On the Stability of Maximal Submanifolds in Pseudo-Riemannian Manifolds

By

Tōru Ishihara
(Received September 10, 1987)

§ 1. Introduction

There are many studies on the stability of minimal submanifolds in Riemannian manifolds (see, for example, [1], [2]). In the present paper, we investigate the stability of spacelike maximal submanifolds in a pseudo-Riemannian manifold. Let $f: M \to N$ be an immersion of a Riemannian manifold into a pseudo-Riemannian manifold. If $f$ is a maximal immersion, then it represents a critical points for the area function on the space of all immersions of $M$ into $N$. To study the stability of maximal immersions, we need a formula related with the second derivatives of the area function, the second variational formula.

In §2, we describe local formulas for immersions into pseudo-Riemannian manifolds. The second variational formula is given in §3. In §4, we will prove the stability of maximal immersions into pseudo-Riemannian manifolds of non-positive sectional curvature. We investigate the instability of a compact Riemannian manifold imbedded into the unit sphere $S^{n+p}_p$ with index $p$ in the last section.

§ 2. Local formulas

Let $N$ be an $(n+p)$-dimensional pseudo-Riemannian manifold with index $p$. Let $M$ be an $m$-dimensional Riemannian manifold isometrically immersed in $N$. As the pseudo-Riemannian metric of $N$ induces the Riemannian metric of $M$, we must assume that $m \leq n$ and we may call it the spacelike immersion. We choose a local field of pseudo-Riemannian orthonormal frames $e_1, e_2, \ldots, e_{n+p}$ in $N$ such that at each point of $M$, $e_1, e_2, \ldots, e_m$ span the tangent space of $M$ and forms an orthonormal frame there. We make use of the following convention on the ranges of indices if otherwise stated:

$$1 \leq A, B, C \leq n+p, 1 \leq i, j, k \leq m, m+1 \leq \alpha, \beta, \gamma \leq n+p$$

and we shall agree that repeated indices are summed over the respective ranges. Let $\{\omega_A\}$ be the coframe field dual to $\{e_A\}$. Then the pseudo-Riemannian metric
of $N$ is given locally by
\[ ds_a^2 = \sum_{a=1}^n \omega_a^2 - \sum_{s=n+1}^{n+p} \omega_s^2 = \sum e_A \omega_A^2, \]
where $e_a = 1$ for $1 \leq a \leq n$ and $e_s = -1$ for $n+1 \leq s \leq n+p$. The structure equations of $N$ are given by
\[ d\omega_A = \sum e_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \]
\[ d\omega_{AB} = \sum e_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum e_C e_B K_{ABCD} \omega_C \wedge \omega_D. \]
(2.1)
We denote by $\theta_A$, $\theta_{AB}$ the restrictions of $\omega_A$, $\omega_{AB}$ to $M$. Then $\theta_s = 0$ for $m+1 \leq s \leq n+p$ and the Riemannian metric is given by $ds_M^2 = \sum \theta_i^2$. We may put
\[ \omega_{si} = \sum h_{sij} \omega_j. \]
(2.2)
Then $h_{sij}$ are the components of the second fundamental form of the immersion. From (2.1), we obtain the structure equations of $M$;
\[ d\theta_i = \sum \theta_{ij} \wedge \theta_j, \]
\[ d\theta_{ij} = \sum \theta_{ik} \wedge \theta_{kj} - \frac{1}{2} \sum R_{ijkl} \theta_k \wedge \theta_l \]
(2.3)
and the Gauss formula
\[ R_{ijkl} = K_{ijkl} + \sum e_a (h_{sik} h_{sjl} - h_{sil} h_{sjk}). \]
(2.4)
We call $H = \frac{1}{m} \sum_{i} (\sum_{s} h_{si}) e_s$ the mean curvature normal. In the present paper, we study an immersion with vanishing mean curvature, that is, $H = 0$. When $n = m$, an immersion with vanishing mean curvature is said to be maximal.

