Surfaces of Revolution in the Lorentzian 3-Space

BY

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§1. Introduction

Let \( L^3 = (R^3, dx^2 + dy^2 - dz^2) \) be the Lorentzian 3-space. Surfaces of revolution are obtained by rotating about their axes the generating curves. There are three types of surfaces of revolution in \( L^3 \), that is, surfaces rotating about a space-like axis, a time-like axis and a null axis. By using a method similar to Kenmotsu's in [7], for a given continuous function \( H(s) \), we can solve the differential equations about the generating curves of surfaces of revolution whose mean curvature is \( H(s) \).

Using these solutions, we can give a Delaunay's characterization of the surfaces of revolution in \( L^3 \) with constant mean curvature, which may be stated roughly as the following: Generating curves of surfaces of revolution in \( L^3 \) with constant mean curvature are roulettes of conics. This problem was already studied by Hano and Nomizu in [5]. But they use the method of Hsiang and Yu [6] and treated only space-like surfaces of revolution. On the other hand, in the present paper, we depend on the Kenmotsu's method and deal with space-like and time-like surfaces together. This gives better geometric interpretation of generating curves of surfaces of revolutions.

In the Lorentz 2-space \( L^2 = (R^2; \ dy^2 - dz^2) \), there are two kinds of conics, horizontal conics and vertical conics. Moreover, for a given conic we have its roulettes rolling along a space-like line and along a time-like line. The roulette of a vertical (resp., horizontal) conic generates a space-like (resp., time-like) surface of revolution with constant mean curvature, and the roulette of a conic rolling along a space-like (resp., time-like) line is a generating curve of a surface of revolution in \( L^3 \) with constant mean curvature, which rotates about a space-like (resp., time-like) axis (see, for details, Theorem 4).

The Gauss map of a surface in \( L^3 \) with constant mean curvature is also a harmonic mapping. Hence the surfaces of revolution with constant curvature constructed in the paper give the harmonic Gauss maps of the surfaces to the sphere \( S^2 \) or the hyperbolic space \( H^2 \).

§2. The outline

A surface in \( L^3 \) is called a surface of revolution with axis \( l \) if it is invariant under
the action of the group of motions in $L^3$ which fix each point of the line $l$. A surface of revolution with space-like axis is given by

\begin{equation}
S(s, t) = (z(s)\sinh t, y(s), z(s)\cosh t), \ (S\text{-axis}),
\end{equation}

where $(y(s), z(s))$ is a curve in $L^2 = \{R^2, dy^2 - dz^2\}$ which is parametrized by the arc length and defined on some open interval $I$. Hence it holds

\begin{equation}
y'^2 - z'^2 = \varepsilon,
\end{equation}

where $\varepsilon$ is 1 or $-1$ according to the space-like curve or the time-like curve. The curve is called the generating curve of the surface. A surface of revolution with time-like axis is given by

\begin{equation}
T(s, t) = (y(s)\cos t, y(s)\sin t, z(t)), \ (T\text{-axis}),
\end{equation}

where $(y(s), z(s)), s \in I,$ is a curve in $L^2$ satisfying (2.2). A surface of revolution with null axis is given by

\begin{equation}
N(s, t) = (y(s) + z(s) - t^2 z(s), -2tz(s), y(s) - z(s) - t^2 z(s)), \ (N\text{-axis}),
\end{equation}

where $(y(s) + z(s), y(s) - z(s))$ is a curve in $L^2$ with

\begin{equation}
4y'z' = \varepsilon.
\end{equation}

Though the true generating curve is $(y(s) + z(s), y(s) - z(s))$ in this case, we call simply the curve $(y(s), z(s))$ the generating curve of the surface. In all cases, surfaces of revolution are space-like or time-like according to $\varepsilon = 1$ or $\varepsilon = -1$.

We can solve the differential equations about the generating curves of surfaces of revolution with given mean curvature function $H(s), s \in I$. In particular, concerning surfaces of revolution with constant mean curvature, we have explicit solutions. For a non-zero constant $H$ and non-negative constant $d$, we put

\begin{equation}
\begin{aligned}
f(s) &= 1 + d^2 - 2d \cosh(2Hs), \quad g(s) = d \cosh(2Hs) - 1, \\
h(s) &= 1 - d^2 - 2d \sinh(2Hs), \quad k(s) = d \sinh(2Hs) - 1, \\
u_1(s) &= d \sin^{-1}(s/d), \quad u_{-1}(s) = d \log(s + \sqrt{d^2 - s^2}), \\
v_1(t) &= (t + \log((t - 1)/(t + 1)))/(2|H|), \\
v_{-1}(t) &= (t - 2\tan^{-1}t)/(2|H|).
\end{aligned}
\end{equation}

\textbf{Theorem 1.} The generating curves $(y(s), z(s))$ of surfaces of revolution with constant mean curvature $H$ are the following, corresponding to rotating about $S$-axis, $T$-axis and $N$-axis:

