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A Note on the Deformations of Hamiltonian Systems on Nilmanifolds

By

Ruishi KUWABARA
College of General Education, The University of Tokushima
Tokushima 770, JAPAN
E-mail: f00886@sinet. ad. jp
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Abstract

This note deals with the isospectral deformations of metrics on nilmanifolds from the view point of Hamiltonian dynamical systems. It is shown in some examples that the associated Hamiltonian system (the system of geodesic flow) is left invariant under such deformations without a nowhere dense subset of the phase space.

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1. Introduction

Given a Riemannian manifold (M, g). Then we have the natural Hamiltonian dynamical system, the system of geodesic flow, $\mathcal{H} = (T^*M, \omega, H)$, where T^*M is the cotangent bundle over M, ω the natural symplectic two form on it, and $H = H_g$ is the Hamiltonian defined from the metric g as

$$H(x, \xi) = \frac{1}{2} g_x(\xi^\#, \xi^\#)$$

for $(x, \xi) \in T^*M$, $\xi^\#$ being the tangent vector at $x \in M$ satisfying $\xi(v) = g_x(\xi^\#, v)$ for every $v \in T_xM$. On the other hand, the metric g defines the natural elliptic self-adjoint differential operator, the Laplace-Beltrami operator, $\Delta = \Delta_g$ acting on functions on M, which is regarded in a sense as a quantum system corresponding to $\mathcal{H}$. It is interesting to consider relationships between geometric or dynamical properties of the Hamiltonian system $\mathcal{H}$ and analytic properties, especially the spectrum (denoted by Spec(Δ)), of the operator Δ.

Two compact Riemannian manifolds are said to be isospectral when their associated Laplace-Beltrami operators have the same spectra. In 1984 C.S. Gordon and E. Wilson [8] exhibited for the first time non-trivial isospectral deformations, i.e., continuous one-parameter families of metrics on a compact manifold which are isospectral but not isometric. Their examples are constructed on solvmanifolds or nilmanifolds, i.e., manifolds whose universal Riemannian coverings are solvable or nilpotent Lie groups with left-invariant metrics, on the basis of the new concept called almost-inner automorphisms of a Lie group. It is worthwhile to analyze their isospectral deformations from the viewpoint of (classical) Hamiltonian systems or systems of geodesic flows.

Now let $g_t(-\varepsilon < t < \varepsilon)$ be a one parameter family of Riemannian metrics (which we call a *deformation* of g_0) on a smooth manifold M. We introduce the following definitions.

Definitions. (0) A deformation g_t of g_0 is said to be *trivial* if there is a one parameter family φ_t of diffeomorphisms of M such that $g_t = \varphi_t^* g_0$.

(1) We denote by $\mathrm{Spec}_L(M, g)$ the collection of lengths of closed geodesics with the multiplicity of $\lambda \in \mathrm{Spec}_L(M, g)$ defined to be the total number of closed geodesics of length λ. A deformation g_t of g_0 is called an *L-isospectral deformation* if $\mathrm{Spec}_L(M, g_t) = \mathrm{Spec}_L(M, g_0)$ for every t.

(2) A deformation g_t of g_0 is called a *symplectic deformation* (abbreviated by *Symp-deformation*) if $(\dot{T}^*M, \omega, H_t) \cong (\dot{T}^*M, \omega, H_0)$ holds for every t, i.e., there is a one parameter smooth family χ_t of homogeneous symplectic diffeomorphisms on $\dot{T}^*M = T^*M \setminus 0$ such that

$$\chi_t^* H_0 = H_t (= H_{g_t}). \tag{1.1}$$

Here a homogeneous symplectic diffeomorphism χ_t means a diffeomorphism of $\dot{T}^*M$ which satisfies

$$\chi_t^* \omega = \omega, \tag{1.2}$$

and the homogeneity condition

$$\chi_t(x, \xi) = (y, \eta) \Longleftrightarrow \chi_t(x, c\xi) = (y, c\eta) \quad (c > 0). \tag{1.3}$$

The symplectic diffeomorphism χ_t maps each integral curve of the Hamiltonian system $(\dot{T}^*M, \omega, H_t)$ to that of $(\dot{T}^*M, \omega, H_0)$. Let Z_t be the infinitesimal generator of χ_t, i.e., $d\chi_t/dt = Z_t \circ \chi_t$. Then, Z_t is a smooth vector field on $\dot{T}^*M$, and satisfies

$$\mathcal{L}_{Z_t} \omega = 0, \tag{1.4}$$

($\mathcal{L}$ being the Lie derivative), the homogeneity condition

$$(j_c)_* Z_t = Z_t \quad (c > 0), \tag{1.5}$$

(j_c being the dilation map $(x, \xi) \mapsto (x, c\xi)$), and

$$Z_t H_t = H_t' (= \frac{d}{dt} H_t). \tag{1.6}$$

These three conditions are lead from (1.2), (1.3) and (1.1), respectively.