§ 3. Variational formulas

We will follow the method in [3]. Let $f: M \to N$ be an immersion as in §2. If $M$ is compact, possibly with boundary, its total volume is given by the integral
\[ V = \int_M \theta_1 \wedge \cdots \wedge \theta_m. \]
(3.1)
Let $I$ be the interval $-\frac{1}{2} < t < \frac{1}{2}$. Let $F: M \times I \to N$ be a differentiable mapping such that its restriction to $M \times t$, $t \in I$, is an immersion and that $F(m, 0) = f(m)$, $m \in M$. Put $f_t(m) = F(m, t)$. We call $f_t$ the variation of $f$. Let $\{e_A(m, t)\}$ be a local frame field over $M \times I$ such that for every $t \in I$, $e_A(m, t)$ are tangent to $F(M \times t)$ and hence $e_s(M, t)$ are normal vectors. The forms $\omega_A$, $\omega_{AB}$ can be written
(3.1) \[ \omega_i = \theta_i + a_i dt, \quad \omega_2 = a_2 dt, \quad \omega_3 = \theta_3 + a_3 dt, \]
where \( \theta_i, \theta_3 \) are linear differential forms in \( M \) with coefficients which may depend on \( t \). For \( t = 0 \) they are reduced to the forms with the same notation on \( M \). The vector \( \sum e_A a_A e_A \) at \( t = 0 \) is called the deformation vector. The operator \( d \) on \( M \times I \) is written as

(3.2) \[ d = d_M + dt \frac{\partial}{\partial t}. \]

Using (2.1), we get

(3.3) \[ d(\omega_1 \wedge \cdots \wedge \omega_m) = \sum e_a \omega_a \Omega_a, \]
where

(3.4) \[ \Omega_a = - \sum \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge \omega_i \wedge \omega_{i+1} \wedge \cdots \wedge \omega_m. \]

Substituting (3.1) into (3.3) and considering the coefficient of \( dt \), we get

(3.5) \[ \frac{\partial}{\partial t} (\theta_1 \wedge \cdots \wedge \theta_m) = d_M \sum (-1)^{i-1} a_i \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m \]
+ \[ \sum e_a a_a \Theta_a, \]
where

(3.6) \[ \Theta_a = - \sum \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_i \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m. \]

Thus the first variation of volume is written as

(3.7) \[ V'(0) = \int_M \sum e_a a_a \Theta_a + \int_{\partial M} \sum (-1)^{i-1} a_i \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m. \]

If the deformation vector is normal to \( M \) along the boundary \( \partial M \), the second term at the right hand side of (3.5) vanishes. This condition is satisfied if the boundary \( \partial M \) remains fixed. The first integral is zero for arbitrary \( a_x \) if and only if the mean curvature normal \( H \) vanishes.

To obtain the second variational formula, we also follow Chern (see §8 in [3]). By exterior differentiation of \( \Omega_a \), we have

(3.8) \[ -d \Omega_a = - \sum \sum e_\beta \omega_\beta \wedge \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge \omega_{j+1} \wedge \cdots \wedge \omega_{i-1} \wedge \omega_i \wedge \omega_{i+1} \]
+ \[ \cdots \wedge \omega_m + \sum e_\beta \omega_\beta \wedge \omega_1 \wedge \cdots \wedge \omega_m \]
+ \[ \sum e_\beta \tilde{R}_{s_\beta} \omega_\beta \wedge \omega_1 \wedge \cdots \wedge \omega_m + \text{terms quadratic in } \omega_s, \omega_\beta, \]
where

(3.9) \[ \tilde{R}_{s_\beta} = \sum_i K_{i s i \beta}. \]
Substituting (3.1) into (3.4), we get

\begin{equation}
\Omega_2 = \Theta_2 + dt \wedge \Phi_2,
\end{equation}

where

\begin{equation}
\Phi_2 = \sum (-1)^l a_{i_1} \theta_{i_1} \wedge \cdots \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m + \sum (-1)^l a_{j} \theta_{j} \wedge \cdots \wedge \theta_{j-1} \wedge \theta_{j+1} \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m.
\end{equation}

From (3.10), it follows

\begin{equation}
d\Omega_2 = dt \wedge \frac{\partial \Theta_2}{\partial t} - dt \wedge d_M \Phi_2 + d_M \Theta_2.
\end{equation}

