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On Some Structures Defined on Dominant Vector Bundles

By

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§ 0. Introduction.

About ten years ago, Miron, the second author of the present paper, put forward to study the differential geometry of vector bundles ([6], [7])[1]. He and his collaborators investigated it actively and made clear the structure of the modern Finsler manifolds. Now, in the present paper, we will investigate some structures defined on dominant vector bundles. Here the dominant vector bundle means a vector bundle whose dimension of each fiber is greater than or equals to the dimension of the base manifold. In this case, the theory of f-structures and the classical König connections play an important role.

Now, throughout the paper, we settle to use the indices as follows:

$$\begin{cases} a, b, c, \dots, s, t, u \text{ run over the range } \{1, 2, 3, \dots, n\}, \\ x, y \text{ and } z \text{ run over the range } \{n+1, n+2, \dots, m\}, \\ \alpha, \beta, \gamma, \dots, \lambda, \mu, \nu, \dots \text{ run over the range } \{\bar{1}, \bar{2}, \bar{3}, \dots, \bar{m}\}, \\ A, B, C, \dots, P, Q, R, \dots \text{ run over the range } \{1, 2, 3, \dots, n; \bar{1}, \bar{2}, \bar{3}, \dots, \bar{m}\}. \end{cases}$$

§ 1. Almost dominant tangent structures and non-linear connections.

Let $\xi=(E, \pi, M)$ be a dominant vector bundle ([6], [7]). That is to say, M is an n-dimensional manifold, E is a vector bundle over M whose fibres are m $(\geqq n)$-dimensional vector spaces and π is the natural projection from E to M. Let $\{U, x^i\}$ be any local coordinate neighbourhood of M and $(x^A)=(x^i, y^\alpha)$ is a canonical coordinate system of $\pi^{-1}(U)$. Let $\{\bar{U}, \bar{x}^i\}$ be another local coordinate neighbourhood of M and $(\bar{x}^A)=(\bar{x}^i, \bar{y}^\alpha)$ is that of $\pi^{-1}(\bar{U})$. If $U \cap \bar{U} \neq \phi$, then, in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$, we have $\bar{x}^i=\bar{x}^i(x^a)$ and $\bar{y}^\alpha=M^\alpha_\beta(x)y^\beta$, where M^α_β is the function of x^i alone and $\det|M^\alpha_\beta| \neq 0$. Let us put $(M^\alpha_\beta)^{-1}=(\overset{-1}{M}{}^\alpha_\beta)$. The Jacobian matrix is given by

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1) Numbers in brackets refer to the references at the end of the paper.

$$J=\left(\frac{\partial \bar{x}^A}{\partial x^B}\right)=\begin{pmatrix} \partial_j \bar{x}^i, & 0 \\ y^\sigma \partial_j M^\alpha_\sigma, & M^\alpha_\beta \end{pmatrix}^{1)}.$$

Let $B^\lambda_i \partial_\lambda \otimes dx^i$ be a tensor field on E. This is a tensor field on E if and only if B^λ_i, which is defined in each $\pi^{-1}(U)$, satisfies the relation $\bar{B}^\lambda_m \partial_i \bar{x}^m = M^\lambda_\sigma B^\sigma_i$ in $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$. Here, (B^λ_i) is called the composite tensor on M. In the following, to avoid the confusion of notations, we denote by $E(M)$ the total space E of the dominant vector bundle ξ, and call $E(M)$ simply a vector bundle.

In this paper we write the partition of a $(m+n)$-squared matrix as

$$(m^A_B)=\begin{pmatrix} m^i_j, & m^i_{\bar{j}}, & m^i_{\bar{y}} \\ m^{\bar{i}}_j, & m^{\bar{i}}_{\bar{j}}, & m^{\bar{i}}_{\bar{y}} \\ m^{\bar{x}}_j, & m^{\bar{x}}_{\bar{j}}, & m^{\bar{x}}_{\bar{y}} \end{pmatrix} \quad \text{or} \quad (m^A_B)=\begin{pmatrix} m^i_j, & m^i_\mu \\ m^\lambda_j, & m^\lambda_\mu \end{pmatrix}.$$

Now we assume that the manifold M admits a composite tensor field (B^λ_i) such that the rank of the matrix (B^λ_i) is n. If we put $B=(B^\lambda_i)$ and $Q=\begin{pmatrix}0, & 0\\ B, & 0\end{pmatrix}$ then Q is the matrix of components of the tensor field $B^\lambda_i \dot{\partial}_\lambda \otimes dx^i$ with respect to the canonical coordinate (x^i, y^λ) and Q satisfies $Q^2=0$ and rank $Q=n$.

From the analogy of the case of tangent bundles ([2], [4]), we call Q an *almost dominant tangent structure* and abbreviate it by *a.d.t.str.* hereafter. Now Q gives $E(M)$ a (1, 1)-tensor field. So we may calculate the Nijenhuis tensor $N_Q(U, V)$ of Q defined by

$$N_Q(U, V)=[QU, QV]+Q^2[U, V]-Q[QU, V]-Q[U, QV],$$

U and V being arbitrary vector fields in $E(M)$. By using the relations $Q^2=0$, $Q(\partial_i)=B^\lambda_i \dot{\partial}_\lambda$ and $Q(\dot{\partial}_\lambda)=0$, we have $N_Q(\partial_i, \partial_j)=(B^\lambda_i \dot{\partial}_\lambda B^\mu_j - B^\lambda_j \dot{\partial}_\lambda B^\mu_i)\dot{\partial}_\mu$, $N_Q(\partial_i, \dot{\partial}_\lambda)=0$ and $N_Q(\dot{\partial}_\lambda, \dot{\partial}_\mu)=0$. Hence we obtain

Theorem 1. *Let* $Q=\begin{pmatrix}0, & 0\\ B, & 0\end{pmatrix}$ *be an almost dominant tangent structure assigned on a vector bundle* $E(M)$. *Then, the structure* Q *is integrable if and only if* $B^\lambda_i \dot{\partial}_\lambda B^\mu_j - B^\lambda_j \dot{\partial}_\lambda B^\mu_i = 0$ *holds good. If the components of the composite tensor* B^λ_i *are functions of* x^i *alone, the structure* Q *is always integrable.*

If an a.d.t.str. is integrable, it is called a *dominant tangent structure*.

Putting $Y^*_i = B^\lambda_i \dot{\partial}_\lambda$ in each $\pi^{-1}(U)$, we see that the condition $N_Q=0$ is equivalent to $[Y^*_i, Y^*_j]=0$. In $\pi^{-1}(U) \cap \pi^{-1}(\bar{U})$, it is evident that $\bar{Y}^*_i = \frac{\partial x^a}{\partial \bar{x}^i} Y^*_a$. Then the distribution spanned by $\{Y^*_i\}$ is a global n-dimensional distribution on $E(M)$. Thus we obtain

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1) The notations ∂_i and $\dot{\partial}_\lambda$ stand for $\partial/\partial x^i$ and $\partial/\partial y^\lambda$ respectively.