(3) A deformation g_t of g_0 is called a *Hamiltonian deformation* (abbr. *Ham-deformation*)

if g_t is a Symp-deformation and Z_t (defined above) is a Hamiltonian vector field, that is, there exists a one parameter family of smooth functions F_t on $\dot{T}^*M$ such that $i_{Z_t}\omega = dF_t$, where i_{Z_t} is the interior product with respect to Z_t. Note that in this case the conditions (1.4) and (1.6) are equivalent to the unified condition:

$$\{F_t, H_t\} = H_t', \tag{1.7}$$

where $\{\ ,\ \}$ is the Poisson bracket.

Notice the following diagram about the definitions above, where $A \longrightarrow B$ means that A implies B:

$$\begin{tikzcd} & \text{trivial deform.} \arrow[dl] \arrow[dr] & \\ \text{Ham-deform.} \arrow[d] & & \text{isospectral deform.} \\ \text{Symp-deform.} \arrow[d] & & \\ \text{L-isospectral deform.} & & \end{tikzcd}$$

We have the following results concerning these deformations of a metric.

(1) The Zoll deformations constructed by odd functions on the sphere are Ham-deformations (cf. [2, pp. 121–123], [11]) but not isospectral. The non-isospectrality of the Zoll deformations follows from the result of Tanno [15] claiming the non-existence of non-trivial isospectral deformations of the canonical metric on the sphere.

(2) Colin de Verdière [3] showed that if the metric g on the compact manifold M satisfies a certain generic property, then $\mathrm{Spec}(\Delta_g)$ determines $\mathrm{Spec}_L(M, g)$, and accordingly, an isospectral deformation is an L-isospectral deformation.

(3) If the associated geodesic flow is Anosov, Livcic's theorem [13] asserts that an L-isospectral deformation is a Ham-deformation with F_t to be a C^1 function.

(4) Moreover, Guillemin and Kazhdan [9], [10] proved for the metrics of negative curvature with a pinching property that any Ham-deformation is trivial , and consequently the non-existence of non-trivial isospectral deformations.

(5) Known examples of non-trivial isospectral deformations are those which are constructed by Gordon and Wilson [8] using almost-inner automorphisms of nilpotent or solvable Lie groups. In [6] Gordon proved that these isospectral deformations are L-isospectral deformations.

(6) Gordon [7] showed that a particular isospectral deformation on a nilmanifold M is not a Symp-deformation, more precisely, the geodesic flows are not conjugate under any continuous family of homeomorphisms of $\dot{T}^*M$.

(7) A Hamiltonian system is decomposed into a family of reduced systems by the reduction procedure formulated by Marsden and Weinstein [14] if the system has a "symmetry". The previous paper [12] showed that a certain class of isospectral deformations on nilmanifolds induce the trivial deformations on each reduced systems.

In contrast to Gordon's result (5) above we here in this note show in some examples that the isospectral deformation g_t on M by an almost-inner automorphism derives an isomorphism $(U, \omega, H_0) \cong (U, \omega, H_t)$ for an open and dense subset U of T^*M.

2. Dynamical systems on Lie groups

Let G be a Lie group endowed with a left invariant Riemannian metric. Consider the Hamiltonian dynamical system on the cotangent bundle T^*G with the Hamiltonian function defined by the Riemannian metric.

For each element g of G, let $L_g(R_g)$ denote the left (right) translation on G by g, and set $I_g = L_g \circ R_{g^{-1}}$ (the inner automorphism of G). As the differentials (and their dual operators) of these diffeomorphisms we define the following linear isomorphisms of the tangent (and cotangent) spaces for each $h \in G$:

$$\begin{array}{ll} L_{g*} : T_hG \longrightarrow T_{gh}G, & R_{g*} : T_hG \longrightarrow T_{hg}G, \\ L_g^* : T_{gh}^*G \longrightarrow T_h^*G, & R_g^* : T_{hg}^*G \longrightarrow T_h^*G, \\ Ad(g) := (I_{g*})_e : \mathrm{g} \longrightarrow \mathrm{g}, & Ad^*(g) := (I_g^*)_e : \mathrm{g}^* \longrightarrow \mathrm{g}^*, \end{array}$$

where $\mathrm{g} = T_eG$ is the Lie algebra of G and g^* is the dual space of g.