Substituting (3.1) into (3.8), we have

\begin{equation}
-d\Omega_2 = -dt \wedge (\sum \varepsilon_{\beta} \omega_{\beta_2} \wedge \Phi_2 + \Lambda_2) + \text{other terms},
\end{equation}

where

\begin{equation}
\Lambda_2 = -\sum \varepsilon_{\beta} n_{\beta_2} \omega_{\beta_2} \wedge \cdots \wedge \theta_m + \sum \varepsilon_{\beta} a_{\beta} \theta_1 \wedge \cdots \wedge \theta_{j-1} \wedge \theta_{j+1} \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m.
\end{equation}

From (3.12) and (3.13), we obtain

\begin{equation}
\frac{\partial \Theta_2}{\partial t} = d_M \Phi_2 + \sum \varepsilon_{\beta} \omega_{\beta_2} \wedge \Phi_2 + \Lambda_2.
\end{equation}

Taking the exterior derivative of the second equation of (3.1) and using (2.1), we get

\begin{equation}
d_M a_2 = -\sum (a_{i} \theta_{i_2} - a_{i_2} \theta_i) - \sum \varepsilon_{\beta} a_{\beta} \omega_{\beta_2}.
\end{equation}

Combining (3.15) and (3.16), we have

\begin{equation}
\frac{\partial}{\partial t} (\sum \varepsilon_2 a_2 \Theta_2) = \sum \varepsilon_2 \frac{\partial a_2}{\partial t} \Theta_2 + d_M (\sum \varepsilon_2 a_2 \Phi_2) + \sum \varepsilon_2 a_2 \Lambda_2
\end{equation}

\begin{equation}
+ \sum \varepsilon_2 (a_{i} \theta_{i_2} - a_{i_2} \theta_i) \wedge \Phi_2.
\end{equation}

In the sequel, we assume that \( f: M \to N \) is an immersion with vanishing mean curvature normal, that is, \( \Theta_2 |_{t=0} = 0 \). Differentiating (3.5) and setting \( t=0 \), we get

\begin{equation}
V''(0) = \int \frac{\partial^2}{\partial t^2} \left( \theta_1 \wedge \cdots \wedge \theta_m \right) |_{t=0}
\end{equation}

\begin{equation}
= \int \frac{\partial}{\partial t} \left( \sum (-1)^{l-1} a_{l} \theta_{l_1} \wedge \cdots \wedge \theta_{l-1} \wedge \theta_{l+1} \wedge \cdots \wedge \theta_m + \sum \varepsilon_2 a_2 \Phi_2 \right)
+ \int_M (\sum \varepsilon_2 (a_{i} \theta_{i_2} - a_{i_2} \theta_i) \wedge \Phi_2 + \sum \varepsilon_2 a_2 \Lambda_2).
\end{equation}
Moreover assume that the variation vector is normal and the boundary $\partial M$ is fixed, that is, $a_i = 0$ and $a_s(m, 0) = 0$, $m \in \partial M$. Then on $M$, we have

$$\Phi_x = \sum (-1)^i a_{is} \theta_i \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_m,$$

and

$$\sum \varepsilon_x a_s A_x = - \left( \sum_{\sigma, \beta} \varepsilon_x \varepsilon_\beta a_s a_\beta \bar{K}\sigma_\beta + \sum \varepsilon_x \varepsilon_\beta a_s a_\beta \sigma_\beta \right) dM$$

where $dM = \theta_1 \wedge \cdots \wedge \theta_m$ and

(3.19) \hspace{1cm} \sigma_\beta = \sum_{i, j} h_{i, j} h_{\beta, j}.$$

Thus we obtain

(3.20) \hspace{1cm} V'(0) = \int \left( \sum \varepsilon_x a_s^2 - \sum \varepsilon_x \varepsilon_\beta a_s a_\beta (\bar{K}\sigma_\beta + \sigma_\beta) \right) dM.$$

From (3.16), we have on $M$

(3.21) \hspace{1cm} \sum a_{is} \theta_i = d a_s + \sum \varepsilon_\beta a_\beta \omega_{\beta s}.$$

Hence $a_{is}$ are the coefficients of the covariant derivatives of $a = \sum \varepsilon_s a_s e_s$. Its second covariant derivative is given by