Corollary. *If a vector bundle $E(M)$ admits a dominant tangent structure, the n-dimensional distribution spanned by $\{Y_i^*\}$ is integrable.*

Next, let there be given a non-linear connection N on our vector bundle $E(M)$ ([6]). That is to say, $E(M)$ admits the quantities $N_i^\lambda(x, y)$ in each $\pi^{-1}(U)$ which satisfy $\partial_a\bar{x}^i\bar{N}_i^\lambda = M_\sigma^\lambda N_a^\sigma - \partial_a M_\sigma^\lambda y^\sigma$ in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$. In this case we can put $X_i = \partial_i - N_i^\sigma\dot{\partial}_\sigma$ in each $\pi^{-1}(U)$, which satisfy $X_i = \partial_i\bar{x}^m\bar{X}_m$ in each $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$. Hence the distribution spanned by $\{X_i\}$ is a global n-dimensional distribution, which we call the horizontal distribution. If we put $Y_\lambda = \dot{\partial}_\lambda$ the distribution spanned by $\{Y_\lambda\}$ is an m-dimensional distribution and is nothing but a fibre itself and is called the vertical distribution. Of course, $\{X_i, Y_\lambda\}$ becomes a local frame in $\pi^{-1}(U)$, which we call the fundamental frame.

Now let us put

$$G_1 = \left\{ \begin{pmatrix} a, & 0, & 0 \\ 0, & a, & b \\ 0, & 0, & c \end{pmatrix} \middle| a \in \mathrm{GL}(n, R),\ c \in \mathrm{GL}(m-n, R),\ b \in M_{n,m-n} \right\}.$$

Then G_1 is a linear Lie group.

Now, in each $\pi^{-1}(U)$, we can choose $\{Y_x^*\} = \{Y_{n+1}^*, \ldots, Y_m^*\}$ out of $\{Y_\lambda\}$ such that $\{Y_1^*, \ldots, Y_n^*, Y_{n+1}^*, \ldots, Y_m^*\}$ forms a local frame of the vertical distribution. Then $\{X_i, Y_\lambda^*\}$ becomes a local frame in $\pi^{-1}(U)$, and satisfies, in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$,

$$\bar{X}_i = \frac{\partial x^m}{\partial \bar{x}^i} X_m,\ \bar{Y}_i^* = \frac{\partial x^m}{\partial \bar{x}^i} Y_m^* \quad\text{and}\quad \bar{Y}_x^* = b_x^m Y_m^* + C_x^y Y_y^*.$$

Thus we obtain "*A vector bundle which is assigned a composite tensor B_i^λ and a non-linear connection N_i^λ admits the G_1-structure in the sence of the theory of G-structures*". *In the next section we shall consider the converse of this fact.*

§2. *G*-structures defined on *E*(*M*).

On a vector bundle $E(M)$, the natural frame $\{\partial_i, \dot{\partial}_\lambda\}$ in each local coordinate neighbourhood satisfies

$$\partial/\partial x^i = \partial_i\bar{x}^m\partial/\partial\bar{x}^m + \partial_i M_\sigma^\lambda y^\sigma\partial/\partial\bar{y}^\lambda,\quad \partial/\partial y^\lambda = M_\lambda^\sigma\partial/\partial\bar{y}^\sigma$$

in each $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$. Therefore, if we put

$$G_0 = \left\{ \begin{pmatrix} A, & 0 \\ C, & B \end{pmatrix} \middle| A \in \mathrm{GL}(n, R),\ B \in \mathrm{GL}(m, R),\ C \in M_{m,n} \right\},$$

then we see that G_0 is a linear Lie group and $E(M)$ admits a G_0-structure in the sense of G-structures and, moreover, the G_0-structure is integrable. So we denote by $\mathscr{P}_0$ this integrable G_0-structure and call it the standard G_0-structure on $E(M)$.

Now let us consider, on $E(M)$, a G-structure which is a reduction of $\mathscr{P}_0$. That is, let G be a Lie subgroup of G_0 and let $E(M)$ admit such the G-structure that the local frame adapted to the G-structure is, at the same time, adapted to the structure $\mathscr{P}_0$. In the paper [4] we have named these G-structures G-structures depending on $\mathscr{P}_0$. In order that an adapted frame $\{W_A\}$ to the G-structure under consideration be depending on $\mathscr{P}_0$ it is necessary and sufficient that $W_i=a_i^j\partial/\partial x^j+b_i^\lambda\partial/\partial y^\lambda$ and $W_\lambda=C_\lambda^\sigma\partial/\partial y^\sigma$ hold good in each $\pi^{-1}(U)$.

In this paper we treat, for G-structures on $E(M)$, only the G-structures depending on $\mathscr{P}_0$.

Now we put

$$Q_0=\begin{pmatrix}0, & 0, & 0\\ E_n, & 0, & 0\\ 0, & 0, & 0\end{pmatrix}, \qquad P_0=\begin{pmatrix}E_n, & 0, & 0\\ 0, & -E_n, & 0\\ 0, & 0, & -E_{m-n}\end{pmatrix}.$$

The straightforward calculation gives us

Lemma 1. *Let P be an element of* $\mathrm{GL}(n+m, R)$. *The conditions $PQ_0=Q_0P$ and $PP_0=P_0P$ are satisfied if and only if $P\in G_1$.*

Now, let us assume that $E(M)$ admits a G_1-structure depending on $\mathscr{P}_0$, which we denote by $\mathscr{P}_1$ hereafter. By the definition, the local frame $\{W_A\}$ adapted to $\mathscr{P}_1$ is also adapted to $\mathscr{P}_0$. So, $\{W_A\}$ can be written, in each $\{\pi^{-1}(U), (x^\lambda, y^\lambda)\}$, as $W_A=\Phi_A^B\partial_B$ where $(\Phi_A^B)\in G_0$, that is, $W_a=\gamma_a^\lambda\partial_i+\sigma_a^\lambda\dot{\partial}_\lambda$ and $W_\alpha=\tau_\alpha^\sigma\dot{\partial}_\sigma$. We put $\Phi=(\Phi_A^B)=\begin{pmatrix}\gamma & 0\\ \sigma & \tau\end{pmatrix}$ where $\gamma=(\gamma_a^i)$, $\sigma=(\sigma_a^\lambda)$ and $\tau=(\tau_\sigma^\lambda)$. Then we have $\Phi^{-1}=\begin{pmatrix}\gamma^{-1} & 0\\ -\tau^{-1}\sigma\gamma^{-1} & \tau^{-1}\end{pmatrix}$. Putting $E'_{(m,n)}=\begin{pmatrix}E_n\\ 0\end{pmatrix}$ and $Q=\Phi Q_0\Phi^{-1}$, we see $Q=\begin{pmatrix}0 & 0\\ \tau E'\gamma^{-1} & 0\end{pmatrix}$. Since $E(M)$ admits the G_1-structure $\mathscr{P}_1$, as is well-known, Q is independent to the choice of the adapted frame $\{W_A\}$ and gives $E(M)$ a global (1, 1)-tensor field. Now we put $\tau E'\gamma^{-1}=B=(B_i^\lambda)$. The relation $\bar{Q}J=JQ$ shows us that B_i^λ is a composite tensor field of rank n. Of course, $Q^2=0$ holds true. Next, let us put $P=\Phi P_0\Phi^{-1}$. It is evident that P is also a global (1, 1)-tensor field on $E(M)$ and satisfies $P^2=E_{n+m}$, that is, an almost product structure. Moreover it is easy to see that $P=\begin{pmatrix}E_n & 0\\ 2\sigma\gamma^{-1} & -E_m\end{pmatrix}$ with respect to the caninical coordinate $\{\pi^{-1}(U), (x^i, y^\alpha)\}$. And the relation $\bar{P}J=JP$ shows us that $N=-\sigma\gamma^{-1}$ is a non-linear connection. Thus, combining these results with the one stated in the last of §1, we obtain

