On the Connectivity Structures of Spaces

By

Tadashi Tanaka

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1. Introduction

If $\sigma$ is a topology on a set $X$, the resulting space will be denoted by $(X, \sigma)$. The family of all connected subsets of $(X, \sigma)$ is called the connectivity structure of $(X, \sigma)$ and denoted by $C(X, \sigma)$. A function $f: (X, \sigma) \rightarrow (Y, \tau)$ is a connected function if for every connected subset $C$ of $(X, \sigma)$, $f(C)$ is connected. Also $f$ is a connectivity function if the graph function $g: (X, \sigma) \rightarrow (X, \sigma) \times (Y, \tau)$, defined by $g(p) = (p, f(p))$, is a connected function [1], [2]. In [3], S. K. Hildebrandt and D. E. Sanderson have shown that $f: (X, \sigma) \rightarrow (Y, \tau)$ is a connectivity function if and only if $C(X, \sigma) = C(X, \sigma \cup f^{-1}(\tau))$ where $\sigma \cup f^{-1}(\tau)$ is the topology on $X$ generated by $\sigma$ and $f^{-1}(\tau)$.

In this paper we investigate what conditions, if we have such a space $(X, \sigma_2)$ that is finer than $(X, \sigma_1)$, will lead the relation $C(X, \sigma_1) = C(X, \sigma_2)$. Some results concerned with this problem may be found [4] and [5].

2. Definitions and preliminary results

Let $(X, \sigma_1)$ be a $T_1$-space and denote it by $(U(p))$, the family of all neighbourhoods of a point $p$ in $(X, \sigma_1)$. Let $F$ be a family of subsets of $X$ having the following properties:

(F1) The empty set $\emptyset$ belongs to $F$.

(F2) If $F_a$ and $F_b$ belong to $F$, then the sum $F_a \cup F_b$ belongs to $F$.

Let $(V(p))$ be the family of all subsets $V(p)$ of $X$, $V(p)$ of which takes the form $(U(p) - F) \cup p$ where $U(p) \in \{U(p)\}$ and $F \in F$. Then the following proposition holds.

Proposition. If to each point $p$ of $X$ there corresponds the family $(V(p))$ of subsets of $X$, then there is a unique $T_1$-space (denoted by $(X, \sigma_2)$) such that, for each point $p$ of $X$, $(V(p))$ is a base of neighbourhoods of $p$ in $(X, \sigma_2)$. And moreover $(X, \sigma_2)$ is finer than $(X, \sigma_1)$.

Proof. To prove this, we shall show that $(V(p))$ corresponded to each point $p$ of $X$ satisfies the three conditions on the bases of neighbourhoods of
a point.

First, it is obvious from the definition of \( \{V(p)\} \) that any set belonging to \( \{V(p)\} \) contains \( p \).

Second, if \( V_\alpha(p) \) and \( V_\beta(p) \) belong to \( \{V(p)\} \), then \( V_\alpha(p) \cap V_\beta(p) \) belongs to \( \{V(p)\} \). For, since \( V_\alpha(p) = (U_\alpha(p) - F_\alpha) \cup p \) and \( V_\beta(p) = (U_\beta(p) - F_\beta) \cup p \), where \( U_\alpha(p) \) and \( U_\beta(p) \) belong to \( \{U(p)\} \) and \( F_\alpha \) and \( F_\beta \) belong to \( F \), it follows that

\[
V_\alpha(p) \cap V_\beta(p) = \{(U_\alpha(p) - F_\alpha) \cap (U_\beta(p) - F_\beta)\} \cup p
\]

\[
= \{(U_\alpha(p) \cap F_\alpha) \cap (U_\beta(p) \cap F_\beta)\} \cup p
\]

\[
= \{(U_\alpha(p) \cap U_\beta(p)) \cap (F_\alpha \cap F_\beta)\} \cup p
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\[
= \{(U_\alpha(p) \cap U_\beta(p)) \cap (F_\alpha \cup F_\beta)\} \cup p
\]

\[
= \{(U_\alpha(p) \cap U_\beta(p)) \cap (F_\alpha \cup F_\beta)\} \cup p
\]

Therefore \( V_\alpha(p) \cap V_\beta(p) \) belongs to \( \{V(p)\} \).

Third, if \( V_\alpha(p) \) belongs to \( \{V(p)\} \) and \( q \) is any point of \( V_\alpha(p) \), then there exists \( V_\beta(q) \) belonging to \( \{V(q)\} \) such that \( V_\beta(q) \subset V_\alpha(p) \). For, let \( V_\alpha(p) = (U_\alpha(p) - F_\alpha) \cup p \) exactly as before and let \( U_\beta(q) \) be a set belonging to \( \{U(q)\} \) such that \( U_\beta(q) \subset U_\alpha(p) \). Then it follows that

\[
(U_\beta(q) - F_\alpha) \cup q \subset U_\alpha(p) - F_\alpha \subset (U_\alpha(p) - F_\alpha) \cup p = V_\alpha(p).
\]

Hence \( (U_\beta(q) - F_\alpha) \cup q \) is a set satisfying the required condition, which belongs to \( \{V(q)\} \).

