Arithmetic Genera of Some Local Rings

By

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Introduction

Let $A$ be a zero dimensional ideal of a local ring $\mathcal{O}$. The constant term of the Hilbert-Samuel's function of $A$ is called the arithmetic genus of $A$ and is denoted by $p_a(A)$. In this note, we investigate this less studied invariant of $A$ in connection with the properties of complete ideals. We show that if $\mathcal{O}$ is analytically unramified and of (Krull) dimension 1, $p_a(A)$ is the same for any zero dimensional normal ideal $A$ and that the similar result holds for an analytically irreducible normal local domain of dimension 2 which satisfies the condition $N[2]$.

It is to be mentioned that these results are considered, in a sense, as an arithmetic analog of those obtained by Muhly and Zariski [3]. The possibility of the study of the arithmetic genus in this direction was suggested to me by H. T. Muhly, to whom I would express my sincere gratitude. The theorem 1 in §1 is essentially due to him.

All rings (resp. local rings) considered here are assumed commutative (resp. commutative Noetherian) with identity. By a local ring $(\mathcal{O}, m)$ we mean that $\mathcal{O}$ is a local ring and $m$ is its maximal ideal.

§1. Arithmetic genus of one dimensional local ring

Let $(\mathcal{O}, m)$ be an analytically unramified local ring and let $\bar{\mathcal{O}}$ be its integral closure in its total quotient ring. Then, $\bar{\mathcal{O}}$ is a finite $\mathcal{O}$-module, say, $\bar{\mathcal{O}}=\mathcal{O}\omega_1+\cdots+\mathcal{O}\omega_r$ and there is a non zero divisor $c$ in $\mathcal{O}$ such that $c \bar{\mathcal{O}} \subset \bar{\mathcal{O}}$. Denote by $\mathfrak{c}$ the conductor of $\bar{\mathcal{O}}$ with respect to $\mathcal{O}$, then $c \in \mathfrak{c}$ so that $\mathfrak{c}$ is an m-primary ideal if $\mathcal{O}$ is one dimensional.

**Lemma 1.** Let $A_1, \ldots, A_r$ be ideals in $\mathcal{O}$, then there is an integer $k$ such that $(A_1^{r-k} \cdots A_r) = (A_1^{r-k} \cdots A_r)(A_1^{r-k} \cdots A_r)_a$ for $n_i \geq k$ ($i=1, \ldots, r$), where $B_a$ is the integral closure of $B$. In particular, for an ideal $A$ of $\mathcal{O}$, $(A^d)_a$ is normal for some integer $d$. We call this ideal a derived normal ideal of $A$.

**Proof.** The proof of this lemma is quite similar to the one given in
Theorem 1 in [8], so that we shall state it briefly in the case when $r=2$. Let $A=(a_1, \ldots, a_r)$ and $B=(b_1, \ldots, b_s)$ and let $\mathcal{O}(A, B)=\mathcal{O}[a_1 t, \ldots, a_r t, t^{-1}, b_1 u, \ldots, b_s u, u^{-1}]$ where $t$ and $u$ are indeterminates. Then, $\mathcal{O}(A, B)$ is a bigraded subring of $\mathcal{O}[t, t^{-1}, u, u^{-1}]$. If $\mathcal{O}^* (A, B)$ is the integral closure of $\mathcal{O}(A, B)$ in $\mathcal{O}[t, t^{-1}, u, u^{-1}]$, then $\mathcal{O}^* (A, B)$ is again bigraded. Since $\mathcal{O}$ is analytically unramified, there is an integer $k$ such that $(A^k B^m)_a \subset A^{e_k} B^{m_k}$ for all $n, m \geq k$. Hence, by the argument similar to Lemma 1 and 2 in [8], we obtain our lemma.

**Lemma 2.** $A \mathcal{O} \cap \mathcal{O} \subset A$. If $A$ is a principal ideal generated by a non zero divisor, we have $A \mathcal{O} \cap \mathcal{O} = A_a$.

**Proof.** If $x \in A \mathcal{O} \cap \mathcal{O}$, then we can write $x \omega_i$ as $x \omega_i = \sum_{j=1}^s a_{ij} \omega_j$ with $a_{ij} \in A$. Therefore, $\sum_{j=1}^s (a_{ij} - \delta_{ij} x) \omega_j = 0$ $(i = 1, \ldots, s)$ and whence $\det(a_{ij} - \delta_{ij} x) = 0$. Consequently, $x \in A_a$. Suppose $A = \langle x \rangle$, $x$ is a non zero divisor of $\mathcal{O}$. If $y \in (x)_a$, then $y$ satisfies the equation of the form, $y^s + c_1 x y^{s-1} + \cdots + c_s x^s = 0$, where $c_i \in \mathcal{O}$. Dividing by $x^s$, we get $(y/x)^s + c_1(y/x)^{s-1} + \cdots + c_s = 0$ and hence $y/x \in \mathcal{O}$.