$S$-axis, $H = 0$, $S_0(s) = (u_0, \sqrt{d^2 - s^2})$, $(0 \leq s \leq d)$,
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\[ H \neq 0, \quad S_1(H, d; s) = (\delta \int g/\sqrt{\epsilon f} \, ds, \sqrt{\epsilon f}/(2|H|)), \text{ if } \epsilon = -1, \quad d \geq 0, \]

\[ S_2(H, d; s) = (\delta \int k/\sqrt{\epsilon h} \, ds, \sqrt{\epsilon h}/(2|H|)), \text{ if } \epsilon = -1, \quad d \neq 0, \]

\[ S_3(H; t) = (\pm \nu, \ t/(2|H|)), \]

**T-axis, \( H = 0, \quad T_0(s) = (\sqrt{d^2 - s^2}, u_-, \text{ if } 0 \leq s \leq d), \)**

\[ H \neq 0, \quad T_1(H, d; s) = (\sqrt{-\epsilon f}/(2|H|), \delta \int g/\sqrt{-\epsilon f} \, ds), \text{ if } \epsilon = 1, \quad d \geq 0, \]

\[ T_2(H, d; s) = (\sqrt{-\epsilon h}/(2|H|), \delta \int k/\sqrt{-\epsilon h} \, ds), \text{ if } \epsilon = 1, \quad d \neq 0, \]

\[ T_3(H; t) = (t/(2|H|), \pm \nu_-), \]

**N-axis, \( H = 0, \quad N_0(s) = (ea^{3/2}/(3d), d s^{3/2}, \quad s > 0, \quad d > 0, \)**

\[ H \neq 0, \quad N_1(H, d; t) = \left( \frac{\epsilon}{16H^2} \left( \frac{2t}{1-t^2} - \log \frac{(1-t)/(1+t)}{(1+t)/(1-t)} \right), \quad dt \right), \quad d > 0, \]

\[ N_2(H, d; t) = \left( \frac{e^{-\epsilon t}}{8H^2} \left( \tan^{-1} t - \frac{y^2}{1+y^2} \right), \quad dt \right) \quad d > 0, \]

\[ N_3(N, t) = \left( -\frac{e^{-\epsilon t}}{4H^2}, \quad e^{-\epsilon t} \right), \]

where \( \delta \) is the sign of \( H \). The arc-length parameter \( s \) and the proper parameter \( t \) are taken on open intervals so that the functions in consideration have meaning. If \( \epsilon = 1 \) (resp., \(-1\)), the above curves and the corresponding surfaces of revolution are space-like (resp., timelike).

A space-like surface with vanishing mean curvature is said to be a maximal surface. The surfaces corresponding to \( S_0, T_0 \) and \( N_0 \) are the same as those constructed by O. Kobayashi in [8].

The curve \( S_1(H, d; s) = (y(s), z(s)) \) satisfies

\[ \pm y' = \frac{d \cosh (2Hs) - 1}{2|H|z}, \quad \epsilon (2|H|z)^2 = 1 + d^2 - 2d \cosh (2Hs). \]

From these, we get

\[ z^2 \pm \frac{z}{H} \frac{dy}{ds} + \frac{d^2 + 1}{4H^2} = 0. \]

For other cases, similarly we can obtain the corresponding differential equations.
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{i} & \textbf{S} & \textbf{T} & \textbf{N} \\
\hline
0 & \(z = d(a(1 - \frac{dz}{dy})^2)^{1/2}\) & \(y = d(a(\frac{dy}{dz})^2 - 1)^{1/2}\) & \(\frac{dz}{dx} = \frac{d^2}{y^2}\) \\
\hline
1 & \(z^2 + \frac{z}{H}y' + e - \frac{d^2}{4H^2} = 0\) & \(y^2 + \frac{y}{H}y' - \frac{1 - d^2}{4H^2} = 0\) & \(\frac{dy}{dz} = \frac{e}{4H^2(d^2 - y^2)^2}\) \\
\hline
2 & \(z^2 + \frac{z}{H}y' + e + \frac{d^2}{4H^2} = 0\) & \(y^2 + \frac{y}{H}y' - e - \frac{1 + d^2}{4H^2} = 0\) & \(\frac{dy}{dz} = \frac{e}{4H^2(d^2 - y^2)^2}\) \\
\hline
3 & \(z^2 + \frac{z}{H}y' + \frac{e}{4H^2} = 0\) & \(y^2 + \frac{y}{H}y' - \frac{e}{4H^2} = 0\) & \(\frac{dy}{dz} = \frac{e}{4H^2y^2}\) \\
\hline
\end{tabular}

Lemma 2. The generating curves \(S_i, T_i, N_i, S'_i, T'_i, N'_i\) \((i = 1, 2, 3, 4)\) satisfy the following differential equations respectively.