Finally, Since \( (X, \sigma_1) \) is a \( T_1 \)-space and \( F \) contains the empty set, it is obvious that \( (X, \sigma_2) \) is a \( T_1 \)-space and is finer than \( (X, \sigma_1) \).

Thus our proposition is proved.

The space \( (X, \sigma_2) \) defined above is said to be the refined space of \( (X, \sigma_1) \) by \( F \).

Let \( A \) be any set of \( X \). Then "\( A \) is \( \sigma_i \)-P" means that \( A \) has the property P in \( (X, \sigma_i) \), and \( Cl_\epsilon A \) denotes the closure of \( A \) in \( (X, \sigma_i) \) where \( \epsilon = 1, 2 \).

3. Connectivity structures of \( (X, \sigma_i) \)

**Theorem 1.** Let \( (X, \sigma_2) \) be the refined space of \( (X, \sigma_1) \) by \( F \) and \( C \) any nondegenerate subset of \( X \). In order that \( C \) be \( \sigma_2 \)-connected, it is necessary and sufficient that (1) \( C \) be \( \sigma_1 \)-connected and (2) if \( F_\alpha \) is any set belonging to \( F \) then \( Cl_\epsilon (C - F_\alpha) \supset C \).

**Proof.** The condition is necessary. For let \( C \) be \( \sigma_2 \)-connected. Then \( C \)
is $\sigma_1$-connected since $(X, \sigma_2)$ is finer than $(X, \sigma_1)$. To show (2), suppose, on
the contrary, that there exist a point $p$ of $C$ and a set $F_a$ belonging to $F$ such
that $p$ is not in $Cl_{\sigma_1}(C-F_a)$. Then there exists a set $U_a(p)$ belonging to
$\{U(p)\}$ such that $U_a(p)$ is disjoint from $C-F_a$. Let set $V_a(p) = (U_a(p)-\n F_a) \cup p$, where $F_a$ and $U_a(p)$ are the sets defined above. Then it follows that

$$C \cap V_a(p) = C \cap \{(U_a(p)-F_a) \cup p\} = \{C \cap \{(U_a(p)-F_a) \cup p\} \cup \{C \cap p\} \cup \{C \cup U_a(p) \cap F_a\} \cup p = U_a(p) \cap (C-F_a) \cup p = p.$$

Hence in the case in which $C$ is a subspace of $(X, \sigma_2)$, $p$ is both open and
closed in $C$. Therefore the nondegenerate set $C$ is not $\sigma_2$-connected, contrary
to the supposition, and thus (2) is proved.

The condition is sufficient. For suppose, on the contrary, that $C$ satisfies
the conditions (1) and (2) in this theorem, and that $C$ is not $\sigma_2$-connected.
Let $C = A \cup B$ be a $\sigma_2$-separation of $C$. Then by (1), we assume, there exists a
point $a$ of $A$ such that $a$ is in $Cl_{\sigma_1}B$ without losing generality. Let $V_a(a) = (U_a(a)-F_a) \cup a$ be a set belonging to $\{V(a)\}$ that is disjoint from $B$. Then
there exists a point of $b$ of $B$ such that $b$ is in $U_a(a)$. Let $V_b(b) = (U_b(b)-F_b) \cup b$ be a set belonging to $\{V(b)\}$ such that $U_b(b) \subset U_a(a)$ and $V_b(b)$ is
disjoint from $A$. Then it follows that

$$B \cap U_b(b) \subset B \cap U_a(a) \subset B \cap \{(U_a(a)-F_a) \cup F_a\} = \{B \cap \{(U_a(a)-F_a) \cup F_a\} \cup (B \cap F_a) = \phi \cup (B \cap F_a) \subset F_a$$

and

$$A \cap U_b(b) \subset A \cap \{(U_b(b)-F_b) \cup F_b\} = \{A \cap \{(U_b(b)-F_b) \cup F_b\} \cup (A \cap F_b) = \phi \cup (A \cap F_b) \subset F_b.$$  

Hence

$$C \cap U_b(b) = (A \cap U_b(b)) \cup (B \cap U_b(b)) \subset F_a \cup F_b.$$ 

Therefore the point $b$ of $B$ is not in $Cl_{\sigma_1}(C-(F_a \cup F_b))$. This contradicts the
condition (2) since $F_a \cup F_b$ belongs to $F$.

Thus the sufficiency is proved.

**Lemma 1.** If $F_a$ is any set belonging to $F$, then $F_a$ is either empty or $\sigma_2$-
totally disconnected.
Proof. Let $F_a$ be any set belonging to $F$, $p$ be any point of $F_a$, and $U_a(p)$ be any set belonging to $\{U(p)\}$. Let us define $V_a(p) = (U_a(p) - F_a) \cup p$. Then we have

$$F_a \cap V_a(p) = F_a \cap \{(U_a(p) - F_a) \cup p\} = p.$$ 

Therefore, in the case in which $F_a$ is a subspace of $(X, \sigma_2)$, $p$ is open in $F_a$ and hence $p$ is a component of $F_a$.