**Lemma 3.** Let $\alpha$ and $\beta$ be non zero divisors in $\mathcal{O}$ such that $\alpha^{m_0} \in \mathcal{O}$, then we have $(\alpha^m \beta^n)_a = (\alpha^{m-m_0} \beta^n)(\alpha^{m_0})_a$ for $n \geq 0$ and $m \geq m_0$.

**Proof.** We have $(\alpha^m \beta^n)_a = (\alpha^m \beta^n) \mathcal{O} \cap \mathcal{O} = (\alpha^m \beta^n \mathcal{O}) = (\alpha^{m-m_0} \beta^n)(\alpha^{m_0}_a)$. 

**Lemma 4.** If $\mathcal{O}$ is one dimensional and if $A$ is a normal $m$-primary ideal, then for any $m$-primary ideal $B$, there exist positive integers $p$, $q$ and non-negative integer $m_0$ such that $A^m B^n$ is complete for $n \geq 0$ and $m \geq m_0$ where $A = A^p$ and $B = B^q$.

**Proof.** For suitable powers $A^p = A'$ and $B^q = B'$ of $A$ and $B$, $A'$ and $B'$ have minimal reductions of order 1, i.e., $A' = A'_a = (\alpha)_a$ and $B' = (\beta)_a$ for some non zero divisors $\alpha$ and $\beta$ [4, 5]. If $m_0$ is an integer such that $\alpha^{m_0} \in \mathcal{O}$, then, from the relation, $(A^m B^n)_a = (\alpha^m \beta^n)_a = (\alpha^{m-m_0} \beta^n)(\alpha^{m_0})_a \subset A^{m-m_0} B^n A^{m_0} = A^m B^n$ (lemma 3), our assertion follows.

**Lemma 5.** Let $A$ and $B$ be $m$-primary ideals in an one dimensional local ring $\mathcal{O}$ and assume that $A$ and $B$ have minimal reductions of order 1 and that $A$ is normal, then we have $A^m B^{n-1}/A^m B^n = A^{m_0}/A^{m_0} B$ for $n > 0$ if $m \geq m_0$ where $m_0$ is some integer depending on $A$ and $B$. Consequently, $l(A^m B^{n-1})$ is constant for $n > 0$ if $m \geq m_0$, where $l(A)$ denotes the length of an $m$-primary ideal of $A$.  

Proof. By lemma 4, we have \( A^m B^n = (\alpha^{m-m_0} \beta^n) \alpha^{m_0} = \alpha^{m-m_0} \beta^n A^{m_0} \). Hence \( A^m B^{n-1} / A^m B^n = \alpha^{m-m_0} \beta^{n-1} A^{m_0} / \alpha^{m-m_0} \beta^n A^{m_0} \). Since \( \alpha \) and \( \beta \) are non-zero divisors in \( \mathcal{O} \), we see immediately that \( A^m B^{n-1} / A^m B^n \approx A^{m_0} / \beta A^{m_0} = A^{m_0} / BA^{m_0} \).

For fixed \( m \)-primary ideals \( A \) and \( B \), we denote by \( r(m, n) \) the length of \( (A^m B^n)_a \) and by \( s(m, n) \) the length of \( A^m B^n \). By a theorem, due to Bhattacharya, \( s(m, n) \) becomes polynomial in \( m \) and \( n \) if \( m \geq M \) and \( n \geq N \) for some integers \( M \) and \( N \) \([1]\). We denote this polynomial by \( \rho(m, n) = \hat{\alpha} m + \hat{\beta} n + \hat{\tau} \).

Lemma 6. In the situation of lemma 5, we have \( r(m, n) = \rho(m, n) \) if \( m \geq \text{Max} \{m_0, M\} \) and \( n \geq 0 \).

Proof. Both \( r(m, n) \) and \( \rho(m, n) \) are integer valued functions and coincide if \( m \) and \( n \) are sufficiently large (lemma 4). Since \( r(m, n) - r(m, n - 1) \) is constant, say \( b \) if \( m \geq m_0 \) (lemma 5) and \( \rho(m, n) - \rho(m, n - 1) = \hat{\beta} \). Hence \( \beta = b \) and \( r(m, n) = \rho(m, n) \) for \( m \geq \text{Max} \{m_0, M\} \) and \( n \geq 0 \).

Theorem 1. If \( A \) and \( B \) are normal \( m \)-primary ideals in an analytically unramified local ring of dimension 1, then \( p_a(A) = p_a(B) \) where \( p_a(C) \) is the arithmetic genus of \( C \).

