J. Math. Tokushima Univ.
Vol. 23 (1989), 39–40

On Uniform Limit of Quasiperiodic Functions

By

Atsuhito KOHDA and Yoshitane SHINOHARA

(Received September 14, 1989)

In the papers [1], [2], we use the proposition which states that the uniform limit $f(t)$ of a sequence of quasiperiodic functions $f_n(t)$ is also quasiperiodic. We say a function $f(t)$ quasiperiodic with periods $\omega_1,\dots, \omega_m$, if $f(t)$ is represented as

$$f(t) = g(t,\dots, t)$$

for some continuous function $g(u_1,\dots, u_m)$ which is periodic with period ω_i in each u_i. We assume here that every ω_i is positive and $\omega_1^{-1},\dots, \omega_m^{-1}$ are rationally linearly independent.

The present paper is concerned with an alternative proof of the above fact and gives some examples.

Proposition. *Let* $\{f_n(t)\}$ $(n = 1,2,\dots)$ *be a sequence of quasiperiodic functions with periods* $\omega_1,\dots, \omega_m$, *and let* $f(t)$ *be the uniform limit of* $f_n(t)$ *as* $n \to \infty$, *then* $f(t)$ *is quasiperiodic with the same periods.*

PROOF. From the quasiperiodicity of $f_n(t)$ there exists a continuous periodic function $g_n(u_1,\dots, u_m)$ with periods ω_i in each u_i such that $f_n(t) = g_n(t,\dots, t)$.

By Kronecker's theorem, for arbitrary $u_1,\dots, u_m$ and for any positive number ε correspond a real number t and integers $p_1,\dots, p_m$ such that

$$|u_i - p_i\omega_i - t| < \varepsilon \quad (i = 1,\dots, m).$$

Since $g_n(t + p_1\omega_1,\dots, t + p_m\omega_m) = g_n(t,\dots, t)$, $g_n(u_1,\dots, u_m)$ converges uniformly on the dense subset $\{(t + p_1\omega_1,\dots, t + p_m\omega_m)\}$, so converges uniformly on $\boldsymbol{R}^m$. It is evident that the uniform limit $g(t,\dots, t)$ of $g_n(t,\dots, t) = f_n(t)$ equals to $f(t)$. From the periodicity of $g_n(u_1,\dots, u_m)$ we have

$$\begin{aligned} g(\dots, u_i + \omega_i,\dots) &= \lim_{n\to\infty} g_n(\dots, u_i + \omega_i,\dots) \\ &= \lim_{n\to\infty} g_n(\dots, u_i,\dots) \\ &= g(u_1,\dots, u_m). \end{aligned}$$

This completes the proof.

Remark. In the proposition, we fix the set of periods $\{\omega_1,\dots,\omega_m\}$ for every $f_n(t)$. This is essential as the following example shows.

Example 1. Let $\{c_j\}$ be a sequence such that $\sum_{j=1}^{\infty} c_j$ is absolutely convergent, and let $\{\omega_j\}$ be a set of positive numbers such that any finite subset has the property that the reciprocals of the elements are rationally linearly independent. Set

$$g_m(u_1,\dots,u_m) = \sum_{j=1}^{m} c_j \sin\frac{2\pi u_j}{\omega_j},$$

then $f_m(t) = g_m(t,\dots,t)$ is quasiperiodic with periods $\omega_1,\dots,\omega_m$. It is clear that $f_m(t)$ converges uniformly to

$$f(t) = \sum_{j=1}^{\infty} c_j \sin\frac{2\pi t}{\omega_j}$$

but $f(t)$ is of course not quasiperiodic.

The next example shows that the set of periods $\omega_1,\dots,\omega_m$ of $f(t)$ may contain irrelevant elements.

Example 2. Let $f_n(t) = \sin t + \frac{1}{n}\sin 2\pi t \quad (n = 1,2,\dots)$, then $f_n(t)$ is quasiperiodic with periods $1, 2\pi$. It is obvious that $f_n(t)$ converges uniformly to $f(t) = \sin t$ and $f(t)$ is (quasi-)periodic with period 2π only. In the same way, let

$$f_n(t) = \sin 2t + \sin 2\pi t + \frac{1}{n}\sin t \quad (n = 1,2,\dots)$$

which is quasiperiodic with periods 1, 2π. It converges uniformly to $f(t) = \sin 2t + \sin 2\pi t$ and $f(t)$ is quasiperiodic with periods $1,\pi$ rather than with periods $1, 2\pi$.

Department of Applied Mathematics,
Faculty of Engineering,
Tokushima University

Department of Applied Mathematics,
Faculty of Engineering,
Tokushima University

References

[1] M. Kurihara, Quasiperiodic solutions of quasiperiodic differential difference equations, Reports Fac. Eng. Yamanashi Univ., **37** (1986), 65–69.

[2] Y. Shinohara, M. Kurihara and A. Kohda, Numerical analysis of quasiperiodic solutions to nonlinear differential equations, Japan J. Appl. Math., **3** (1986), 315–330.