Theorem 2. *In order that a vector bundle $E(M)$ admits a G_1-structure depending on $\mathscr{P}_0$, it is necessary and sufficient that $E(M)$ admits an almost dominant* *tangent structure Q (or a composite tensor B_i^λ) and a non-linear connection N_i^λ.*

It is clear that the Lie algebra $\mathscr{G}_1$ of the Lie group G_1 is given by

$$\mathscr{G}_1=\left\{\begin{pmatrix} p & 0 & 0 \\ 0 & p & r \\ 0 & 0 & q \end{pmatrix} \middle| \, p \in \mathfrak{gl}(n, R),\ q \in \mathfrak{gl}(m-n, R),\ r \in M_{n,m-n} \right\}.$$

Let Γ be a G-connection relative to the structure $\mathscr{P}_1$, denote by ∇ the covariant derivative with respect to the Γ and let $\{W_A\}$ be a local frame adapted to the structure $\mathscr{P}_1$. Then we see that $\nabla_U W_A=\Gamma_{AB}^C W_C U^B$ where $\Gamma_{AB}^C U^B \in \mathscr{G}_1$ for any $U=U^B W_B$. From the last equation, we have $\Gamma^i_{jA}=\Gamma^{\bar{i}}_{\bar{j}A}$, $\Gamma^{\bar{i}}_{jA}=0$, $\Gamma^{i}_{\bar{x}A}=0$, $\Gamma^{\bar{i}}_{jA}=0$, $\Gamma^{\bar{x}}_{jA}=0$ and $\Gamma^{\bar{x}}_{\bar{j}A}=0$. Moreover the tensor Q and P have such components as $Q=\begin{pmatrix} 0 & 0 & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P=\begin{pmatrix} E_n & 0 \\ 0 & -E_m \end{pmatrix}$ with respect to any adapted frame $\{W_A\}$. So, using these relations, we can easily verify that $\nabla Q=0$ and $\nabla P=0$ hold true. Hence, Γ leaves the vertical distribution parallel and the horizontal distribution, too.

Conversely, let Γ be a linear connection on $E(M)$ satisfying $\nabla Q=0$ and $\nabla P=0$. Then, first, $\nabla Q=0$ leads us to $\Gamma^{i}_{\bar{j}A}=0$ $\Gamma^{\bar{x}}_{\bar{j}A}=0$, $\Gamma^{i}_{\bar{y}A}=0$ and $\Gamma^{i}_{jA}=\Gamma^{\bar{i}}_{\bar{j}A}$. Moreover the condition $\nabla P=0$ leads us to $\Gamma^{\lambda}_{jA}=0$. Therefore, Γ becomes a G-connection relative to the structure $\mathscr{P}_1$. Thus we obtain

Theorem 3. *Let us assume that a vector bundle $E(M)$ admits a G_1-structure depending on $\mathscr{P}_0$. A linear connection on $E(M)$ is a G-connection relative to the G_1-structure if and only if $\nabla Q=0$ and $\nabla P=0$ hold good.*

[Remark] The theory of G-structures teaches us that there always exists a G-connection relative to a G-structure ([10]). So, if $E(M)$ admits the structure $\mathscr{P}_1$, $E(M)$ also admits, at least, a linear connection satisfying $\nabla Q=0$ and $\nabla P=0$.

§ 3. Local expressions.

First, let us examine the components Γ^A_{BC} of a linear connection Γ on $E(M)$ with respect to the fundamental frame $\{Z_A\}=\{X_i, Y_\lambda\}$ in $\pi^{-1}(U)$. Of course, $\nabla_{Z_C} Z_B=\Gamma^A_{BC} Z_A$. In $\pi^{-1}(\bar{U})$, also is the relation $\nabla_{\bar{Z}_C}\bar{Z}_B=\bar{\Gamma}^A_{BC}\bar{Z}_A$. Hence, using the relations $X_j=\partial_j \bar{x}^C \bar{X}_C$ and $Y_\lambda=M^\sigma_\lambda \bar{Y}_\sigma$ in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, we obtain

$$\begin{cases} \bar{\Gamma}^a_{bc}\partial_j\bar{x}^b\partial_k\bar{x}^c=\Gamma^m_{jk}\partial_m\bar{x}^a-\partial_j\partial_k\bar{x}^a, \\ \bar{\Gamma}^\alpha_{\sigma a}\partial_j\bar{x}^a M^\sigma_\lambda=\Gamma^\sigma_{\lambda j}M^\alpha_\sigma-\partial_j M^\alpha_\lambda, \\ \bar{\Gamma}^\alpha_{bc}\partial_j\bar{x}^b\partial_k\bar{x}^c=\Gamma^\sigma_{jk}M^\alpha_\sigma,\ \bar{\Gamma}^h_{\sigma m}M^\sigma_\lambda\partial_j\bar{x}^m=\Gamma^m_{\lambda j}\partial_m\bar{x}^h, \\ \bar{\Gamma}^a_{b\sigma}\partial_j\bar{x}^b M^\sigma_\mu=\Gamma^h_{j\mu}\,\partial_h\bar{x}^a,\ \bar{\Gamma}^\alpha_{b\sigma}\partial_j\bar{x}^b M^\sigma_\mu=\Gamma^\lambda_{j\mu}M^\alpha_\lambda, \\ \bar{\Gamma}^h_{\sigma\tau}M^\sigma_\lambda M^\tau_\mu=\Gamma^m_{\lambda\mu}\partial_m\bar{x}^h,\ \bar{\Gamma}^\alpha_{\sigma\tau}M^\sigma_\lambda M^\tau_\mu=\Gamma^\sigma_{\lambda\mu}M^\alpha_\sigma. \end{cases}$$

These relations, however, have been already shown by Miron ([6]). From these we see that $\Gamma^h_{ij}(x, y)$ undergoes the transformation law of a linear connection of a base manifold, $\Gamma^\lambda_{\mu i}(x, y)$ undergoes the transformation law of a König connection (or a dominant affine connection or a hyperconnection ([1], [3], [5], [6], [8], [10])) and the other components are the composite tensors.