Proof. Since \( p_a(A) = p_a(A^p) \) for any positive integer \( p \), we can assume that both \( A \) and \( B \) have minimal reductions of order 1. By lemma 6, there exist integers \( m^* \) and \( n^* \) such that

\[ r(m, n) = \rho(m, n) \quad \text{for} \quad m \geq m^* \quad \text{and} \quad n \geq 0, \quad \text{or for} \quad m \geq 0 \quad \text{and} \quad n \geq n^*. \]

Hence, \( r(0, 0) = \rho(0, 0) = \hat{\alpha} m + \hat{\beta} n + \hat{\tau} = l(A^m) = e(A)m + p_a(A) \) and \( r(0, n) = \rho(0, n) = \hat{\beta} n + \hat{\tau} = l(B^n) = e(B)n + p_a(B) \) where \( e(C) \) is the multiplicity of \( C \). Consequently, we have \( p_a(A) - \hat{\tau} = p_a(B) \).

We denote this common arithmetic genus of normal \( m \)-primary ideals by \( p_a(\mathcal{O}) \) and call it the arithmetic genus of the local ring \( \mathcal{O} \).

Theorem 2. For any \( m \)-primary ideal \( A \) of \( \mathcal{O} \), we have \( p_a(A) = p_a(\mathcal{O}) \).

Proof. Let \( A'_g = (A^e)_a \) be a derived normal ideal of \( A \) (lemma 1). Then, by theorem 1, \( p_a(A'_g) = p_a(\mathcal{O}) \). Hence our assertion follows from the relations:

\[ e(A^e)n + p_a(A^e) = l(A^e) \geq l((A'_g)^e) = e(A'_g)n + p_a(\mathcal{O}), \quad e(A^e) = e(A_g) \quad \text{and} \quad p_a(A) = p_a(A^e), \quad \text{q.e.d.} \]
§2. Arithmetic genus of two dimensional local ring which satisfies the condition $N$

First, we recall the following definitions which will be needed in this section [2]. Let $(\mathcal{O}, m)$ be a normal local domain and let $v$ be a discrete rank 1 valuation of the quotient field $F$ of $\mathcal{O}$ which dominates $\mathcal{O}$. We say that $v$ is a divisor of second kind if the residue field $R_v/M_v$ of $v$ is a finitely generated extension of transcendence degree $r-1$ over the residue field $\mathcal{O}/m$ of $\mathcal{O}$ where $r=$dimension $\mathcal{O}$. Let $A_0=\mathcal{O}$ and $A_i=\{x \in \mathcal{O} \mid v(x)>v(A_{i-1})\}(i>0)$, then the sequence $A_0=\mathcal{O} \supset A_1 \supset A_2 \supset \ldots$ of valuation ideals has the property that $A_i A_j \subseteq A_{i+j}$. If $B_0=\mathcal{O} \supset B_1 \supset \ldots$ is any subsequence of $A_0 \supset A_1 \supset \ldots$ such that $v(B_{i+j})=v(B_i)+v(B_j)$, then the direct sum $G(B)=\Sigma B_i/B_{i+1}$ becomes a graded ring. The divisor $v$ is said to be Noetherian if there exists a subsequence $B_0 \supset B_1 \supset \ldots$ such that $G(B)$ is Noetherian. If every divisor of second kind is Noetherian, we say that $\mathcal{O}$ satisfies the condition $N$. If $\bar{v}_A$ is the homogeneous pseudo valuation associated with powers of an $m$-primary ideal $A$, $\bar{v}_A$ can be represented as a subvaluation, $\bar{v}_A=\min\{\frac{v_1}{e_1}, \ldots, \frac{v_s}{e_s}\}$, $e_i=v_i(A)$, and $v_i$ is a divisor of 2nd kind if $\mathcal{O}$ is analytically irreducible. The valuations $v_1, \ldots, v_s$ are uniquely determined by $A$ and are called the Rees valuations associated with $A$. We denote the set $\{v_1, \ldots, v_s\}$ by $S(A)$. An ideal $W$ in $\mathcal{O}$ is called asymptotically irreducible if all powers $W^n$ of $W$ are the valuation ideals of some divisor of 2nd kind relative to $\mathcal{O}$ [2, §2].

Lemma 7. Let $(\mathcal{O}, m)$ be an analytically irreducible 2 dimensional normal local domain which satisfies the condition $N$ and let $A$ and $B$ be $m$-primary ideals. Suppose $S(A) \subset S(B)$ and $B$ is normal, then for some powers $A'=A^p$ and $B'=B^q$ of $A$ and $B$, $A^m B^n$ is complete for $m \geq 0$ and $n \geq 1$ where $S(A)$ and $S(B)$ are the sets of Rees valuations associated with $A$ and $B$ respectively.

