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On Numerically Integrable Solutions of Ordinary Differential Equations

By

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Abstract. The paper notes that the numerically integrable solutions of ordinary differential equations must not be numerically ill conditioned on the interval in question.

On the process of numerical computation of the solution of the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \tag{1}$$

on the interval $J = [t_0, T)$ by a step by step method

$$y_{k+1} = y_k + h_k \varphi(t_k, y_k; h_k), \; h_k = t_{k+1} - t_k \quad (k = 0, 1, 2, \dots, N-1),$$

$$t_N = T,$$

the local errors, that is , the local discretization error and the round off error, constitute a sequence of perturbations $\{r(t_k)\}$ that shift computed solutions to neighboring integral curves, as illustrated in Figure 1, where $y \in \boldsymbol{R}^m$, $f \in C^1(J \times D \to \boldsymbol{R}^m)$ and D is a domain in $\boldsymbol{R}^m$.

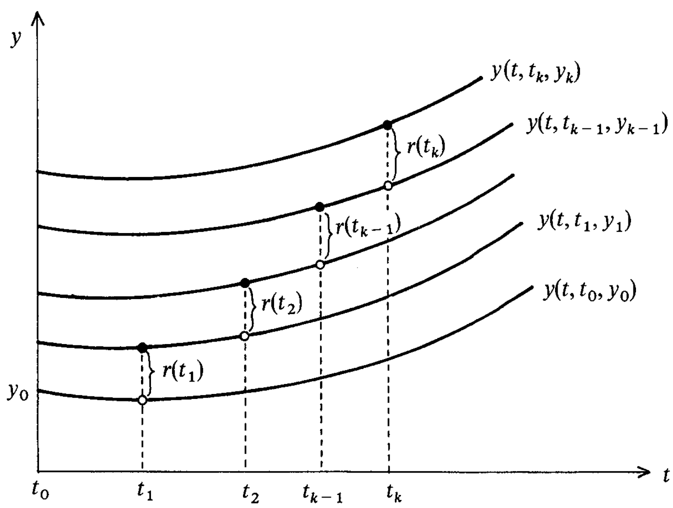

Figure 1

Our purpose is to follow the particular curve $y(t, t_0, y_0)$ but, due to local errors, we may leave it and begin following near curves in the family of solutions of the differential equation. Thus, on the numerical process of solution of the initial value problem (1), the local errors $\{r(t_k)\}$ oblige us to trace the discontinuous solution $x(t)$ of the following initial value problem

$$(2) \qquad \frac{dx}{dt} = f(t, x), \ t \neq t_k,$$

$$x(t_k) = x(t_k + 0) = x_k,$$

$$x(t_k + 0) - x(t_k - 0) = r(t_k).$$

for $k = 0, 1, 2, 3, \ldots, x_k = y_k$, and $r(t_0) = \mathbf{0}$ but $r(t_k) \neq \mathbf{0}$ for $k \geq 1$, where $x \in R^m$.

If all the solution curve $y(t, t_k, y_k)$ tend to $y(t, t_0, y_0)$ as the integral proceeds to ∞, the effect of errors is damped out and we say the solution $y(t, t_0, y_0)$ is stable. In the contrary case of the solution curves diverging from $y(t, t_0, y_0)$, the effect of errors may grow as the integration progresses to ∞ and we say the solution $y(t, t_0, y_0)$ is unstable.

On the other hand, on the numerical process of solution with sufficiently small $h = \max_k |h_k|$ on the finite interval $[t_0, T)$, the norm $\|x(t) - y(t, t_0, y_0)\|$ may grow infinite before T is reached or as t tends to T.

We can say that the solution $y(t, t_0, y_0)$ of the initial value problem (1) is *numerically ill conditioned* on the *finite* interval $[t_0, T)$ if $\|x(t) - y(t, t_0, y_0)\| \to \infty$ with sufficiently small $h = \max_k |h_k|$ as $t \to T$ or as $t \to t^* < T$.

There are many studies on the asymptotic behaviour and the stability of solutions of differential equations (see, for example, [1], [2], [3]). But the concept of numericallly ill conditioning of solutions of the initial value problem is novel and important in numerical analysis of differential equations.

Both of the stability and the numerically ill conditioning depend on the behavior of the difference $x(t) - y(t, t_0, y_0)$ which is seen from the following theorem.

Theorem. Let $y(t, t_0, y_0)$ be the solution of (1), then

$$x(t) = y(t, t_0, y_0) + \sum_{k=1}^{n} \int_0^1 \Phi(t, t_k, x_k + (\theta - 1)r(t_k))r(t_k)d\theta$$

is the solution of (2) for t such that $t_0 < t_1 < t_2 < \cdots < t_n \leq t < T (n \leq N - 1)$, where $\Phi(t, t_k, x_k)$ is the fundamental matrix of the linear system

$$\frac{dz}{dt} = \frac{\partial f}{\partial w}(t, w(t, t_k, x_k))z$$

with respect to the solution $w(t, t_k, x_k)$ of the initial value problem

$$\frac{dw}{dt} = f(t, w), \ w(t_k) = x_k,$$

and $\Phi(t_k, t_k, x_k) = \mathrm{I}$(unit matrix).

PROOF. As is easily seen, we have

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds + \sum_{k=1}^{n} \Delta x_k$$

for t such that $t_0 < t_1 < t_2 < \cdots < t_n \leq t$, where

$$\Delta x_k = w(t, t_k, x_k) - w(t, t_k, x_k - r(t_k)).$$

Put $X(\theta) = w(t, t_k, x_k + (\theta - 1)r(t_k))$, then we have

$$\begin{aligned} \Delta x_k &= \int_0^1 \frac{d}{d\theta} X(\theta) d\theta \\ &= \int_0^1 \frac{\partial w}{\partial x_k}(t, t_k, x_k + (\theta - 1)r(t_k))r(t_k) d\theta \\ &= \int_0^1 \Phi(t, t_k, x_k + (\theta - 1)r(t_k))r(t_k) d\theta. \end{aligned}$$

Q.E.D.

Corollary. The solution $y(t, t_0, y_0)$ of (1) is numerically ill conditioned on the interval $[t_0, T)$ if

$$\left\| \sum_{k=1}^{n} \int_0^1 \Phi(t, t_k, x_k + (\theta - 1)r(t_k))r(t_k) d\theta \right\| \to \infty$$

with sufficiently small $h = \max_k |h_k|$ as $t \to T$ or as $t \to t^* < T$.

The numerically integrable solutions of ordinary differential equations must not be numerically ill conditioned on the interval in question.

Example. Consider the solution $y(t, -1, \xi) = t^2 + \frac{\xi - 1}{t^2}$ of the initial value problem

$$\frac{dy}{dt} = -\frac{2}{t} y + 4t, \quad y(-1) = \xi,$$

for any $\xi \in \boldsymbol{R}$.

The continuous solution $y(t, -1, 1)=t^2(-\infty<t<\infty)$ is numerically ill conditioned on the interval $[-1, 0)$, because in general

$$\sum_k \int_0^1 \Phi(t, t_k, x_k+(\theta-1)r(t_k))r(t_k)d\theta=\frac{1}{t^2}\sum_k t_k^2 r(t_k)\to \pm\infty \text{ as } t\to 0$$

for any h.

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