We consider the cotangent bundle T^*G. Using the left translations we get a bundle isomorphism
$\lambda : T^*G \to G \times \mathrm{g}^*$ as

$$T_h^*G \ni \xi \longmapsto (h, L_h^*\xi) \in G \times \mathrm{g}^*.$$

Then,

$$\lambda \circ L_g^* \circ \lambda^{-1}(h, \mu) = (g^{-1}h, \mu), \quad \lambda \circ R_g^* \circ \lambda^{-1}(h, \mu) = (hg^{-1}, Ad^*(g^{-1})\mu), \tag{2.1}$$

holds for $(h, \mu) \in G \times \mathrm{g}^*$, and we denote these mappings on $G \times \mathrm{g}^*$ by the same notations L_g^* and R_g^*. Let θ_0 be the canonical one form on the cotangent bundle T^*G, and set $\omega_0 = -d\theta_0$. The two form ω_0 is the natural symplectic structure on T^*G. By virtue of the isomorphism λ we obtain the forms $\theta = (\lambda^{-1})^*\theta_0$ and $\omega = (\lambda^{-1})^*\omega_0$ on $G \times \mathrm{g}^*$. Thus we have the symplectic manifold $(G \times \mathrm{g}^*, \omega)$.

Proposition 2.1(cf. [1, p.315]). *Let* $(g, \mu) \in G \times \mathrm{g}^*$, *and* $(v, \rho), (w, \sigma) \in T_{(g,\mu)}(G \times \mathrm{g}^*) = T_gG \times \mathrm{g}^*$. *Then,*

(i) $\theta(g, \mu)(v, \rho) = \mu(L_{g^{-1}*}v)$.

(ii) $\omega(g, \mu)((v, \rho), (w, \sigma)) = -\rho(L_{g^{-1}*}w) + \sigma(L_{g^{-1}*}v) + \mu([L_{g^{-1}*}v, L_{g^{-1}*}w])$.

Let $G \ni g \mapsto \langle , \rangle_g$ be a left-invariant Riemannian metric on G, which is uniquely induced from the inner product $\langle , \rangle = \langle , \rangle_e$ in g. Let H_0 be the Hamiltonian function on T^*G defined by the metric, and let $H = (\lambda^{-1})^*H_0$ be a Hamiltonian on $G \times \mathrm{g}^*$. It is obvious that the function H is invariant under the left translation L_g^* for every $g \in G$, and

$$H(g, \mu) = \frac{1}{2}\langle \mu, \mu \rangle^* := \frac{1}{2}\langle \mu^\#, \mu^\# \rangle \tag{2.2}$$

by means of (2.1). Let X_H is the Hamiltonian vector field on $(G \times g^*, \omega)$ defined by H, i.e. $i_{X_H}\omega = dH$ (i_X: the interior product with respect to X). Then,

Proposition 2.2. *Let* $(g, \mu) \in G \times g^*$. *Then,*

$$X_H(g,\mu) = (L_{g*}(\mu^\#), ad^*(\mu^\#)\mu) \in T_gG \times g^*,$$

where $ad^*(\mu^\#)$ *is the dual operator of* $ad(\mu^\#) : g \to g; w \mapsto [\mu^\#, w]$.

Proof. Direct calculation using Proposition 1.1 and (1.2). ∎

It is to be noted in the above proposition that the g^*-component of X_H is independent of $g \in G$.

Dynamical systems on $\Gamma\backslash G$. Suppose G has a discrete subgroup Γ. A left-invariant Riemannian metric on G induces a metric on the manifold $M = \Gamma\backslash G$. Associated with the isomorphism $T^*G \cong G \times g^*$, we have the isomorphism $T^*M \cong M \times g^*$ by left translations by G, and the objects ω, H and X_H on $G \times g^*$ are identified with those on $M \times g^*$ because they are invariant under any left translation by $\gamma \in \Gamma$.

3. A dynamical property of isospectral deformations by Gordon-Wilson

In this section we let g be an n-dimensional nilpotent Lie algebra and $G = \exp g$ the corresponding Lie group, which is diffeomorphic to $\boldsymbol{R}^n$. Suppose G has a uniform discrete subgroup Γ, that is equivalent to the existence of a basis of g relative to which the structure constants are rational (cf. [3]). We consider the Riemannian manifold $M = \Gamma\backslash G$ endowed with the metric $\langle , \rangle$ which is induced from a left-invariant metric, also denoted by $\langle , \rangle$, on G, and the associated Hamiltonian system $\mathcal{H} = (T^*M = M \times g^*, \omega, H)$.