(3.22) \hspace{1cm} \sum a_{isi} \theta_i = d a_s + \sum a_{js} \theta_{ji} + \sum \varepsilon_\beta a_{is} \omega_{\beta s}$

and the Laplacian of $a$

(3.23) \hspace{1cm} \Delta a_s = \sum a_{isi}.$$

Hence we have

(3.24) \hspace{1cm} d(\sum \varepsilon_x a_s a_{is} \theta_i) = (\sum \varepsilon_x a_s^2 + \langle \Delta a, a \rangle) dM.$$

Thus, by putting

(3.25) \hspace{1cm} L a_s = - \Delta a_s - \sum \varepsilon_\beta a_\beta (\bar{K}\sigma_\beta + \sigma_\beta),$$

we obtain

(3.26) \hspace{1cm} V'(0) = \int_M \langle L a, a \rangle dM.$$

The operator introduced in (3.25) is a strongly elliptic operator and has distinct real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty$. Moreover, the dimension of each eigenspace is finite. The index of $M$ is the sum of the dimensions of the eigenspaces which correspond to negative eigenvalues. The nullity to $M$ is the dimension of the null eigenspace.
§ 4. The stability of spacelike maximal immersions

In this section we assume that \( n = m \). Then we have \( e_{\alpha} = -1 \) for all \( n + 1 \leq \alpha \leq n + p \). The condition \( H = 0 \) implies that the immersion is maximal. In this case, the second variational formula (3.20) is reduced to

\[
V''(0) = -\int (\sum a_{i\alpha}^2 + \sum (\tilde{K}_{\alpha\beta} + \sigma_{\alpha\beta})a_{\alpha}a_{\beta})dM.
\]

We can put

\[
\|h(a)\|^2 = \sum \sigma_{\alpha\beta}a_{\alpha}a_{\beta}
\]

and

\[
\sum \tilde{K}_{\alpha\beta}a_{\alpha}a_{\beta} = -\sum_{i} K(e_i, a)\|a\|^2,
\]

where \( \|a\|^2 = \sum a_{\alpha}^2 \) and each \( K(e_i, a) \) is the sectional curvature of \( N \) for the plane spanned by \( e_i \) and \( a \). Now we let \( f: M \to N \) be a maximal isometric immersion. Let \( D \) be a domain on \( M \) with \( \overline{D} \) compact. Then the domain is called stable if \( V''_D(0) \leq 0 \), where \( V''_D(0) \) is the second variation for the immersion \( f|_D: D \to N \), the restriction of \( f \) to \( D \). The immersion is stable if every such domain is stable. Thus from (4.1), we have

\[
V''_D(0) = -\int_D (\sum a_{i\alpha}^2 + \|h(a)\|^2 - \sum_{i} K(e_i, a)\|a\|^2) dM.
\]

Thus we obtain

**Proposition 4.1.** Let \( f: M \to N \) be an maximal isometric immersion of a Riemannian manifold \( M \) into a pseudo-Riemannian manifold with non-positive sectional curvature. Then the maximal immersion is stable.

Let \( H^{n+p}_p(r) \) be the pseudo-hyperbolic space of radius \( r(>0) \) (see, for example, [4] or [5]). We constructed the maximal isometric immersion of \( H^{2}(\sqrt{3}) \) into \( H^{2}_2(1) \) and the maximal isometric immersion of \( H^{n}(\sqrt{n_1/n}) \times \cdots \times H^{n+p+1}(\sqrt{n_{p+1}/n}) \) into \( H^{n+p}_p(1) \), where \( n_1 + \cdots + n_{p+1} = n \). From the above proposition, it is evident that these immersions are stable. Let \( M \) be a compact hypersurface of \( N \), that is, \( p = 1 \). In this case, a deformation vector \( a \) is written as \( a = u e_{n+1} \), where \( e_1, \ldots, e_{n+1} \) is a local orthonormal frame field of \( N \) such that on \( M \), \( e_{n+1} \) is a timelike normal unit vector field. Let \( D \) be a relative compact domain on \( M \). From (4.4), it follows

\[
V''_D(0) = -\int_D (|\nabla u|^2 + (\|h\|^2 - \sum_{i} K(e_i, e_{n+1}))u^2) dM
\]

\[
= \int_D (u \Delta u - (\|h\|^2 - \sum_{i} K(e_i, e_{n+1}))u^2) dM,
\]
where $\Delta$ is the Laplacian on $D$. As it is well known, the first eigenvalue $\lambda_1(D)$ satisfies
\[ \int_D |\nabla u|^2 dM \geq \lambda_1(D) \int_D u^2 dM. \]
Hence, from (4.5), we obtain