The a.d.t.str. Q operates as $Q(Y_\alpha)=Q(\dot{\partial}_\alpha)=0$ and $Q(X_i)=Q(\partial_i)-N^\sigma_i Q(\dot{\partial}_\sigma)=Q(\partial_i)=B^\lambda_i\dot{\partial}_\lambda$. So Q has the components $\begin{pmatrix}0 & 0\\ B^\lambda_i & 0\end{pmatrix}$ with respect to the fundamental frame $\{X_i, Y_\alpha\}$. Now, calculating the components of ∇Q, we have

$$\begin{cases} \nabla_{X_j}Q^h_i=\Gamma^h_{\sigma j}B^\sigma_i, & \nabla_{X_j}Q^h_\mu=0,\\ \nabla_{X_j}Q^\lambda_i=X_j(B^\lambda_i)+\Gamma^\lambda_{\sigma j}B^\sigma_i-\Gamma^m_{ij}B^\lambda_m, & \nabla_{X_j}Q^\lambda_\mu=-\Gamma^m_{\mu j}B^\lambda_m,\\ \nabla_{Y_\tau}Q^h_i=\Gamma^h_{\sigma\tau}B^\sigma_i, & \nabla_{Y_\tau}Q^h_\mu=0,\\ \nabla_{Y_\tau}Q^\lambda_i=Y_\tau(B^\lambda_i)+\Gamma^\lambda_{\sigma\tau}B^\sigma_i-\Gamma^m_{i\tau}B^\lambda_m, & \nabla_{Y_\tau}Q^\lambda_\mu=-\Gamma^m_{\mu\tau}B^\lambda_m. \end{cases}$$

From the fact that the rank of (B^λ_i) is n, we obtain

Theorem 4. *Let us assume that a vector bundle $E(M)$ admits a non-linear connection N and an almost dominant tangent structure Q. Let Γ^A_{BC} be the components of a linear connection Γ on $E(M)$ with respect to the fundamental frame $\{X_i, Y_\alpha\}$. Then Γ satisfies $\nabla Q=0$ if and only if $\Gamma^h_{\lambda j}=0$, $\Gamma^h_{\lambda\mu}=0$, $X_j(B^\lambda_i)+\Gamma^\lambda_{\sigma j}B^\sigma_i-\Gamma^m_{ij}B^\lambda_m=0$ and $Y_\mu(B^\lambda_i)+\Gamma^\lambda_{\tau\mu}B^\tau_i-\Gamma^m_{i\mu}B^\lambda_m=0$ hold good.*

By virtue of Theorem 2, the assumption in Theorem 4 means that $E(M)$ admits a G_1-structure depending on $\mathscr{P}_0$, which we have denoted by $\mathscr{P}_1$. For the inteligibility, however, we call the structure $\mathscr{P}_1$ the *(N, Q)-structure* and the G-connection relative to the structure $\mathscr{P}_1$ the *(N, Q)-connection*. Of course, according to Theorem 3, the (N, Q)-connection is a linear connection on $E(M)$ satisfying $\nabla Q=0$ and $\nabla P=0$.

Here, we examine the condition $\nabla P=0$ with respect to the frame $\{Z_A\}=\{X_i, Y_\alpha\}$. Since $P(\partial_i)=\partial_i-2N^\lambda_i\dot{\partial}_\lambda$ and $P(\dot{\partial}_\lambda)=-\dot{\partial}_\lambda$, we see $P(Y_\lambda)=-Y_\lambda$ and $P(X_i)=X_i$. Namely, P has the components $\begin{pmatrix}E_n & 0\\ 0 & -E_m\end{pmatrix}$ with respect to $\{Z_A\}=\{X_i, Y_\alpha\}$. Now calculating the components of ∇P, we obtain $\nabla_A P^h_i=0$, $\nabla_A P^h_\lambda=-2\Gamma^h_{\lambda A}$, $\nabla_A P^\alpha_i=2\Gamma^\alpha_{iA}$ and $\nabla_A P^\alpha_\lambda=0$ where we put $\nabla_{Z_A}=\nabla_A$. Therefore the condition $\nabla P=0$ can be written as $\Gamma^h_{\lambda A}=0$ and $\Gamma^\alpha_{iA}=0$. Hence we obtain

Theorem 5. *Let $E(M)$ be a vector bundle admitting an (N, Q)-structure and let Γ^A_{BC} be components of a linear connection Γ on $E(M)$ with respect to the fundamental frame $\{X_i, Y_\alpha\}$. The condition for the Γ to be an (N, Q)-connection is given by*

$$\begin{cases} \Gamma^h_{\lambda i}=0, \quad \Gamma^h_{\lambda\mu}=0, \quad \Gamma^\alpha_{ij}=0, \quad \Gamma^\alpha_{i\lambda}=0, \\ X_j(B^\lambda_i)+\Gamma^\lambda_{\sigma j}B^\sigma_i-\Gamma^m_{ij}B^\lambda_m=0, \\ Y_\mu(B^\lambda_i)+\Gamma^\lambda_{\sigma\mu}B^\sigma_i-\Gamma^m_{i\mu}B^\lambda_m=0. \end{cases}$$

Concerning composite tensors, for example $T^{h\lambda}_{i\mu}$, we use the following notations:

$$T^{h\lambda}_{i\mu|j}=X_j(T^{h\lambda}_{i\mu})+\Gamma^h_{mj}T^{m\lambda}_{i\mu}+\Gamma^\lambda_{\sigma j}T^{h\sigma}_{i\mu}-\Gamma^m_{ij}T^{h\lambda}_{m\mu}-\Gamma^\sigma_{\mu j}T^{h\lambda}_{i\sigma},$$

$$T^{h\lambda}_{i\mu}|_\alpha=Y_\alpha(T^{h\lambda}_{i\mu})+\Gamma^h_{m\alpha}T^{m\lambda}_{i\mu}+\Gamma^\lambda_{\sigma\alpha}T^{h\sigma}_{i\mu}-\Gamma^m_{i\alpha}T^{h\lambda}_{m\mu}-\Gamma^\sigma_{\mu\alpha}T^{h\lambda}_{i\sigma}.$$

§4. An f-structure.

In this section we use the notations defined in §2. First, we put

$$J_0=\begin{pmatrix} 0 & -E_n & 0 \\ E_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_2=\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| a\in \mathrm{GL}(n, R),\ b\in \mathrm{GL}(m-n, R)\right\}.$$

The direct calculation and Lemma 1 lead us to

Lemma 2. *Let P be an element of* $\mathrm{GL}(n+m, R)$. *The conditions $PQ_0=Q_0P$, $PP_0=P_0P$ and $PJ_0=J_0P$ are satisfied if and only if $P\in G_2$.*

