Remarks on the Structure of Power Semigroups

By

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Dedicated to Professor M. Yamada on his 60th birthday

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1. Introduction

Let $S$ be a semigroup. The power semigroup $P(S)$ of $S$ is the set of all non-empty subsets of $S$ with the operation defined by

$$XY = \{xy | x \in X, y \in Y\}$$

for $X, Y$ in $P(S)$. This concept is old as is found in Dubriel [1], Liapin [4] and Tamura [9], but precise studies have begun recently (see, for example, Gould and Iskra [2], Tamura [7]). Even if $S$ has a simple structure, the structure of $P(S)$ can be very complicated. This is especially so if $S$ is infinite. Suppose that $S$ is a commutative semigroup with a non-preiodic element. In this paper we show that (i) $P(S)$ has uncountably many incomparable archimedean components, and (ii) $P(S)$ contains uncountably many free generators. The first result answers to a question posed by Tamura [8]. The second result may be interesting in connection with the embedding problem in power semigroups.

The set of positive integers will be denoted by $\mathbb{P}$.

2. Archimedean components

If $S$ is a commutative semigroup, then so is $P(S)$. A standard way to investigate a commutative semigroup is to decompose it into a semilattice of archimedean semigroups. Let $T$ be a commutative semigroup. The relation $\rho$ on $T$ defined by

$$x \rho y \text{ if } x^n = yz \text{ and } y^n = xw \text{ for some } n \in \mathbb{P} \text{ and } z, w \in T,$$

is a congruence of $T$. The $\rho$-classes are archimedean subsemigroups of $T$ and are the archimedean components of $T$. The archimedean component containing $x \in T$

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is denoted by $\mathcal{A}_x$. The quotient $T/\rho$ is a (lower) semilattice and is the greatest semilattice image of $T$. For $\mathcal{A}_x, \mathcal{A}_y \in T/\rho$, $\mathcal{A}_x \leq \mathcal{A}_y$ if and only if $x^n = yz$ for some $n \in \mathbb{P}$ and $z \in T$.

Now we shall show the semilattice decomposition of $P(S)$ is intricate in general.

**Theorem 1.** Let $S$ be a commutative semigroup with a non-periodic element. Then $P(S)$ has uncountably many incomparable archimedean components.

To prove the theorem we need the following easy lemma.

**Lemma 1.** Let $X$ be a countable infinite set. Then there is an uncountable family $\{X_x\}_{x \in I}$ of subsets $X_x$ of $X$ such that the difference $X_x \setminus X_y$ is infinite for any different $x$ and $y$ in $I$.

**Proof.** We may suppose $X$ is the set of rational numbers and $I$ is the set of real numbers. For $x \in I$ define a subset $X_x$ of $X$ by $X_x = \{x \in X | x \leq x \leq x + 1\}$. Then the family $\{X_x\}_{x \in I}$ satisfies the desired property.

**Proof of Theorem 1.** Let $a$ be a non-periodic element of $S$. First we choose an infinite sequence $X = \{n(i) | i \in \mathbb{P}\}$ of positive integers such that $n(1) = 1$ and $n(i + 1) > N(i)^2$ for $i \in \mathbb{P}$. By Lemma 1 we can find an uncountable family $\{X_x\}_{x \in I}$ of subsequences of $X$ such that $X_x \setminus X_y$ is infinite for any different $x, y \in I$. We may assume that every $X_x$ contains $1$. Let $A_x$ be an element of $P(S)$ defined by $A_x = \{a^n | n \in X_x\}$ and let $\mathcal{A}_x$ be the archimedean component of $A_x$ in $P(S)$. We shall show that $\mathcal{A}_x$ and $\mathcal{A}_y$ are incomparable if $x \approx y$.

Assume to the contrary that $\mathcal{A}_x \leq \mathcal{A}_y$, that is,

$$A_x^+ = A_y^{-1}$$

for some $m_1, \ldots , m_t \in X_x$. It follows from (2) and (3) that $a^{m_1 + \cdots + m_t} = a^{m_1 + \cdots + m_t + n - 1}$ or

$$a^{n_1} a^{n_2} \cdots a^{n_t} = a^n c$$

for some $n_1, \ldots , n_t \in X_x$. Since $a \in A_y$, again by (1) we have

$$a^{m_1} a^{m_2} \cdots a^{m_t} = a c$$

for some $m_1, \ldots , m_t \in X_x$. It follows from (2) and (3) that $a^{m_1 + \cdots + m_t} = a^{m_1 + \cdots + m_t + n - 1}$ or

$$n_1 + \cdots + n_t = m_1 + \cdots + m_t + n - 1.$$ 

Since $n$ is not in $X_x$, $n$ is different from any of $n_1, \ldots , n_t$ and $m_1, \ldots , m_t$.