**Proposition 4.2.** If $M$ is a $n$-dimensional spacelike maximal hypersurface of a Lorentzian manifold $N$. Assume that the sectional curvature is bounded above by the positive constant $c$. Let $D$ be a relative compact domain on $M$. If the first eigenvalue of the Laplacian of $D$ satisfies $\lambda_1(D) \geq nc$, then $D$ is stable.

Now assume that $M$ is a maximal spacelike surface immersed in a 3-dimensional Lorentzian manifold. Then from the Gauss formula (2.4), we get
\[ 2(R-K(e_1, e_2)) = \|h\|^2, \]
where $R$ is the curvature of $M$. If the sectional curvature of $N$ is bounded above by a positive constant $c$, we have from (4.6), for a relative compact domain $D$ on $M$,
\[ V''_D(0) \leq -\int_D (|\nabla u|^2 + 2(R-2c)) dM. \]
Thus we obtain

**Proposition 4.3.** Let $M$ be a maximal spacelike surface immersed in a 3-dimensional Lorentzian manifold. Assume that the sectional curvature is bounded above by a positive constant $c$. If the curvature of $M$ is bounded below by the constant $2c$, the immersion is stable.

§ 5. **Compact maximal submanifolds in $S^{n+p}_p$**

Let $N = S^{n+p}_p$ be the pseudosphere with index $p$, that is, $S^{n+p}_p = \{ x \in R^{n+p+1}_p : \langle x, x \rangle = x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 - \cdots - x_{n+p+1}^2 \}$. In this case, $\tilde{R}_{ab}$ defined by (3.9) are given by $\tilde{R}_{ab} = ne_{a\beta} \delta_{\alpha \beta}$. In this section, if a Riemannian manifold $M$ is immersed in $S^{n+p}_p$ with vanishing mean curvature normal, we say that $M$ is immersed maximally in $S^{n+p}_p$, though it is an abuse of language. Now, we consider the standard totally geodesic imbedding of $S^m$ into $S^{n+p}_p$ ($m \leq n$). Then the corresponding operator defined by (3.25) is reduced to
\[ La_\alpha = -\Delta a_\alpha - na_\alpha. \]
Thus, by using the same argument in the proof of Proposition 5.1.1 in [6], we have