Let us suppose that a vector bundle $E(M)$ admits a G_2-structure depending on $\mathscr{P}_0$ and $\{W_A\}$ is a local frame adapted to the G_2-structure, and put $W_A=\Phi^B_A\partial_B$. Then $F=\Phi J_0\Phi^{-1}$ is independent to the choice of the local frame $\{W_A\}$ adapted to the G_2-structure, and F becomes a global (1, 1)-tensor field on $E(M)$. Moreover we can easily see that $F^3+F=0$ and rank $F=2n$. That is, $E(M)$ admits an f-structure of rank $2n$ ([8]). Since $G_2\subset G_1$, if $E(M)$ admits a G_2-structure depending on $\mathscr{P}_0$, $E(M)$ also admits an (N, Q)-structure. Thus $W_a=\gamma^i_a\partial_i+\sigma^\alpha_a\dot\partial_\alpha=\gamma^i_a\partial_i-N^\alpha_m\gamma^m_a\dot\partial_\alpha=\gamma^m_aX_m$ and $W_\alpha=\tau^\beta_\alpha\dot\partial_\beta=\tau^\beta_\alpha Y_\beta$. Hence if we put $W_A=\Psi^B_AZ_B$ and $\Psi=(\Psi^B_A)$, we see $\Psi=\begin{pmatrix}\gamma^i_a & 0\\ 0 & \tau^\alpha_\beta\end{pmatrix}$. Then, F is written, with respect to the fundamental frame $\{X_i, Y_\alpha\}$, as $F=\Psi J_0\Psi^{-1}$. And we see $F=\begin{pmatrix} 0 & -\gamma^tE'\tau^{-1} \\ \tau E'\gamma^{-1} & 0\end{pmatrix}$ with respect to $\{X_i, Y_\alpha\}$. In §2, we have put $\tau E'\gamma^{-1}=B=(B^\lambda_i)$ and we have seen B^λ_i is a composite tensor. Here let us put $\gamma^tE'\tau^{-1}=C=(C^i_\lambda)$. Then rank $C=n$. Since F is a tensor field on $E(M)$ and the Jacobian matrix from $\{X_i, Y_\alpha\}$ to $\{\bar X_i, \bar Y_\alpha\}$ is given by $J=\begin{pmatrix}\partial_j\bar x^i & 0\\ 0 & M^\lambda_\mu\end{pmatrix}$, so relation $\bar FJ=JF$ leads us to $\bar C^i_\sigma M^\sigma_\lambda=\partial_m\bar x^iC^m_\lambda$, that is, C^i_λ is a composite tensor. From the definition, it follows that $CB=\gamma^tE'\tau^{-1}\tau E'\gamma^{-1}=E_n$, that is, $C^i_\sigma B^\sigma_j=\delta^i_j$.

Conversely, let there be given a non-linear connection N^λ_i and composite tensors B^λ_i and C^i_λ satisfying $C^i_\sigma B^\sigma_j=\delta^i_j$ and rank $(B^\lambda_i)=$rank $(C^i_\lambda)=n$. We define a (1, 1)-

tensor F by $F=(F^A_B)=\begin{pmatrix}0 & -C^i_\mu\\ B^\lambda_j & 0\end{pmatrix}$ with respect to the frame $\{X_i, Y_\alpha\}$ in each $\pi^{-1}(U)$. Then F becomes a global tensor field on $E(M)$ and operates as $F(X_i)=B^\lambda_i Y_\lambda=Y^*_i$, $F(Y_\lambda)=-C^m_\lambda X_m$, from which it follows that $F^3+F=0$. Thus F gives $E(M)$ an f-structure of rank $2n$. Now consider the following simultaneous linear equation $C^i_\lambda p^\lambda=0$. Since the rank of (C^i_λ) is n, there exist $(m-n)$ linearly independent solutions p^λ_x. On putting $Y^*_x=p^\lambda_x Y_\lambda$, we can show that $\{Y^*_i, Y^*_x\}$ is a local frame of each vertical distribution. Then $\{X_i, Y^*_i, Y^*_x\}$ is a local frame in $\{\pi^{-1}(U), (x^i, y^\alpha)\}$. Denoting the local frame in $\{\pi^{-1}(\bar{U}), (\bar{x}^i, \bar{y}^\alpha)\}$ by $\{\bar{X}_i, \bar{Y}^*_i, \bar{Y}^*_x\}$, we have, in $\pi^{-1}(U)\cap\pi^{-1}(\bar{U})$, that $X_i=\partial_i\bar{x}^m\bar{X}_m$, $Y^*_i=\partial_i\bar{x}^m\bar{Y}^*_m$. Moreover, from $\bar{C}^i_\lambda\bar{p}^\lambda_y=0$, we see $C^j_\sigma(\overset{-1}{M}{}^\sigma_\lambda\bar{p}^\lambda_y)=0$. Then we have $\overset{-1}{M}{}^\sigma_\lambda\bar{p}^\lambda_y=k^x_y p^\sigma_x$ for certain k^x_y. That is, $\bar{p}^\lambda_y=M^\lambda_\sigma p^\sigma_x k^x_y$. Then we see

$$\bar{Y}^*_y=\bar{p}^\lambda_y\bar{Y}_\lambda=M^\lambda_\sigma p^\sigma_x k^x_y\bar{Y}_\lambda=p^\sigma_x k^x_y Y_\sigma=k^x_y Y^*_x.$$

Namely, $\bar{X}_i=\dfrac{\partial x^m}{\partial\bar{x}^i}X_m$, $\bar{Y}^*_i=\dfrac{\partial x^m}{\partial\bar{x}^i}Y^*_m$, $\bar{Y}^*_y=k^x_y Y^*_x$ and $X_i=\partial_i-N^\lambda_i\dot{\partial}_\lambda$, $Y^*_i=B^\lambda_i\dot{\partial}_\lambda$, $Y^*_x=p^\lambda_x\dot{\partial}_\lambda$ hold. Therefore $E(M)$ admits a G_2-structure depending of $\mathscr{P}_0$. Thus we obtain

Theorem 6. *In order that a vector bundle $E(M)$ admits a G_2-structure depending on $\mathscr{P}_0$, it is necessary and sufficient that $E(M)$ admits a non-linear connection N^λ_i and composite tensors B^i_λ and C^i_λ satisfying $C^i_\sigma B^\sigma_j=\delta^i_j$ and rank $(B^\lambda_i)=\mathrm{rank}(C^i_\lambda)=n$. In this case, $E(M)$ admits an f-structure of rank $2n$.*

For the inteligibility, we call the structure stated in Theorem 6 an (N, Q, F)-*structure* hereafter.

For the f-structure F, as is well-known ([9], [11]), $\mathscr{L}=-F^2$ and $\mathscr{M}=E_{n+m}-\mathscr{L}$ satisfy $\mathscr{L}^2=\mathscr{L}$, $\mathscr{M}^2=\mathscr{M}$, $\mathscr{L}\mathscr{M}=0$, $\mathscr{M}\mathscr{L}=0$ and $\mathscr{L}+\mathscr{M}=E_{n+m}$. That is, $\mathscr{L}$ and $\mathscr{M}$ are projection tensors. We denote by D_l and D_m the distributions defined by $\mathscr{L}$ and $\mathscr{M}$ respectively. Of course, $\dim D_l=2n$ and $\dim D_m=m-n$ and $T_P(E(M))=(D_l)_P\oplus(D_m)_P$ for any point P of $E(M)$.