We will consider the conics in the Lorentz 2-space \(L^2 = (R^2; dy^2 - dz^2)\). Let \(F\) be a fixed point and \(D\) a fixed line in \(L^2\). Put \(E = e, e > 0\) or \(E = ie, e \neq 0\). The conic \(C\) of focus \(F\), directrix \(D\) and eccentricity \(E\) is, by definition, the locus of a point such that \(\|PF\|_L/\|PD\|_L = E\), where \(\|PF\|_L\) is the Lorentzian distance between \(P\) and \(F\), and \(\|PD\|_L\) is the Lorentzian distance from \(P\) to \(D\). Put \(A = \|PD\|/2\) for \(E = 1\) and \(A = E\|FD\|_L|E^2 - 1|\) for other \(E\). \(C\) is said to be horizontal (resp., vertical) if \(A\) is a real (resp., pure imaginary) number. According to \(E = 1, 0 < E < 1\) or \(1 < E\), the conic is a parabola, an ellipse or a hyperbola. We derive equations describing the roulette of a conic \(C\), that is, the trace of a focus \(F\) of a conic \(C\) as \(C\) rolls along a line. Let \(s\) be the arc-length of the roulette. We put

\begin{equation}
S = \begin{cases} 
\frac{s}{y} & \text{if the roulette is a space-like curve,} \\
\frac{s}{z} & \text{if the roulette is a time-like curve.}
\end{cases}
\end{equation}

Lemma 3. Let \(C\) be a conic with \(E\) and \(A\). Let \(\Gamma_y\) (resp., \(\Gamma_z\)) be the roulette of the conic \(C\) as \(C\) rolls along the \(y\)-axis (resp., \(z\)-axis). The roulette \(\Gamma_y\) and \(\Gamma_z\) are space-like curves if and only if \(C\) is vertical (resp., horizontal). If \(C\) is a parabola, that is, \(E = 1\), the roulette \(\Gamma_y\) (resp., \(\Gamma_z\)) satisfies

\begin{equation}
A = iz\frac{dy}{dS} \quad \text{(resp., } A = iy\frac{dz}{dS} \text{)}.
\end{equation}

If \(E = e\) \((e > 1\) or \(1 > e > 0\)) or \(ie(e > 0)\), the roulette \(\Gamma_y\) (resp., \(\Gamma_z\)) satisfies

\begin{equation}
z^2 + 2iAz\frac{dy}{dS} + (E^2 - 1)A^2 = 0 \quad \text{(resp., } y^2 + 2iAy\frac{dz}{dS} - (E^2 - 1)A^2 = 0 \text{)}.
\end{equation}

From Theorem 1, Lemma 2 and Lemma 3, we obtain
Theorem 4. The generating curves of surfaces of revolution with constant mean curvature, rotating about \( S \)-axis or \( T \)-axis are characterized as follows.

(I) Space-like surfaces
   Rotating about \( S \)-axis (resp., \( T \)-axis)
   (1) The roulette of the vertical parabola rolling along \( y \)-axis (resp., \( z \)-axis), which is exactly \( S_0(s) \) (resp., \( T_0(s) \)).

   (2) The undulatory, the roulette of an vertical ellipse with \( E = d \) (\( 0 < d < 1 \)) and \( A = i \frac{1}{2H} \) rolling along \( y \)-axis (resp., \( x \)-axis), which is exactly \( S_1(d, H; s) \) (resp., \( T_1(d, H; s) \)) for \( 0 < |d| < 1 \).

   (3) The nodary, the roulette of a vertical hyperbola with \( E = d \) (\( d > 1 \)) and \( A = i \frac{1}{2H} \) rolling along \( y \)-axis (resp., \( x \)-axis), which is exactly \( S_1(d, H; s) \) (resp., \( T_1(d, H; s) \)) for \( 1 < |d| \).

   (4) The roulette of the vertical conic with \( E = id \) (\( d > 0 \)) and \( A = i \frac{1}{2H} \) rolling along \( y \)-axis (\( x \)-axis), which is exactly \( S_2(d, H; s) \) (resp., \( T_2(d, H; s) \)).

   (5) The curve \( S_3(d, H; s) \) (resp., \( T_3(d, H; s) \)).
   "Rotatinsg about \( S \)-axis,

   (6) The circle \( y^2 - z^2 = \left( \frac{1}{H} \right)^2 \) with radius \( H \), which is exactly \( S_1(1, H; s) \).

   (7) The line \( z = \frac{1}{2H} \), which is exactly \( S_1(0, H; s) \).

(II) Time-like surfaces
   Rotating about \( S \)-axis (resp., \( T \)-axis)
   If we replace vertical conics by horizontal conics in (1), (2), (3) and (4) of (I) respectively, we get corresponding generating curves of time-like surfaces of revolution. Moreover, we have

   (8) The curve \( S_3^t(d, H; s) \) (resp., \( T_3^t(d, H; s) \)).
   "Rotatinsg about \( T \)-axis,

   (9) The circle \( y^2 - z^2 = \left( \frac{1}{H} \right)^2 \) with radius \( iH \), which is exactly \( T_1^t(1, H; s) \).

   (10) The line \( y = \frac{1}{2H} \) which is exactly \( T_1^t(0, H; s) \).
§3. Surfaces of revolution

The first and second fundamental forms of the surfaces given by (2.1) are
\[ \varepsilon ds^2 + z'^2(s)dt^2 \text{ and } (y''(s)z'(s) - y'(s)z''(s))ds^2 - y'(s)z(s)dt^2, \]
respectively. Hence the mean curvature \( H(s) \) satisfies
\[ 2H(s)z(s) + y'(s) + \varepsilon z(s) (y'(s)z''(s) - y''(s)z'(s)) = 0, \text{ (S-axis)}. \]

(3.1)