**Proposition 5.1.** When $S^m$ is regarded as a maximal submanifold of $S^{n+p}_p$, its index is $n+p-m$ and its nullity is $(m+1)(n+p-m)$.
Let \( \{ f_A \} \) be the frame field on \( R^{n+p+1}_p \) given by the parallel translation of the standard base of \( R^{n+p+1}_p \). It satisfies \( \langle f_{n+p+1}, f_{n+p+1} \rangle = 1 \) and
\[
(5.2) \quad \langle f_A, f_B \rangle = e_A \delta_{AB}, \quad e_i = 1 \quad \text{for} \quad 0 \leq i \leq n + p, \\
e_{n+1} = -1 \quad \text{for} \quad n + 1 \leq \alpha \leq n + p.
\]
Let \( \{ \theta_A \} \) be the dual coframe field. Then we have
\[
(5.3) \quad \theta_A(f_B) = e_A \delta_{AB}, \quad \text{for} \quad 1 \leq A', B' \leq n + p + 1.
\]
Take a local frame field \( \{ e_A \} \) on \( S^{n+p}_p \) such that \( \langle e_A, e_B \rangle = e_A \delta_{AB} \) and for each \( x \in S^{n+p}_p, e_{n+p+1}(x) = x \). Its dual coframe field \( \{ \omega_A \} \) satisfies
\[
(5.4) \quad \omega_A(e_B) = e_A \delta_{AB}.
\]
Put
\[
(5.5) \quad f_A = \sum_{B' = 1}^{n+p+1} e_{B'} \lambda_{A'B'} e_{B'}, \quad \theta_A = \sum_{B' = 1}^{n+p+1} e_{B'} \mu_{A'B'} \omega_{B'}.
\]
Then we have
\[
(5.6) \quad \sum e_{B'} \lambda_{A'B'} \mu_{B'C} = e_A \delta_{AC}.
\]
We can put
\[
(5.7) \quad dx = \sum e_A \omega_A e_A, \quad d\omega_A = \sum e_B \omega_{A'B'} e_{B'}.
\]
By the exterior differentiation of the first equation at (5.5), we get
\[
d\lambda_{A'B'} + \sum e_C \lambda_{A'C} \omega_{C'B'} = 0.
\]
Using (5.6), we have
\[
(5.8) \quad \omega_{A'B'} = -\sum e_C \mu_{C'A'} d\lambda_{C'B'}.
\]
As we have \( \langle dx, x \rangle = 0 \), we have \( \omega_{n+p+1} = 0 \) on \( S^{n+p}_p \). From (5.7), it follows
\[
d e_{n+p+1} = \sum e_A \omega_{n+1A} e_A = \sum e_A \omega_A e_A.
\]
Thus we have \( \omega_{n+1A} = \omega_A \) on \( S^{n+p}_p \). Hence the second fundamental form \( H_{AB} \) of \( S^{n+p}_p \) in \( R^{n+p+1}_p \) is given by
\[
(5.9) \quad H_{AB} = e_A \delta_{AB}.
\]
Let \( \vec{Z} = \sum e_A \lambda_{A'} f_{A'} \) be a parallel vector field on \( R^{n+p+1}_p \) where \( c_{A'} \) are constants. Denote by \( Z = \sum e_A z_A e_A \) the tangential projection onto \( S^{n+p}_p \) of \( \vec{Z} \). Then we have
\[
(5.10) \quad z_B = \sum e_A \lambda_{A'B}.
\]
The coefficients of the covariant derivative of \( Z \) are given by
(5.11) \[ \sum \xi_A z_{BA} \omega_A = d z_B + \sum \xi_A z_A \omega_{AB}. \]

Then, using (5.8), we obtain

(5.12) \[ z_{AB} = \lambda \xi_A \delta_{AB}, \]

where we put \( \lambda = - \sum \varepsilon_{A' A} \lambda_{A' A + p + 1}. \)

Let \( M \) be a compact \( m \)-dimensional Riemannian manifold imbedded maximally in \( S^{n+p}_{p+p} \). We may assume that the frame field satisfies that \( e_1, \ldots, e_m \) are tangent to \( M \). Now put on \( M \)

(5.13) \[ Z^T = \sum z_i e_i, \quad Z^N = \sum e_a z_a e_a. \]

The covariant derivative of \( Z^T \) is given by

(5.13) \[ \sum z^N_i \omega_i = d z_a + \sum e_{\beta} z_{\beta} \omega_{\beta}. \]

From (5.12), it follows

(5.14) \[ z^N_i = - \sum z_j h_{zi j}. \]

Similarly, the covariant derivative of \( Z^T \) is given by

(5.15) \[ \sum z^T_i \omega_i = d z_j + \sum z_i \omega_{ij}. \]

Hence, we have

(5.16) \[ z^T_j = \lambda \delta_{ij} + \sum \xi_a z_a h_{ai j}. \]

The second covariant derivative of \( Z^N \) is also given by

(5.17) \[ \sum z^N_{ai j} \omega_j = d z_{ai} + \sum z^N_{ai j} \omega_{ji} + \sum e_{\gamma} z_{\gamma} \omega_{\beta}. \]

By a calculation, we get

(5.18) \[ Z^N_{ai j} = - \sum z_a h_{zk i j} - \lambda h_{ai j} + \sum e_{\beta} z_{\beta} h_{zk i j}. \]

where \( h_{zk i j} \) are the components of the covariant derivative of the second fundamental form \( h \) of \( M \) and satisfy

(5.19) \[ h_{zk i j} = h_{zk i j}, \quad h_{zk i j} = h_{zik j}. \]

As \( M \) is a maximal submanifold, that is, \( \sum h_{ai i} = 0 \), it follows from (5.19) that \( \sum h_{zk i j} = 0 \). Thus we obtain

(5.20) \[ \Delta z_a = \sum z^N_{ai i} = - \sum e_{\gamma} z_{\gamma} z_{\beta}. \]

The operator \( L \) defined (3.25) is reduced to

(5.21) \[ L z_a = - \Delta z_a - n z_a - \sum e_{\gamma} z_{\gamma} z_{\beta} = - n z_a. \]
In other words, we obtain $LZ^N = -nZ^N$. Thus we have

**Lemma 5.2.** Let $M$ be a $m$-dimensional compact Riemannian manifold imbedded maximally in $S^p_{n+p}$. Then it holds the index of $M \geq n + p - m$.