With respect to $\{X_i, Y_\alpha\}$, F is represented by $F=\begin{pmatrix}0 & -C\\ B & 0\end{pmatrix}$. So, $\mathscr{L}=\begin{pmatrix}E & 0\\ 0 & BC\end{pmatrix}$ and $\mathscr{M}=\begin{pmatrix}0 & 0\\ 0 & E-BC\end{pmatrix}$. Now we put $\mathfrak{A}=\dfrac{1}{2}\begin{pmatrix}E & C\\ B & BC\end{pmatrix}$ and $\mathscr{B}=\dfrac{1}{2}\begin{pmatrix}E & -C\\ -B & BC\end{pmatrix}$. Then, these satisfy $\mathfrak{A}^2=\mathfrak{A}$, $\mathscr{B}^2=\mathscr{B}$, $\mathfrak{A}\mathscr{B}=0$, $\mathscr{B}\mathfrak{A}=0$, $\mathfrak{A}+\mathscr{B}=\mathscr{L}$. Thus $\mathfrak{A}$ and $\mathscr{B}$ are also projection tensors. Denoting by D_a and D_b the distributions defined by $\mathfrak{A}$ and $\mathscr{B}$ respectively, we have $D_a\oplus D_b=D_l$ and $T_P(E(M))=(D_a)_P\oplus(D_b)_P\oplus(D_m)_P$.

§5. (N, Q, F)-connections.

We consider successively the (N, Q, F)-structure in this section. The f-structure F defined in the preceding section has the following components with respect to $\{X_i, Y_\alpha\}$: $F=\begin{pmatrix}F^h_i & F^h_\mu\\ F^\lambda_i & F^\lambda_\mu\end{pmatrix}=\begin{pmatrix}0 & -C^h_\mu\\ B^\lambda_i & 0\end{pmatrix}$.

Now, let Γ be an (N, Q)-connection on $E(M)$. Calculating ∇F with respect to this Γ, we obtain

$$\nabla_{Xj}F_i^h=0, \quad \nabla_{Xj}F_\mu^h=-C_{\mu|j}^h, \quad \nabla_{Xj}F_i^\lambda=B_{i|j}^\lambda=0, \quad \nabla_{Xj}F_\lambda^\lambda=0,$$

$$\nabla_{Y\alpha}F_i^h=0, \quad \nabla_{Y\alpha}F_\mu^h=-C_\mu^h|_\alpha, \quad \nabla_{Y\alpha}F_i^\lambda=B_i^\lambda|_\alpha=0, \quad \nabla_{Y\alpha}F_\mu^\lambda=0.$$

Therefore, on a vector bundle $E(M)$ admitting an (N, Q, F)-structure, a necessary and sufficient condition for the (N, Q)-connection to satisfy $\nabla F=0$ is that $C_{\mu|j}^h=0$ and $C_\mu^h|_\lambda=0$ hold.

Here we call a linear connection on $E(M)$ satisfying $\nabla P=0$, $\nabla Q=0$, $\nabla F=0$ as an *(N, Q, F)-connection*. Then, according to Theorem 5 and the above result, we obtain

Theorem 7. *Let $E(M)$ be a vector bundle admitting an (N, Q, F)-structure, and let Γ_{BC}^A be components of a linear connection Γ on $E(M)$ with respect to the fundamental frame $\{X_i, Y_\alpha\}$. The condition for Γ to be an (N, Q, F)-connection is given by*

$$\begin{cases} \Gamma_{\lambda i}^h=0, \quad \Gamma_{\lambda\mu}^h=0, \quad \Gamma_{ij}^\alpha=0, \quad \Gamma_{i\mu}^\alpha=0, \\ B_{i|j}^\lambda=0, \quad B_i^\lambda|_\sigma=0, \quad C_{\mu|j}^i=0, \quad C_\mu^i|_\sigma=0. \end{cases}$$

[Remark] From the last 4 conditions in Theorem 7, Γ_{ij}^h and $\Gamma_{i\lambda}^h$ are determined as follows:

$$\begin{cases} \Gamma_{ij}^h=C_\sigma^h(X_j(B_i^\sigma)+\Gamma_{\tau j}^\sigma B_i^\tau), \\ \Gamma_{i\lambda}^h=C_\sigma^h(Y_\lambda(B_i^\sigma)+\Gamma_{\tau\lambda}^\sigma B_i^\tau). \end{cases}$$

§6. König structures.

If an (N, Q, F)-structure satisfies, in each $\pi^{-1}(U)$,

$$B_i^\alpha=B_i^\alpha(x), \quad C_\lambda^i=C_\lambda^i(x) \quad \text{and} \quad N_i^\alpha=\Gamma_{\sigma i}^\alpha(x)y^\sigma,$$

then the structure is said to be a *König structure.* That is to say, as for the König structure, B_i^α, C_α^i and $\Gamma_{\beta i}^\alpha$ are functions of x^i alone and the non-linear connection is given by $N_i^\alpha=\Gamma_{\sigma i}^\alpha y^\sigma$. The last condition is globally well-defined. Because, taking account of $\bar{\Gamma}_{\sigma a}^\alpha\partial_j\bar{x}^a M_\lambda^\sigma=\Gamma_{\lambda j}^\sigma M_\sigma^\alpha-\partial_j M_\lambda^\alpha$, we have $(\bar{\Gamma}_{\sigma a}^\alpha\bar{y}^\sigma)\partial_j\bar{x}^a=(\Gamma_{\tau j}^\sigma y^\tau)M_\sigma^\alpha-\partial_j M_\sigma^\alpha y^\sigma$, that is, $\Gamma_{\sigma j}^\alpha y^\sigma$ is a globally defined non-linear connection. In this case, $\Gamma_{\beta i}^\alpha$ is a so-called König connection ([1], [3], [8], [10]) (or a dominant affine connection). Due to Theorem 1, Corollary and Theorem 6, it follows directly that

Theorem 8. *In a vector bundle admitting a König structure, the almost dominant tangent structure Q is integrable and the distribution spanned by $\{Y_i^*\}$ is also integrable.*

Theorem 9. *A vector bundle admitting a König structure admits an f-structure.*

From now on, we call this f-structure the *f-structure derived from the König structure*. Now let us examine the integrability conditions of this f-structure F. The Nijenhuis tensor N_F of F is given by

$$N_F(U, V)=[FU, FV]+F^2[U, V]-F[FU, V]-F[U, FV],$$

U and V being arbitrary vector fields in $E(M)$.

First, the condition for the distribution D_m to be integrable is given by $N_F(\mathscr{M}U, \mathscr{M}V)=0$ ([8], [10]). On the other hand, $F\mathscr{M}=F(E+F^2)=0$, $\mathscr{M}X_i=(E+F^2)X_i=0$ and $\mathscr{M}Y_\lambda=(E+F^2)Y_\lambda=Y_\lambda-B^\sigma_m C^m_\lambda Y_\sigma$. So it follows that $N_F(\mathscr{M}X_i, \mathscr{M}X_j)=0$, $N_F(\mathscr{M}X_i, \mathscr{M}Y_\lambda)=0$ and $N_F(\mathscr{M}Y_\lambda, \mathscr{M}Y_\mu)=[Y_\lambda-B^\sigma_m C^m_\lambda Y_\sigma, Y_\mu-B^\tau_r C^r_\mu Y_\tau]=0$. Hence we obtain

Theorem 10. *The distribution D_m defined by the f-structure derived from a König structure is always integrable.*