Similarly, as the first and second fundamental forms of the surface given by (2.3) are
\[ \varepsilon ds^2 + y^2(s)dt^2 \text{ and } (y'(s)z''(s) - y''(s)z'(s))ds^2 + y(s)z'(s)dt^2, \]
respectively, we have
\[ 2H(s)y'(s) - z'(s) - \varepsilon y(s) (y'(s)z''(s) - y''(s)z'(s)) = 0, \text{ (T-axis)}. \]

(3.2)

For the surface given by (2.4), those forms are \( \varepsilon ds^2 + 4z^2(s)dt^2 \) and \( 2(y''(s)z'(s) - y'(s)z''(s))ds^2 - 4z(s)z'(s)dt^2. \) Hence we get
\[ 2H(s)z(s) + z'(s) + \varepsilon 2z(s) (y'(s)z''(s) - y''(s)z'(s)) = 0, \text{ (S-axis)}. \]

(3.3)

From the facts described above, it is evident that surfaces of revolution are space-like (resp., time-like) if and only if their generating curves are space-like (resp., time-like). Multiplying (2.1) by \( y'(s) \) and using (1.2), we get
\[ 2H(s)z(s)y'(s) + (z(s)z(s))y' + \varepsilon = 0. \]

If we multiply (2.1) by \( z'(s) \) and use (1.2), we have
\[ 2H(s)z(s)z'(s) + (z(s)y'(s))' = 0. \]

From these equation, it follows
\[ u'(s) + 2H(s)u(s) - \varepsilon = 0, \ v'(s) - 2Hv'(s) - \varepsilon = 0 \text{ (S-axis, T-axis)}, \]
where
\[ (3.5) \ u(s) = -z(s) (y'(s) + z'(s)), \ v(s) = z(s) (y'(s) - z'(s)) \text{ (S-axis)}. \]

Similarly, from (3.2) and (2.2) it follows that the generating curves of surfaces of revolution with \( T \)-axis satisfy the same differential equations (3.4). But in this case, we put
\[ u(s) = y(s) (y'(s) - z'(s)), \ v(s) = y(s) (y'(s) + z'(s)) \text{ (T-axis)}. \]

(3.6)

Using (2.5), we get from (3.3) the following equations.
\[ u'(s) - 2H(s)u(s) - \varepsilon = 0, \ v'(s) + 2H(s)v(s) = 0 \text{ (N-axis)}, \]
where
\[ (3.7) \ u(s) = 2z(s)y'(s), \ v(s) = 2z(s)z'(s). \]

(3.8)
To solve the above differential equations, we introduce the following functions.

\[ F(s) = \int_0^s \sinh\left(2 \int_0^t H(u) \, du\right) \, dt, \quad G(s) = \int_0^s \cosh\left(2 \int_0^t H(u) \, du\right) \, dt. \]  

The general solutions of (3.4) are given by

\[ u = (G' - F') \left( b_1 + e(G + F) \right), \quad v = (G' + F') \left( b_2 + e(G - F) \right), \]

where \( b_1 \) and \( b_2 \) are integral constants. Similarly, as the general solutions of (3.7), we have

\[ u = (G' + F') \left( b_1 + e(G - F) \right), \quad v = b_2 (G' - F'). \]

As we get \( z^2 = -\epsilon uv \) and \( y' = (v - u)/(2z) \) from (2.2) and (3.5), for the generating curve of a surface with \( S\)-axis, by putting \( b_1 + b_2 = 2c_1, \quad b_1 - b_2 = 2c_2 \) and

\[ H(c_1, c_2) = ((F + c_1)^2 - (G + c_2)^2), \quad I(c_1, c_2) = (F(G + c_2) - G'(F + c_1)). \]

we have

\[ y = \int I(c_1, c_2) / \sqrt{\epsilon H(c_1, c_2)} \, ds, \quad z = \sqrt{\epsilon H(c_1, c_2)}, \]  

(\( S\)-axis).

Similarly, the generating curve of a surface with \( T\)-axis is given by

\[ y = \sqrt{-\epsilon H(c_1, c_2)}, \quad z = \int I(c_1, c_2) / \sqrt{-\epsilon H(c_1, c_2)} \, ds, \]  

(\( T\)-axis).

The generating curve of a surface with \( N\)-axis has the following expression.

\[ y = \epsilon \int (G' + F') (c_1 + G - F)/(2\sqrt{K}) \, ds, \quad z = \sqrt{K}, \]  

(\( N\)-axis),

where we put \( K = c_2 (c_1 + G - F) \) and \( c_1, c_2 \) are integral constants. Set

\[ S(H, \epsilon) = \{(c_1, c_2) \in \mathbb{R}^2, \epsilon H(c_1, c_2) > 0 \text{ for all } s \in I\}, \]

\[ T(H, \epsilon) = \{(c_1, c_2) \in \mathbb{R}^2, -\epsilon H(c_1, c_2) > 0 \text{ for all } s \in I\}, \]

\[ N(H, \epsilon) = \{(c_1, c_2) \in \mathbb{R}^2, K(c_1, c_2) > 0 \text{ for all } s \in I\}. \]

For a given continuous function \( H(s) \) on some interval \( I \), the sets defined in (3.16) may be empty. Now we have the following theorem corresponding to the main result in [7].