By the same argument in the proof of Proposition 5.1.6 in [6], we have

**Lemma 5.3.** Under the same assumption of Lemma 5.2, the index of $M$ is $n + p - m$ if and only if $M$ is isometric to $S^n$, and imbedded in the standard way as a totally geodesic submanifold.

A Killing vector field $X$ on a pseudo-Riemannian manifold is a vector field for which the Lie derivative of the metric tensor vanishes. Let $X = \sum X_A e_A$ be the vector field on $S^p_{n+p}$. Then it is a Killing vector field if and only if it is skew-adjoint relative to the metric, that is,

$$X_{AB} + X_{BA} = 0,$$

where $X_{AB}$ are the components of the covariant derivative of $X$ (see p. 250 of [5]). Let $X_{ABC}$ be the components of the covariant derivative of the killing vector field $X$. Then they satisfy

$$X_{ABC} - X_{ACB} = \sum \epsilon_{DE} X_D K_{EABC},$$

where $K_{EABC} = \epsilon_{DE} (\delta_{DB} \delta_{AC} - \delta_{AD} \delta_{BC})$. Let $M$ be a compact Riemannian manifold immersed maximally in $S^p_{n+p}$. Let $X^N$ be the normal vector field on $M$ by normal projection of a Killing vector field $X$. Then the components of the covariant derivative of $X^N$ are given by

$$X^N_{ai} = X_{ai} - \sum X_j h_{aji},$$

and the components of its covariant derivative are given by

$$X^N_{aij} = X_{aij} + \sum \epsilon_{\beta \gamma} X_{\beta \gamma} h_{pij} + \sum X_{\kappa \lambda} h_{akj} - \sum X_{\kappa \lambda} h_{akt}$$

$$- \sum X_k h_{skij} - \sum \epsilon_{\beta \gamma} X_{\beta \gamma} h_{skj} h_{aki}.$$

Hence the Laplacian of $X^N$ is given by

$$\Delta X^N_{a} = \sum X^N_{a ii} = \sum X_{a ii} - \sum \epsilon_{\beta \gamma} \sigma_{a \beta} X_{a \gamma}.$$

On the other hand, we get from (5.22) and (5.23)

$$\sum X_{a ii} = -nX_a.$$

Thus as similarly as (5.21), we obtain

$$LZ^N = 0,$$

that is, $Z^N$ is a Jacobi field on $M$. 
Lemma 5.4. Let $\Omega$ denote the vector space of Killing vector fields on $S_p^{n+p}$. For any $X \in \Omega$, $X^N$ is a Jacobi field on $M$.

By using the same arguments in the proofs of Lemmas 5.1.8 and 5.1.9 of [6], we have

Lemma 5.5. Put $\Omega^N = \{X^N : X \in \Omega\}$. Then $\dim \Omega^N \geq (n+p-m)(n+1)$. Dim $\Omega^N = (n+p-m)(n+1)$ if and only if $M$ is diffeomorphic to $S^m$ and imbedded in the standard way as a totally geodesic submanifold.

Consequently, we have an analogue of Theorem 5.1.1 of [6].

Theorem 5.6. Let $M$ be a compact $m$-dimensional Riemannian manifold imbedded in $S_p^{n+p}(m \leq n)$ such that the mean curvature normal vanishes. Then the index of $M$ is greater than or equal to $n+p-m$, and equality holds only when $M$ is $S^m$. The nullity of $M$ is greater than or equal to $(n+p-m)(m+1)$ and equality holds only when $M$ is $S^m$.

Department of Mathematics and Computer Sciences,
Faculty of Integrated Arts and Sciences,
Tokushima University

References