Next, the condition for the distribution D_l to be integrable is given by $\mathscr{M}N_F(U, V)=0$ ([9], [11]). By means of $\mathscr{M}F=0$, we can rewrite this condition as $\mathscr{M}[FU, FV]=0$. On the other hand, we see $[X_i, X_j]=R^\beta_{\alpha ij}y^\alpha Y_\beta$ where

$$R^\beta_{\alpha ij}=\partial_j\Gamma^\beta_{\alpha i}-\partial_i\Gamma^\beta_{\alpha j}+\Gamma^\beta_{\sigma j}\Gamma^\sigma_{\alpha i}-\Gamma^\beta_{\sigma i}\Gamma^\sigma_{\alpha j}.$$

This $R^\beta_{\alpha ij}$ is called the curvature tensor of the König connection $\Gamma^\beta_{\alpha i}(x)$. Now, direct calculation shows us

$$\begin{aligned}
\mathscr{M}[FX_i, FX_j]&=\mathscr{M}[B^\alpha_i Y_\alpha, B^\beta_j Y_\beta]=0,\\
\mathscr{M}[FX_i, FY_\alpha]&=\mathscr{M}[B^\alpha_i Y_\alpha, -C^m_\alpha X_m]\\
&=(\Gamma^\tau_{\sigma m}B^\sigma_i C^m_\alpha+\partial_m B^\tau_i C^m_\alpha)\mathscr{M}(Y_\tau)\\
&=C^m_\alpha(\partial_m B^\tau_i+B^\sigma_i\Gamma^\tau_{\sigma m})(\delta^\gamma_\tau-B^\gamma_t C^t_\tau)Y_\gamma,\\
\mathscr{M}[FY_\alpha, FY_\beta]&=\mathscr{M}[C^m_\alpha X_m, C^l_\beta X_l]=C^m_\alpha C^l_\beta\mathscr{M}[X_m, X_l]\\
&=C^m_\alpha C^l_\beta R^\sigma_{\tau ml}(\delta^\gamma_\sigma-B^\gamma_h C^h_\sigma)Y_\gamma.
\end{aligned}$$

Thus we obtain

Theorem 11. *The distribution D_l defined by the f-structure derived from a König structure is integrable if and only if*

$$\begin{cases}
(\delta^\alpha_\sigma-B^\alpha_m C^m_\sigma)(\partial_j B^\sigma_i+\Gamma^\sigma_{\tau j}B^\tau_i)=0,\\
(\delta^\beta_\sigma-B^\beta_m C^m_\sigma)R^\sigma_{\alpha ij}=0,
\end{cases}$$

where $R^\beta_{\alpha ij}$ is the curvature tensor of the König connection $\Gamma^\alpha_{\beta i}(x)$.

When the distribution D_l is integrable and the induced almost complex structure f' is complex analytic on each integral manifold of D_l, we say that the f-structure F to be *partially integrable.* And the condition for the f-structure F to be partially integrable is given by $N_F(\mathscr{L}U, \mathscr{L}V)=0$ ([9], [11]). Since $F\mathscr{L}=F$, we get

$$N_F(\mathscr{L}U, \mathscr{L}V)=F^2[F^2U, F^2V]+[FU, FV]+F[FU, F^2V]+F[F^2U, FV].$$

Then, we see

$$N_F(\mathscr{L}X_i, \mathscr{L}X_j)=\{X_i(B^\alpha_j)+\Gamma^\alpha_{\sigma i}B^\sigma_j-X_j(B^\alpha_i)-\Gamma^\alpha_{\sigma j}B^\sigma_i\}C^h_\alpha X_h-R^\alpha_{\beta ij}B^\gamma_m C^m_\alpha Y_\gamma,$$

$$N_F(\mathscr{L}X_i, \mathscr{L}Y_\alpha)=C^m_\alpha\{X_m(B^\tau_i)+\Gamma^\tau_{\sigma m}B^\sigma_i-B^\tau_l C^l_\gamma(X_i(B^\gamma_m)+\Gamma^\gamma_{\sigma i}B^\sigma_m)+R^\tau_{\sigma im}y^\sigma\}Y_\tau,$$

$$N_F(\mathscr{L}Y_\alpha, \mathscr{L}Y_\beta)=C^h_\tau C^m_\alpha C^l_\beta\{X_l(B^\tau_m)+\Gamma^\tau_{\sigma l}B^\sigma_m-X_m(B^\tau_l)-\Gamma^\tau_{\sigma m}B^\sigma_l\}X_h+C^m_\alpha C^l_\beta R^\tau_{\sigma ml}Y_\tau.$$

Thus the condition under consideration can be written as

$$\begin{cases} R^\beta_{\alpha ij}=0, & \cdots(1)\\ C^h_\alpha\{(\partial_i B^\alpha_j+\Gamma^\alpha_{\sigma i}B^\sigma_j)-(\partial_j B^\alpha_i+\Gamma^\alpha_{\sigma j}B^\sigma_i)\}=0, & \cdots(2)\\ (\partial_j B^\alpha_i+\Gamma^\alpha_{\sigma j}B^\sigma_i)-B^\alpha_m C^m_\gamma(\partial_i B^\gamma_j+\Gamma^\gamma_{\sigma i}B^\sigma_j)=0. & \cdots(3)\end{cases}$$

Substituting (2) into (3), we get

$$(\partial_j B^\alpha_i+\Gamma^\alpha_{\sigma j}B^\sigma_i)-B^\alpha_m C^m_\gamma(\partial_j B^\gamma_i+\Gamma^\gamma_{\sigma j}B^\sigma_i)=0.$$

Namely, we get

$$(\delta^\alpha_\gamma-B^\alpha_m C^m_\gamma)(\partial_j B^\gamma_i+\Gamma^\gamma_{\sigma j}B^\sigma_i). \qquad \cdots(4)$$

Conversely, the condition (2) and (4) lead us to (3). Hence we obtain

Theorem 12. *The f-structure derived from a König structure is partially integrable if and only if*

$$\begin{cases} R^\beta_{\alpha ij}=0,\\ (\delta^\alpha_\beta-B^\alpha_m C^m_\beta)(\partial_j B^\beta_i+\Gamma^\beta_{\sigma i}B^\sigma_i)=0,\\ C^h_\alpha\{(\partial_i B^\alpha_j+\Gamma^\alpha_{\sigma i}B^\sigma_j)-(\partial_j B^\alpha_i+\Gamma^\alpha_{\sigma j}B^\sigma_i)\}=0.\end{cases}$$

Finally we look for the condition for F to be integrable. An f-structure F is said to be integrable when $N_F=0$. Calculating the components of N_F, we get

$$\begin{aligned} N_F(X_i, X_j)=&-R^\tau_{\sigma ij}y^\sigma C^m_\tau B^\alpha_m Y_\alpha\\ &-C^h_\lambda\{(\partial_i B^\lambda_j+B^\sigma_j\Gamma^\lambda_{\sigma i})-(\partial_j B^\lambda_i+\Gamma^\lambda_{\sigma j}B^\sigma_i)\}X_h,\\ N_F(X_i, Y_\mu)=&\{C^m_\mu(\partial_m B^\alpha_i+\Gamma^\alpha_{\sigma m}B^\sigma_i)+B^\alpha_m(\partial_i C^m_\mu-\Gamma^\sigma_{\mu i}C^m_\sigma)\}Y_\alpha\\ &-C^m_\mu R^\tau_{\sigma im}y^\sigma C^h_\tau X_h,\end{aligned}$$

$$N_F(Y_\lambda, Y_\mu) = \{C^m_\lambda(\partial_m C^h_\mu - \Gamma^\sigma_{\mu m} C^h_\sigma) - C^m_\mu(\partial_m C^h_\lambda - \Gamma^\sigma_{\lambda m} C^h_\sigma)\} X_h$$
$$+ C^a_\lambda C^b_\mu R^\sigma_{\tau a b} y^\tau Y_\sigma.$$