**Theorem 5.** If the generating curve of a surface of revolution is parametrized by the arc lengths, its mean curvature is a function of the s. The arc length parametrized generating curve \((y(s), z(s)), s \in I, \) of a surface of revolution with mean curvature \( H(s) \) is given by (3.13), (3.14) or (3.15) for some constants \( c_1, c_2, \) according as it rotates about a
space-like axis, time-like axis or a null axis. A surface of revolution is space-like (resp.,
time-like) if and only if its generating curve is space-like, that is, \( \varepsilon = 1 \) (resp., time-like,
that is, \( \varepsilon = -1 \)). Conversely, for a given continuous function \( H(s) \), \( s \in I \) with \( S(H, \varepsilon) \neq \phi \),
by taking a point \( (c_1, c_2) \in S(H, \varepsilon) \) and using (3.13), we construct a surface of revolution
with mean curvature \( H(s) \), which is space-like or time-like according to \( \varepsilon \). Similarly, if
\( T(H, \varepsilon) \neq \phi \) or \( N(H, \varepsilon) \neq \phi \), we construct a surface of revolution by (3.14) or (3.15).

Now, we will show Theorem 1 in \( \S\) 2. At first, we consider surfaces of revolution
with vanishing mean curvature, that is, \( H = 0 \). Then the functions \( F(s) \), \( G(s) \) given by
(3.9) are reduced to \( F(s) = 0 \), \( G(s) = s \). We get the solutions \( S_0(s) \), \( T_0(s) \) and \( N_0(s) \) in
Theorem 1, from (3.13), (3.14) and (3.15) respectively. Next, for a constant \( H \neq 0 \), the
functions \( F(s) \), \( G(s) \) become

\[
F(s) = \frac{1}{2H} \cosh 2Hs, \quad G(s) = \frac{1}{2H} \sinh 2Hs.
\]

(3.17)

Hence, from (3.12), we get, after some parallel translations of the arc length

\[
H(c_1, c_2) =
\begin{cases}
(1 + d^2 - 2d \cos h 2Hs)/(4H^2), & \text{if } c_1 > c_2, \\
(1 - d^2 - 2d \sin h 2Hs)/(4H^2), & \text{if } c_1 < c_2,
\end{cases}
\]

(3.18)

\[
(1 + 4Hce^{-2Hs})/(4H^2), \quad \text{if } c_1 = c_2 = c,
\]

\[
(1 + 4Hce^{2Hs})/(4H^2), \quad \text{if } c_1 = -c_2 = c,
\]

\[
(d \cos h 2Hs - 1)/(2H), \quad \text{if } c_1 > c_2,
\]

\[
(d \sin h 2Hs - 1)/(2H), \quad \text{if } c_1 < c_2,
\]

(3.18)

\[
(1 + 2Hce^{-2Hs})/(2Hs), \quad \text{if } c_1 = c_2 = c,
\]

\[
(1 + 2Hce^{2Hs})/(2Hs), \quad \text{if } c_1 = -c_2 = c,
\]

where \( d = -\text{sgn}(c_1)2H \sqrt{c_1^2 - c_2^2} \) (resp., \( -\text{sgn}(c_2)2H \sqrt{c_2^2 - c_1^2} \)) if \( c_1 > c_2 \) (resp., \( c_1 > c_2 \)).

Using these, we obtain Theorem 1.

\[\S\) 4. The Lorentzian plane

Let \( L^2 \) be the Lorentzian plane. At first, we describe some facts about angles in
Lorenzian geometry. Details are found in [4]. For a vector \( p = (y, z) \in L^2 \), the Lorentzian
norm \( \|p\|_1 \), is defined to be \( \|p\| \) if \( p \) is space-like or null, and \( i \|p\|_1 \), if \( p \) is time-like, where
\( \|p\| \) is the (absolute) norm of \( p \) and \( i = \sqrt{-1} \). If \( p \) is non-zero and non-null, its polar
coordinates \( R, \Omega \) is defined by \( R = \|p\|, \Omega = \theta + i\omega \), where \( y = R \cos(\theta + i\omega) \), \( iz = R \sin(\theta + i\omega) \), \( \theta = \pi/2, \pi, 3\pi/2 \), \( -\infty < \omega < \infty \). We call \( \Omega \) the angle of the vector \( p \). For non-zero
and non-null vectors \( p, q \in L^2 \), let \( \Omega_p \) and \( \Omega_q \) be the angles of \( p \) and \( q \) respectively.
The oriented angle from \( p \) to \( q \) is defined to be \( \Omega_q - \Omega_p \) (mod 2\( \pi \)) and denote by \( \hat{pq} \). If \( p \) is
moved to \( p' \) counterclockwise with fixing the beginning point of \( p \) so that \( \hat{\mathbf{p}q} = 0 \), we say that \( \hat{\mathbf{p}q} \) is the positively oriented angle. The (unoriented) angle is \( \hat{\mathbf{p}q} \) (resp., \( \hat{\mathbf{q}p} \)) if \( pq \) (resp., \( \hat{\mathbf{q}p} \)) is positively oriented. In the Lorentzian plane, we have

\[
\cos \hat{\mathbf{p}q} = \frac{\langle p, q \rangle}{\|p\|_L\|q\|_L}.
\]

For any points \( A, B \in L^2 \), let \( \mathbf{AB} \) be the corresponding vector. Let \( \Delta ABC \) be a triangle in \( L^2 \) with non-null \( \mathbf{AB}, \mathbf{BC}, \mathbf{CA} \). If we denote by \( \hat{A}, \hat{B}, \hat{C} \) the angles of the triangle, then we have \( \hat{A} + \hat{B} + \hat{C} = \pi \). If \( \hat{C} = \pi / 2 \), it follows from (4.1) that \( \|BC\|_L = \|AB\|_L \cos \hat{B} \) and \( \|AC\|_L = \|AB\|_L \sin \hat{B} \).