If $n > \max(n_1, \ldots , n_t)$ then $\sqrt{n} > \max\{n_1, \ldots , n_t\}$ by the property of the
sequence $X$. So by (4) we have $n \leq n_1 + \cdots + n_\ell < \sqrt{n \cdot \ell} < n$, a contradiction. Hence $n < \max\{n_1, \ldots, n_\ell\}$. We may suppose that $n_1 = \max\{n_1, \ldots, n_\ell\}$ and $m_1 = \max\{m_1, \ldots, m_\ell\}$. If $n_1 > m_1$, then $\sqrt{n_1} > m_1$ for $i = 1, \ldots, \ell$. Noting $\sqrt{n_1} > n$, we get the impossible inequalities

$$n_1 \leq m_1 + \cdots + m_\ell + n - 1 < \sqrt{n_1 \cdot (\ell + 1)} \leq \sqrt{n_1 \cdot n} < n_1.$$

In the same way $n_1 < m_1$ is impossible and we have $n_1 = m_1$. Thus we can cancel $n_1$ and $m_1$ in (4) and we get

$$n_2 + \cdots + n_\ell = m_2 + \cdots + m_\ell + n - 1.$$

Repeating the above argument, we can cancel all the $n_i$ and $m_i$ in (4) and finially we would have $n = 1$, but this is impossible.

Similarly, $\mathcal{A}_x \geq \mathcal{A}_y$ is impossible either. Therefore $\mathcal{A}_x$ and $\mathcal{A}_y$ are incomparable and the proof of the theorem is complete.

What about the cardinality of each archimedean component of $P(S)$? We can show that some of the components are uncountable. In fact, let $S$ be a commutative semigroup with a non-periodic element $a$. Consider the subsets of \{a^i | i \in \mathbb{P}\} containing $a^2$ and $a^i$ for all positive odd integer $i$. There are uncountably many such sets and the square of them are all equal. Therefore they are in the same archimedean component which are uncountable. Thus, $P(S)$ has uncountably many archimedean components some of which are uncountable.

The semilattice decomposition of $P(G)$ for a finite group $G$ was described by Putcha [5]. Tamura [8] studied the archimedean components of $P(G)$ for the infinite cyclic group $G$ and asked how many archimedean components $P(G)$ has. The answer is "uncountable" due to Theorem 1.

3. Free commutative subsemigroups

The embedding problem in power semigroups has been of interest (Gould and Iskra [3], Trnkova [10]). In this section we shall prove a somewhat surprising result that the power semigroup of a semigroup with a non-periodic element has a very large free commutative subsemigroup.

**Theorem 2.** Let $C$ be a infinite cyclic semigroup. Then $P(C)$ contains a subsemigroup isomorphic to a free commutative semigroup on an uncountable set of generators.

We need the following lemma stronger than Lemma 1. The result is due to Sierpinski [6].

**Lemma 2.** Let $X$ be a countable infinite set. Then there is an uncountable
family \( \{X_\alpha\}_{\alpha \in I} \) of infinite subsets \( X_\alpha \) of \( X \) such that \( X_\alpha \cap X_\beta \) is finite for any different \( \alpha \) and \( \beta \) in \( I \).

**Proof.** We may suppose \( X = \mathbb{P} \) and \( I \) is the set of real numbers between \( 1/2 \) and \( 1 \). For \( \alpha \in I \), define \( X_\alpha = \{ \text{Int}(2^n \alpha) | n \in \mathbb{P} \} \), where \( \text{Int}(t) \) for a real number \( t \) is the greatest integer not exceeding \( t \). Let \( \alpha \) and \( \beta \) be different elements in \( I \). Since \( 2^{n-1} < 2^n \alpha < 2^n \) and \( 2^{m-1} < 2^m \beta < 2^m \) for any \( n, m \in \mathbb{P} \), we see that \( \text{Int}(2^n \alpha) \neq \text{Int}(2^n \beta) \) if \( n \neq m \). Moreover, \( \text{Int}(2^n \alpha) \neq \text{Int}(2^n \beta) \) if \( n \geq -\log_2 |\alpha - \beta| \). It follows that \( X_\alpha \cap X_\beta \) is finite.

**Proof of Theorem 2.** We may assume that \( C \) is the additive semigroup of positive integers. The operation of \( P(C) \) is also written additively and \( nA \) denotes the sum of \( n \) \( A \)’s for \( n \in \mathbb{P} \) and \( A \in P(C) \). Let \( X = \{ n(i) | i \in \mathbb{P} \} \) be an infinite sequence of positive integers such that \( n(i + 1) > n(i)^2 \) for all \( i \in \mathbb{P} \). Let \( \{ X_\alpha \}_{\alpha \in I} \) be an uncountable family of subsequences \( X_\alpha = \{ n(\alpha, i) | i \in \mathbb{P} \} \) of \( X \) such that \( X_\alpha \cap X_\beta \) is finite for any different \( \alpha \) and \( \beta \) in \( I \). The existence of such a family is guaranteed by Lemma 2. \( X_\alpha \) are considered to be elements of \( P(C) \). We claim that the subsemigroup generated by \( \{ X_\alpha \}_{\alpha \in I} \) is a free commutative semigroup with the free generating set \( \{ X_\alpha \}_{\alpha \in I} \).