Thus, the condition under consideration is given by

$$\begin{cases} R^\beta_{\alpha i j} \equiv 0, \\ C^m_\lambda(\partial_m C^h_\mu - \Gamma^\sigma_{\mu m} C^h_\sigma) - C^m_\mu(\partial_m C^h_\lambda - \Gamma^\sigma_{\lambda m} C^h_\sigma) = 0, \\ C^m_\lambda(\partial_m B^\sigma_i + \Gamma^\alpha_{\sigma m} B^\sigma_i) + B^\alpha_m(\partial_i C^m_\lambda - \Gamma^\sigma_{\lambda i} C^m_\sigma) = 0, \\ C^h_\lambda\{(\partial_i B^\lambda_j + \Gamma^\lambda_{\sigma i} B^\sigma_j) - (\partial_j B^\lambda_i + \Gamma^\lambda_{\sigma j} B^\sigma_i)\} = 0. \end{cases}$$

Multiplying the second equation by $B^\lambda_i B^\mu_j$ and contracting with λ and μ, we get

$$B^\mu_j(\partial_i C^h_\mu - \Gamma^\sigma_{\mu m} C^h_\sigma) - B^\lambda_i(\partial_j C^h_\lambda - \Gamma^\sigma_{\lambda j} C^h_\sigma) = 0.$$

Taking account of $C^i_\sigma B^\sigma_j = \delta^i_j$, we have

$$C^h_\mu\{(\partial_i B^\mu_j + \Gamma^\mu_{\sigma m} B^\sigma_j) - (\partial_j B^\mu_i + \Gamma^\mu_{\sigma j} B^\sigma_i)\} = 0,$$

that is, the second condition includes the fourthe condition. Therefore we obtain

Theorem 13. *The f-structure derived from a König structure is integrable if and only if*

$$\begin{cases} R^\beta_{\alpha i j} \equiv 0, \\ (\partial_m C^h_\mu - \Gamma^\sigma_{\mu m} C^h_\sigma) C^m_\lambda - (\partial_m C^h_\lambda - \Gamma^\sigma_{\lambda m} C^h_\sigma) C^m_\mu = 0, \\ (\partial_m B^\alpha_i + \Gamma^\alpha_{\sigma m} B^\sigma_i) C^m_\beta + B^\alpha_m(\partial_i C^m_\beta - \Gamma^\sigma_{\beta i} C^m_\sigma) = 0. \end{cases}$$

§7. The standard connection.

Let there be given a König structure on a vector bundle $E(M)$. Then, there exist a König connection $\Gamma^\alpha_{\beta i}(x)$ and a non-linear connection $N^\alpha_i = \Gamma^\alpha_{\beta i}(x) y^\beta$.

Now let Γ be a linear connection on $E(M)$ whose components with respect to the frame $\{X_i, Y_\alpha\}$ are Γ^A_{BC}. If we put

$$\begin{cases} \Gamma^\alpha_{\beta i} = \Gamma^\alpha_{\beta i}(x) \quad (\Gamma^\alpha_{\beta i}(x) \text{ is the given König connection}), \\ \Gamma^h_{ij} = C^h_\sigma(\partial_j B^\sigma_i + \Gamma^\sigma_{\tau j} B^\tau_i), \\ \text{the other components vanish.} \end{cases}$$

Due to the fact stated in §3 and Remark shown in §5, this is well-defined. Thus we can determine one linear connection on $E(M)$. Concerning this linear connection, we get

$$\begin{cases} \nabla_{X_j}X_i=\Gamma^h_{ij}X_h, & \nabla_{X_j}Y_\alpha=\Gamma^\beta_{\alpha i}Y_\beta, \\ \nabla_{Y_\lambda}X_i=0, & \nabla_{Y_\lambda}Y_\mu=0. \end{cases}$$

The linear connection Γ thus defined is called the *standard connection derived from a König structure.* For the standard connection, we get

$$\begin{aligned} B^\lambda_{i|j} &= \partial_j B^\lambda_i + \Gamma^\lambda_{\sigma j} B^\sigma_i - \Gamma^m_{ij} B^\lambda_m \\ &= (\delta^\lambda_\sigma - B^\lambda_m C^m_\sigma)(\partial_j B^\sigma_i + \Gamma^\sigma_{\tau j} B^\tau_i), \\ B^\lambda_i|_\mu &= 0, \\ C^h_{\mu|j} &= \partial_j C^h_\mu - \Gamma^\tau_{\mu j} C^h_\tau + \Gamma^h_{mj} C^m_\mu \\ &= \partial_j C^h_\mu - \Gamma^\tau_{\mu j} C^h_\tau + C^m_\mu C^h_\tau \partial_j B^\tau_m + C^m_\mu C^h_\tau \Gamma^\tau_{\sigma j} B^\sigma_m \\ &= (\delta^\sigma_\mu - B^\sigma_m C^m_\mu)(\partial_j C^h_\sigma - \Gamma^\tau_{\sigma j} C^h_\tau), \\ C^h_\alpha|_\mu &= 0. \end{aligned}$$

Therefore, with respect to the standard connection, we can rewrite Theorem 4, 7, 11 and 12 as follows:

Theorem 14. *The standard connection derived from a König structure is an (N, Q)-connection if and only if*

$$(\delta^\lambda_\sigma - B^\lambda_m C^m_\sigma)(\partial_j B^\sigma_i + \Gamma^\sigma_{\mu j} B^\mu_i) = 0.$$

Theorem 15. *The standard connection derived from a König structure is an (N, Q, F)-connection if and only if*

$$\begin{cases} (\delta^\lambda_\sigma - B^\lambda_m C^m_\sigma)(\partial_j B^\sigma_i + \Gamma^\sigma_{\mu j} B^\mu_i) = 0, \\ (\delta^\sigma_\mu - B^\sigma_m C^m_\mu)(\partial_j C^h_\sigma - \Gamma^\tau_{\sigma j} C^h_\tau) = 0. \end{cases}$$

Theorem 16. *If the distribution D_l defined by the f-structure derived from a König structure is integrable, the standard connection is an (N, Q)-connection.*

Theorem 17. *In order that the f-structure derived from a König structure be partially integrable, it is necessary and sufficient that the standard connection satisfies*

$$\begin{cases} R^\beta_{\alpha ij} = 0, \\ T^h_{ij} := \Gamma^h_{ij} - \Gamma^h_{ji} = 0, \\ (\delta^\alpha_\beta - B^\alpha_m C^m_\beta)(\partial_j B^\beta_i + \Gamma^\beta_{\sigma j} B^\sigma_i) = 0. \end{cases}$$

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