Next, we investigate rolling curves along a line in \( L^2 \) and follow the path of any chosen tracing point on the curve. Let \( C \) be a convex curve which can roll along a line \( l \). The tracing point \( P \) can be placed inside, on, or outside \( C \). If \( P \) is regarded as the origin, then \( C \) can be described by polar coordinates \( R, \Theta \). Assume the tangent line \( l \) is the \( y \)-axis. If \( \Phi \) is the angle between the \( y \)-axis and radial line of \( C \), then \( C \) traces out the curve \( \Gamma \) given by \( y = \sigma - R \cos \Phi, iz = R \sin \Phi \), where \( \sigma \) is the arc-length of the curve \( C \). Let \( C \) be on the moving plane \((u, v)\). Let \( \Theta \) be the angle between the \( y \)-axis and the \( u \)-axis.

Then as we have \( \frac{dv}{du} = \tan \Theta \) and \( \Theta = \pi - \Omega - \Phi \), we get \( \tan \Phi = \frac{d\Omega}{dR} \). Since \( ds^2 = dR^2 + R^2 d\Omega^2 = (1 + \tan^2 \Phi) dR^2 \), we may put \( \frac{ds}{dR} = \frac{1}{\cos \Phi} \). Thus we obtain \( \frac{dz}{dy} = \cot \Phi \).

Hence the radial line is normal to \( \Gamma \). Let \( s \) be the arc length of \( \Gamma \), if \( \Gamma \) is space-like (resp., time-like), we put \( dS = ds \) (resp., \( dS = ds \)). Hence, it follows

\[
\begin{align*}
\frac{dy}{dS} &= \sin \Phi, \\
\frac{dz}{dS} &= \cos \Phi.
\end{align*}
\]

If \( C \) rolls along the \( z \)-line, we have similarly

\[
\begin{align*}
\frac{dy}{dS} &= \cos \Phi, \\
\frac{dz}{dS} &= \sin \Phi.
\end{align*}
\]

We describe some properties of conics in \( L^2 \). (a) \( C \) is a parabola. The standard equation of a vertical parabola is \( y^2 = -4az \). Here \( F = (0, a) \), \( D : z = -a \) and \( A = ia \). Let \( l \) be the tangent to \( C \) at a point \( K \), which intersects \( z \)-axis at \( P \) and \( y \)-axis at \( Q \). Using (4.1), we have

\[
\mathbf{FP} \perp l, \quad \angle FQP = \begin{cases} 
\angle FKP & \text{for } a \leq |z| \\
\pi - \angle FKP & \text{for } |z| \leq a.
\end{cases}
\]

\( F = (0, -a) \) and \( D' : z = a \) are regarded as the focus and the directrix of the parabola. The standard equation of a horizontal parabola is given by \( z^2 = -4ay \), where \( F = (a, 0) \) and
$D$: $y = -a$. This has the same property as the above.

(b) $C$ is a ellipse. The standard equation of a vertical ellipse with focus $F = (0, ea)$ (resp., $F = (0, -ea)$), directrix $D: z = a/e$ (resp., $D': z = -a/e$), eccentricity $E = e$ and $A = ia$ is 

\[(1 - e^2)z^2 - y^2 = a^2(1 - e^2).\]

For a point $K = (y, z)$ on $C$, we have

\[(4.5)\quad KF \sim KF' = 2a \quad \text{for} \quad z \leq a/e, \quad KF + KF' = 2a \quad \text{for} \quad a/e \leq z \leq a/e.

Take a line $l$ tangent to $C$ at a point $K = (y, z)$. Through $F$ (resp., $F'$) we draw a line perpendicular to $l$, intersecting $l$ at $Q$ (resp., $Q'$). Denote by $FQ$ the oriented length of $FQ$, that is, $FQ = \|FQ\|$ or $-\|FQ\|$ according as $FQ$ is positively oriented or negatively oriented. Now we have

\[(4.6)\quad FQ \cdot FQ' = \begin{cases} (1 - e^2)a^2 & \text{for} \quad a/e \leq |z| \leq a/e, \\ -(1 - e^2)a^2 & \text{for} \quad |z| > a/e. \end{cases}

By making use of (4.1), we get

\[(4.7)\quad \angle FKQ = \begin{cases} \angle FKQ' & \text{for} \quad a/e \leq |z|, \\ \pi - \angle FKQ' & \text{for} \quad |z| > a/e. \end{cases}

The standard equation of the horizontal ellipse with focus $F = (ea, 0)$ (resp., $(-ea, 0)$), directrix $D: y = a/e$ (resp., $D': y = -a/e$) and $A = a$ is 

\[(1 - e^2)y^2 - z^2 = a^2(1 - e^2).\]

This curve has the properties similar to the above.