Thus $F_a$ is $\sigma_2$-totally disconnected.

**Lemma 2.** If we have $C(X, \sigma_1) = C(X, \sigma_2)$, then any set belonging to $F$ is either empty or $\sigma_1$-totally disconnected.

**Proof.** Suppose, on the contrary, that there exists a set $F_a$ belonging to $F$ which is neither empty nor $\sigma_1$-totally disconnected. Let $C$ be a nondegenerate $\sigma_1$-connected subset of $F_a$. Then, by the hypothesis $C(X, \sigma_1) = C(X, \sigma_2)$, $C$ is $\sigma_2$-connected. On the other hand, by Lemma 1 $F_a$ is $\sigma_2$-totally disconnected and so is $C$.

This contradiction proves Lemma 2.

**Lemma 3.** Assume that $(X, \sigma_1)$ satisfies the condition as follows:

If $C$ is any nondegenerate $\sigma_1$-connected subset of $X$, $p$ is any point of $C$, and $U_a(p)$ is any set belonging to $\{U(p)\}$, then $C \cap U_a(p)$ is not $\sigma_1$-totally disconnected.

Then if each set belonging to $F$ is either empty or $\sigma_1$-totally disconnected, we have $C(X, \sigma_1) = C(X, \sigma_2)$.

**Proof.** Let $C$ be any nondegenerate $\sigma_1$-connected subset of $X$. To prove this, by Theorem 1 it is only need to show that for any set $F_a$ belonging to $F$ we have $C_{\sigma_1}(C - F_a) \supseteq C$. Suppose, on the contrary, that there exist a point $p$ of $C$ and a set $U_a(p)$ belonging to $\{U(p)\}$ such that $U_a(p) \cap (C - F_a)$ is empty. Then $U_a(p) \cap C \subseteq F_a$. Hence $F_a$ contains a nondegenerate $\sigma_1$-connected set since $U_a(p) \cap C$ contains the same. This is impossible because $F_a$ is $\sigma_1$-totally disconnected. Therefore we have $C_{\sigma_1}(C - F_a) \supseteq C$.

Thus Lemma 3 is proved.

Combining Lemma 2 with Lemma 3 we have the following theorem.

**Theorem 2.** Assume that $(X, \sigma_1)$ satisfies the condition as follows:

(*) If $C$ is any nondegenerate $\sigma_1$-connected subset of $X$, $p$ is any point of $C$, and $U_a(p)$ is any set belonging to $\{U(p)\}$, then $C \cap U_a(p)$ is not $\sigma_1$-totally disconnected.

Then in order that we have $C(X, \sigma_1) = C(X, \sigma_2)$ it is both necessary and sufficient that
Each set belonging to $F$ is either empty or $\sigma_1$-totally disconnected.

On the condition (*) in Theorem 2 used only for the proof of sufficiency, we have the following results.

**Corollary.** In order to have $C(X, \sigma_1) = C(X, \sigma_2)$ for every refined space $(X, \sigma_2)$ of $(X, \sigma_1)$ by any family $F$ of subsets of $X$ satisfying the conditions $(F_1)$, $(F_{ii})$ and (**), the condition (*) is both necessary and sufficient.

**Proof.** By Theorem 2 the condition (*) is sufficient. To prove the necessity suppose, on the contrary, that the condition (*) is not satisfied. Then there exist a nondegenerate $\sigma_1$-connected set $C$, a point $p$ of $C$ and a set $U_\alpha(p)$ belonging to $\{U(p)\}$ such that $C \cap U_\alpha(p)$ is $\sigma_1$-totally disconnected. Then the family of subsets of $X$ consisting of the empty set and $C \cap U_\alpha(p)$ satisfies the conditions $(F_1)$, $(F_{ii})$ and (**). Let $(X, \sigma_2)$ be the refined space of $(X, \sigma_1)$ by the family above. Then $C$ is not $\sigma_2$-connected since if $V_\alpha(p) = \{U_\alpha(p) - (C \cap U_\alpha(p)) \cup p$ we have $C \cap V_\alpha(p) = p$. This contradicts our hypothesis. Thus the corollary is proved.

**Remark.** In [6], Mazurkiewicz has shown the existence of nondegenerate connected set in a plane containing none of bounded nondegenerate connected subsets. Accordingly, no spaces containing a subspace homeomorphic to the plane satisfy the condition (*).

On the other hand any nondegenerate continuum which is locally connected and contains no simple closed curve (called the dendrite) satisfies the condition (*), because every connected subset of any dendrite is arcwise connected [7].

*Faculty of Engineering*
*Tokushima University*

**References**