Let \( \{ m_\alpha \}_{\alpha \in I} \) be a set of non-negative integers indexed by \( I \) such that only a finite number of \( m_\alpha \) are positive. Let

\[
Y = \sum_{\alpha \in I} m_\alpha X_\alpha = m_1 X_{\alpha_1} + \cdots + m_r X_{\alpha_r},
\]

where \( \{ m_i = m_\alpha | i = 1, \ldots, r \} \) is the set of all positive integers in \( \{ m_\alpha \}_{\alpha \in I} \). We have to show that the integer \( m_\alpha \) is determined only by \( Y \) and \( \alpha \) for any \( \alpha \in I \). Let \( \alpha \in I \) and \( i \in \mathbb{P} \), and set

\[
Y(\alpha, i) = \{ n \in Y | n \leq n(\alpha, i)^2 \}.
\]

Let \( n \) be in \( Y(\alpha, i) \), then \( n \) is written as

\[
(5) \quad n = \sum_{j=1}^{r} \sum_{k=1}^{m_j} n(\alpha_{p_j}, i_{j(k)}).
\]

If some of \( n(\alpha_j, i_{j(k)}) \) in (5) were greater than \( n(\alpha, i) \), then \( n > n(\alpha, i)^2 \), a contradiction. So every \( n(\alpha_j, i_{j(k)}) \) in (5) is not greater than \( n(\alpha, i) \). If just \( m \) numbers in \( n(\alpha_j, i_{j(k)}) \) are equal to \( n(\alpha, i) \), then

\[
n = m \cdot n(\alpha, i) + p,
\]

where \( 0 \leq p < M \cdot \sqrt{n(\alpha, i)} \) with \( M = \sum_{\alpha \in I} m_\alpha = m_1 + \cdots + m_r \). By the choice of the family \( \{ X_\alpha \}_{\alpha \in I} \), there exists a positive integer \( N \) such that if \( i \geq N \), then \( n(\alpha, i) \geq M^2 \) and \( n(\alpha, i) \) is not in \( X_{\alpha_j} \) for each \( j = 1, \ldots, r \) with \( \alpha_j \neq \alpha \). Therefore, if \( i \geq N \), then the greatest number in \( Y(\alpha, i) \) is of the form \( m_\alpha \cdot n(\alpha, i) + p \) with \( p < n(\alpha, i) \).
< M \cdot \sqrt{n(x, i)}. \) This implies \( m_x = \text{Int} \left( \max \frac{Y(x, i)}{n(x, i)} \right) \) for \( i \geq N \). Consequently we have

\[
m_x = \lim_{i \to \infty} \text{Int} \left( \frac{\max Y(x, i)}{n(x, i)} \right),
\]

showing that \( m_x \) is determined by \( Y \) and \( x \).

**Corollary 1.** Any commutative semigroup \( S \) whose cardinality is not greater than the cardinality of the real numbers deviates the power semigroup \( P(C) \) of the infinite cyclic semigroup \( C \), that is, \( S \) is a homomorphic image of a subsemigroup of \( P(C) \).

**Corollary 2.** If a semigroup \( S \) contains a non-periodic element, then \( P(S) \) contains an uncountable free commutative semigroup.

The above results imply that \( P(S) \) contains a large cancellative subsemigroup in general. It may be interesting to point out that \( P(S) \) itself is not cancellative at all.

**Proposition.** If \( S \) is a semigroup with at least two elements, then \( P(S) \) is not cancellative.

**Proof.** If \( S \) is a band of order greater than 1, then \( S \) is not cancellative and neither is \( P(S) \). If \( S \) is not a band, then \( S \) has a non-idempotent element \( a \). Let \( A \) be the subsemigroup of \( S \) generated by \( a \). Then we have \( A \cdot \{a\} = A \cdot \{a, a^2\} \) in \( P(S) \). Since \( \{a\} \not\approx \{a, a^2\} \), cancellation does not hold in \( P(S) \).

If \( S \) is a monoid with a non-periodic element \( a \), then \( P(S) \) contains an uncountable null subsemigroup as well as an uncountable free commutative semigroup. In fact, subsets of \( \{a^i \mid i \in \mathbb{P}\} \) containing 1 and \( a^i \) for all positive odd integers \( i \) form an uncountable null subsemigroup of \( P(S) \).

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References