(c) $C$ is a hyperbola. The standard equation of a vertical hyperbola with focus $F = (0, ea)$ (resp., $F = (0, -ea)$), directrix $D: z = a/e$ (resp., $D': z = -a/e$), eccentricity $E = e$ and $A = ia$ is given by the same equation as one of the ellipse. In this case, we obtain for a point $K = (y, z)$ on $C$, we have

\[(4.8)\quad KF + KF' = 2a \quad \text{for} \quad |z| \leq a/e, \quad KF \sim KF' = 2a \quad \text{for} \quad a/e \leq |z| \leq a.

Let $l$ be the tangent line to $C$ at a point $K = (y, z)$. Through $F$ (resp., $F'$) we draw a line perpendicular to $l$, intersecting $l$ at $Q$ (resp., $Q'$). Then we have

\[(4.9)\quad FQ \cdot FQ' = \begin{cases} -(1 - e^2)a^2 & \text{for} \quad a/e < |z| < a \\ (1 - e^2)a^2 & \text{for} \quad |z| < a/e. \end{cases}

\[(4.10)\quad \angle FKQ = \begin{cases} \angle FKQ' & \text{for} \quad |z| < a/e, \\ \pi - \angle FKQ' & \text{for} \quad a/e < |z| < a. \end{cases}

For horizontal hyperbolas, we have the corresponding facts.

(d) $C$ is a conic with imaginary eccentricity. The standard equation of the vertical conic with focus $F = (ea, 0)$ (resp., $F' = (-ea, 0)$), directrix $D: y = -a/e$ (resp., $D': y = a/e$), eccentricity $E = ie$ and $A = ia$ is $z^2 - (1 + e^2)y^2 = (1 + e^2)a^2$. In the present case, we obtain for a point $K = (y, z)$ on $C$
\[(4.12) \quad KF + KF = 2a \text{ for } |y| < a/e, \quad KF \sim KF = 2a \text{ for } a/e < |y|.\]

Let \(l\) be the tangent line to \(C\) at a point \(K = (y, z)\). Through \(F\) (resp., \(F')\) we draw a line perpendicular to \(l\), intersecting \(l\) at \(Q\) (resp., \(Q')\). Then we have

\[(4.13) \quad FQ \cdot FQ' = \begin{cases} (e^2 + 1)a^2 & \text{for } |y| < a/e \\ -(e^2 + 1)a^2 & \text{for } |y| > a/e \end{cases},\]

\[
\angle QKF = \begin{cases} \pi - \angle Q'KF & \text{for } |y| < a/e \\ \angle Q'KF & \text{for } |y| > a/e \end{cases}.
\]

Horizontal conics with eccentricity \(E = i\epsilon\) satisfy properties similar to the above.

\section*{§5. The differential equations of rolling curves}

Let \(F\) be the trace of a focus \(F = (y, z)\) of a conic which rolls along a line \(l\) in \(L^2 = \{(y, z)\}\). We will find a differential equation whose solution gives the curve \(I\), that is, we will prove Lemma 3.

(a) \(C\) is a parabola. \((1) l\) is the \(y\)-axis. At first, let \(C\) be the vertical parabola given in (a) of §4. Assume that \(C\) is tangent to \(y\)-axis at a point \(K\). Let \(B\) be the vertex of \(C\) and \(l\) be the line through \(B\) and perpendicular to the axis of \(C\). Assume that \(y\)-axis intersects \(l\) at \(P\) and the axis of \(C\) at \(Q\). Let \(\Phi\) be the angle between \(y\)-axis and the line through the focus \(F = (y, z)\) of \(C\) and \(K\). We get from (4.4) \(\angle BFP = \angle QOF\) (or \(\pi - \Phi\)). As we have \(\|\bar{FB}\|_L = \|\bar{FP}\|_L \sin \Phi\), we may put \(a = z \sin \Phi\). Hence from (4.2), it holds \(a = z\ dy/\ dx\), that is,

\[z = a\sqrt{1 - (dz/\ dx)^2}, \quad dS = ds.\]

Assume next that \(C\) is horizontal. Let points \(K, B, P\) and \(Q\) be taken as similarly as in the above case. Let \(\Phi\) also the angle between \(z\)-axis and the line through the focus \(F\) and \(K\). Now we have \(\angle FBP = \pi/2 - \angle BFP = \pi/2 - (\angle FQP + \angle QPF) = -\Phi\). Hence, from \(\|\bar{FB}\|_L = \|\bar{FP}\|_L \sin \angle FPB\), we get \(a = -iz \sin \Phi\). In this case, it should be \(dS = ids\) in the formula (4.2). Thus we obtain \(a = z(dy/\ dx), \quad ds^2 = dz^2 - dy^2\). In other word, we show (2.8) for \(I_y, \quad A = ia\) and \(dS = ids\).

