \). Then, there exist a König connection \( \Gamma^\mu_\beta(x) \) and a non-linear connection \( N^\mu_\beta = \Gamma^\mu_\beta(x) y^\beta \).

Now let \( \Gamma \) be a linear connection on \( E(M) \) whose components with respect to the frame \( \{ X_\lambda, Y_\mu \} \) are \( \Gamma^\mu_\beta \). If we put

\[
\begin{align*}
\Gamma^\mu_\beta &= \Gamma^\mu_\beta(x) \quad (\Gamma^\mu_\beta(x) \text{ is the given König connection}), \\
\Gamma^\mu_\beta &= C^\mu_\beta \left( \partial_j B^i_j + \Gamma_{\alpha j}^\lambda B^i_j \right), \\
\text{the other components vanish.}
\end{align*}
\]

Due to the fact stated in §3 and Remark shown in §5, this is well-defined. Thus we can determine one linear connection on \( E(M) \). Concerning this linear connection, we get
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\[
\begin{align*}
&\left\{ \begin{array}{l}
V_{X_j} X_i = \Gamma_{i}^{h_j} X_h, \\
V_{X_j} Y_z = \Gamma_{z}^{h_j} Y_h,
\end{array} \right. \\
&\left\{ \begin{array}{l}
V_{Y_X} X_i = 0, \\
V_{Y_X} Y_{\mu} = 0.
\end{array} \right. \\
\end{align*}
\]

The linear connection \( \Gamma \) thus defined is called the standard connection derived from a König structure. For the standard connection, we get

\[
\begin{align*}
B_{i|j} = & \partial_j B_i^h + \Gamma_{\sigma j}^h B^\sigma - \Gamma_{i}^{\sigma} B^\sigma_m \\
= & (\delta_{\alpha}^\lambda - B^\lambda_m C_{\alpha}^m)(\partial_j B_i^\lambda + \Gamma_{\sigma j}^\lambda B^\sigma), \\
B_{i|j|\mu} = & 0, \\
C_{\mu j}^h = & \partial_j C_{\mu}^{h} - \Gamma_{\mu j}^h C_{\eta}^{c} + \Gamma_{\mu j}^h C_{\eta}^{m} \\
= & \partial_j C_{\mu}^{h} - \Gamma_{\mu j}^h C_{\eta}^{c} + C_{\mu}^{n} C_{\eta}^{c} \partial_j B_{m}^{c} + C_{\mu}^{m} C_{\eta}^{h} \Gamma_{\sigma j}^h B_{m}^{\sigma} \\
= & (\delta_{\alpha}^\lambda - B^\lambda_m C_{\alpha}^m)(\partial_j C_{\mu}^{h} - \Gamma_{\sigma j}^h C_{\eta}^{c}), \\
C_{\alpha}^{h|\mu} = & 0.
\end{align*}
\]

Therefore, with respect to the standard connection, we can rewrite Theorem 4, 7, 11 and 12 as follows:

**Theorem 14.** The standard connection derived from a König structure is an \((N, Q)\)-connection if and only if

\[
(\delta_{\alpha}^\lambda - B^\lambda_m C_{\alpha}^m)(\partial_j B_i^\lambda + \Gamma_{\sigma j}^\lambda B^\sigma) = 0.
\]

**Theorem 15.** The standard connection derived from a König structure is an \((N, Q, F)\)-connection if and only if

\[
\begin{align*}
\left\{ \begin{array}{l}
(\delta_{\alpha}^\lambda - B^\lambda_m C_{\alpha}^m)(\partial_j B_i^\lambda + \Gamma_{\sigma j}^\lambda B^\sigma) = 0, \\
(\delta_{\alpha}^\lambda - B^\lambda_m C_{\alpha}^m)(\partial_j C_{\mu}^{h} - \Gamma_{\sigma j}^h C_{\eta}^{c}) = 0.
\end{array} \right.
\end{align*}
\]

**Theorem 16.** If the distribution \( D_i \) defined by the \( f \)-structure derived from a König structure is integrable, the standard connection is an \((N, Q)\)-connection.

**Theorem 17.** In order that the \( f \)-structure derived from a König structure be partially integrable, it is necessary and sufficient that the standard connection satisfies

\[
\begin{align*}
&\left\{ \begin{array}{l}
R_{\sigma ij}^h = 0, \\
T_{ij}^h := \Gamma_{ij}^h - \Gamma_{ji}^h = 0, \\
(\delta_{\alpha}^\lambda - B^\lambda_m C_{\alpha}^m)(\partial_j B_i^\lambda + \Gamma_{\sigma j}^\lambda B^\sigma) = 0.
\end{array} \right.
\end{align*}
\]
References