Proof. Let $S(A)=\{v_1, \ldots, v_s\}$ and $S(B)=\{v_1, \ldots, v_s, v_{s+1}, \ldots, v_t\}$. Since $\mathcal{O}$ satisfies the condition $N$, each $v_i$ defines an asymptotically irreducible ideal $W_i(i=1, \ldots, t)$ and there exist sets of positive integers $p, \alpha_1, \ldots, \alpha_s$ and $q, \beta_1, \ldots, \beta_t$ such that

$$(A^p)_a=(W_1^{\alpha_1} \ldots W_s^{\alpha_s})_a \quad \text{and} \quad (B^q)_a=(W_1^{\beta_1} \ldots W_t^{\beta_t})_a$$

[2, theorem 4.3]. Moreover, these sets of integers are defined up to proportionality, so that we can assume $\alpha_i \leq \beta_i (i=1, \ldots, s)$. Now, let

$$C=W_1^{\beta_1-a_1} \ldots W_s^{\beta_s-a_s} W_{s+1}^{\beta_{s+1}} \ldots W_t^{\beta_t}.$$ 

Then, $B^q=(B^q)_a=(A^p C)_a$. Hence, if $n$ is sufficiently large, say $n \geq n_0$, we have
\[(A^{m}B^{n})_a = (A^m(A'C)^a)_a = (A^{m+n}C^n)_a = (A^{m+n-n_0}C^{n-n_0}(A'^nC^n)_a = A^{m}B'^n,\]

in view of lemma 1, where \(A' = A^q\) and \(B' = B^q\).

**Lemma 8.** With the same hypothesis as in lemma 7, for suitable powers \(A^*\) and \(B^*\) of \(A\) and \(B\), \(r^*(m, n) = r^*(m, n) = s^*(m, n)\) for \(m \geq 0\) and \(n \geq 1\), where \(r^*(m, n), \ldots\) denote the corresponding functions relative to \(A^*\) and \(B^*\).

**Proof.** Since \((A^mB^n)_a = (A^m(A'C)^a)_a = (A^{m+n-n_0}C^{n-n_0}(A'C)^a)_a\) for \(n \geq n_0\), the length \(l(A^mB^n)_a\) becomes polynomial in \(m\) and \(n\) if \(n - n_0\) is sufficiently large, say \(n - n_0 \geq n_1\). Hence we can take \(A^* = A^{n_0 + n_1}\) and \(B^* = B^{n_0 + n_1}\).

**Theorem 3.** If \(A\) and \(B\) are normal \(m\)-primary ideals in an analytically irreducible two dimensional normal local domain which satisfies the condition \(N\) and if \(S(A) = S(B)\), then \(p_a(A) = p_a(B)\) where \(S(C)\) is the set of Rees valuations associated with \(C\).

**Proof.** Since \(S(A) = S(A^p)\) and \(p_a(A) = p_a(A^p)\) for any positive integer \(p\), we can replace \(A\) and \(B\) by their powers. Hence, by lemma 8, we can assume \(A\) and \(B\) satisfy the following relation:

\[r(m, n) = r(m, n) = s(m, n)\] for \(m \geq 0\), \(n \geq 1\), or for \(m \geq 1\), \(n \geq 0\).

If \(P_B(n)\) is the Hilbert-Samuel's function of \(B\), then \(P_B(n) = \rho(0, n)\). Hence \(p_a(B) = P_B(0) = \rho(0, 0)\). Similarly, we have \(p_a(A) = \rho(0, 0)\), and consequently \(p_a(A) = p_a(B)\).

We know that every two dimensional regular local ring satisfies the condition \(N\) \([2, \text{ coroll. 5. 4}]\) and if a simple ideal \(W\) corresponds to the divisor \(v\) of 2nd kind \([9]\), then \(W\) is the asymptotically irreducible ideal of minimal value for \(v\). From these remarks, we get the following

**Corollary.** Let \(A = \mathbb{P}_1^{a_1} \cdots \mathbb{P}_s^{a_s}\) and \(B = \mathbb{P}_1^{b_1} \cdots \mathbb{P}_s^{b_s}\) be the factorizations of the complete ideals \(A\) and \(B\) in a two dimensional regular local ring into the products of simple ideals. If \(s = t\) and \(\{\mathbb{P}_1, \ldots, \mathbb{P}_s\} = \{\mathbb{P}_1, \ldots, \mathbb{P}_t\}\), then \(p_a(A) = p_a(B)\).

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References