(2) \(l\) is the \(z\)-axis. Let \(C\) be the vertical parabola as in (1). Then as similarly as in the above, we obtain the equation \(i (\dot{z}) = y \sin \Phi\), where \(\Phi\) is the angle between the \(z\)-axis and the tangent line. From (4.2), we get \(i (\dot{z}) = \cos(\pi - \Phi) = \sin \Phi\). Thus we obtain (2.8) for \(I_z, \quad A = ia\) and \(dS = ds\). For the horizontal \(C\), we have \(a = y \sin \Phi\). Hence as we must put \(dS = ids\) in (4.2), we get (2.8) for \(I_z, \quad A = a\) and \(dS = ids\).

(b) \(C\) is an ellipse.

(1) \(l\) is the \(y\)-axis. Let \(C\) be the vertical ellipse given in (b) of §4. Let \(F = (y, z)\) and \(F' = (y', \ z')\).
(z', z') be foci of C. The y-line is tangent to C at a point K. Through F (resp., F'), draw a line perpendicular to y-line and intersecting it at Q (resp., Q'). From (4.6), (4.7) and (4.8), we get $FQ \cdot F'Q' = (1 - e^2)a^2$, $KF + KF' = 2a$ and $\angle QKF = \pi - \angle QPF$. Using (4.2), we have $\pm z/KF = \sin \angle QKF = dy/dS$ and $\pm z'/KF = \sin \angle Q'KF = dy/dS$, where the double sign is for $z, z' > 0$ and $z, z' < 0$, respectively. Thus we obtain (2.9) for $\Gamma_y, E = e (0 < e < 1), A = ia$ and $dS = ds$. Assume that $C$ is horizontal. Take points $F, F', Q, Q'$, and $K$ as above. Then we have $KF \sim KF' = 2a, FQ \cdot F'Q' = -(1 - e^2)a^2, iz/KF = \pm \sin \angle QKF = dy/dS$ and $iz'/KF = \sin \angle Q'KF = dy/dS$. Hence, we must put $dS = ids$ and get $z + z' = \pm 2a dy/ds$ and $zz' = -(1 - e^2)a^2$. Thus we have (2.9) for $\Gamma_y, E = e (0 < e < 1), A = a$ and $dS = ds$.

(2) $l$ is the Z-line. $C$ is vertical. In this case, we have $FQ \cdot F'Q' = -(1 - e^2)a^2, KF \sim KF' = 2a, y = \pm KF(dz/ds)$ and $y' = \pm KF'(dz'/ds)$. Thus we have (2.9) for $\Gamma_z, E = e (0 < e < 1), A = ia$ and $dS = ds$. Similarly, when $C$ is horizontal, we get (2.9) for $\Gamma_z, E = e (0 < e < 1), A = a$ and $dS = ds$.

(c) $C$ is a hyperbola.

(1) $l$ is the y-axis. Let $C$ be vertical. Assume that $C$ is tangent to y-axis at $K$. Let $F = (y, z)$ and $F' = (y', z')$ be the foci of $C$. Take points $Q$ and $Q'$ as in (1) of (b). Then we have $FK \sim FK' = 2a, FQ \cdot F'Q' = -(e^2 - 1)a^2, z/QF = \pm dy/dS$ and $z'/F'Q' = \pm dy/dS$. Hence we obtain (2.9) for $\Gamma_y, E = e (1 < e), A = ia$ and $dS = ds$. If $C$ is horizontal, it holds that $KF \sim KF' = 2a, FQ \cdot F'Q' = (e^2 - 1)a^2, iz/KF = \pm dy/dS$ and $iz'/FK = \pm dy/dS$. Hence, we get (2.9) for $\Gamma_y, A = a, E = e (e > 1)$ and $dS = ds$.

(2) $l$ is the Z-axis. If $C$ is vertical, taking points $K, Q, Q'$ as in (1), we have $KF + KF' = 2a, FQ \cdot F'Q' = (e^2 - 1)a^2, y/(ikF) = \pm idz/dS$ and $y'/iKF = \pm idz/dS$. Hence we get (2.9) for $\Gamma_z, E = ie (1 < e), A = ia$ and $dS = ds$. When $C$ is horizontal, we get $KF \sim FK' = 2a, FQ \cdot F'Q' = -(e^2 - 1)a^2, y/KF = \pm idz/dS$ and $y'/FK = \pm idz/dS$. Thus we obtain (2.9) for $\Gamma_z, E = e (1 < e), A = a$ and $dS = ds$.

(d) $C$ is a conic with imaginary eccentricity.

(1) $l$ is the y-axis. Let $C$ be vertical. Take points $K, Q, Q'$ as above. Then we have $KF + KF' = 2a, FK \cdot FK' = (1 + e^2)a^2, z/KF = \pm dy/dS$ and $z'/KF = \pm dy/dS$. Hence we have (2.9) for $\Gamma_y, E = ie, A = ia$ and $dS = ds$. Similarly, for a horizontal $C$, we get (2.9) for $\Gamma_y, E = ie, A = a$ and $dS = ds$.

(2) $l$ is the z-axis. For a vertical $C$, we have $KF \sim KF' = 2a, yy' = -(1 + e^2)a^2, y/(ikF) = \pm idz/dS$ and $y'/iKF = \pm idz/dS$. Thus we have (2.9) for $\Gamma_z, E = ie, A = ia$ and $dS = ds$. Similarly, for a horizontal $C$, we get (2.9) for $\Gamma_z, E = ie, A = a$ and $dS = ds$.

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