We introduce the notion of almost-inner automorphisms of G by Gordon and Wilson. Let Aut(G) be the group of all automorphisms of G. The map Aut(G)$\ni \Phi \mapsto \Phi_* = (\Phi_*)_e$ gives an isomorphism of Aut(G) onto the group Aut(g) of all automorphisms of the Lie algebra g, which is a Lie subgroup of the general linear group GL(g). Let Der(g) denote the set of all derivations of g. Then Der(g) is a Lie subalgebra of gl(g) (the Lie algebra of GL(g)), and exp(Der(g))=Aut(g) holds good.

Definitions. (1) An automorphism Φ of G is said to be *almost-inner* if for each $g \in G$ there exists $a = a_g \in G$ (depending on g) such that $\Phi(g) = aga^{-1}$, or equivalently $\Phi_*(X) = Ad(a)X$ for $X \in g$ with $\exp X = g$.

(2) A derivation ϕ of g is said to be *almost-inner* if for each $X \in g$ there exists $Y = Y_X \in g$ (depending on X) such that $\phi(X) = [Y, X]$.

We denote the set of all almost-inner automorphisms of G by AIA(G) and the set of all almost-inner derivations of g by AID(g). We notice the following fundamental facts (see [8] and [6]).

Lemma 3.1. (1) *The set* AIA(G) *is a connected nilpotent Lie subgroup of* Aut(G) *with*

Lie algebra AID(g).

(2) *A derivation* ϕ *belongs to* AID(g) *if and only if for each* $\mu \in g^*$ *there exists* $Y = Y(\mu) \in g$ *such that*

$$\phi^*(\mu) = ad^*(Y)\mu, \tag{3.1}$$

where ϕ^* *is the dual operator of* ϕ.

Take a derivation ϕ of g, and let $\Phi_{t*} = \exp(t\phi)$ $(t \in \boldsymbol{R})$ be a one-parameter smooth family of automorphisms of g. Then, we get a one-parameter family $\langle , \rangle_t$ of inner products in g as $\langle X, Y\rangle_t = \langle \Phi_{t*}(X), \Phi_{t*}(Y)\rangle$, which are regarded as left-invariant metrics on on G and induce Riemannian metrics, denoted by the same notation $\langle , \rangle_t$, on $M = \Gamma\backslash G$.

Theorem 3.2(Gordon-Wilson [8]). *If* ϕ *belongs to* AID(g), *then the family* $\langle , \rangle_t (t \in \boldsymbol{R})$ *of metrics on* M *induced from* ϕ *is an isospectral deformation.*

Remark. If ϕ is an inner derivation, then $\langle , \rangle_t$ is a trivial deformation. In fact, if $\phi = ad(Y)$ $(Y \in g)$, then $\langle , \rangle_t = \varphi_t^* \langle , \rangle$ holds for the family of diffeomorphisms φ_t on M which is induced from the right translations $(R_{\exp(tY)})^{-1}$ on G.

Now we analyze the family $\mathcal{H}_t = (M \times g^*, \omega, H_t)$ of Hamiltonian systems associated with the isospectral deformation $\langle , \rangle_t$ induced from an almost-inner derivation ϕ of g.

Let U be an open subset of $\dot{g}^* := g^* \setminus 0$, and take a smooth vector field Z on $M \times U$ given as

$$Z([g], \mu) = (L_{g*}Y(\mu), -\phi^*(\mu)) \in T_{[g]}M \times g^* \tag{3.2}$$

for $([g], \mu) \in M \times U$ ($[g]$ denoting the point of M corresponding to $g \in G$), where Y is a smooth map of U into g. Then,

Lemma 3.3. $ZH_t = H_t'$ *holds for every* $t \in \boldsymbol{R}$.

Proof. We have

$$H_t'([g], \mu) = \frac{1}{2}\langle \mu^\#, \mu^\# \rangle_t' + \langle (\mu^\#)', \mu^\# \rangle_t.$$

Differentiate the equation $\langle \mu^\#, X\rangle_t = \mu(X)$ $(X \in g)$ with respect to t, and we get $\langle (\mu^\#)', X\rangle_t = -\langle \mu^\#, X\rangle_t'$. On the other hand,

$$\langle X, X\rangle_t' = \frac{d}{dt}\langle \Phi_{t*}(X), \Phi_{t*}(X)\rangle_0 = 2\langle \phi(X), X\rangle_t$$

holds. Therefore we get

$$\begin{aligned} & H_t'([g], \mu) \\ = \; & -\frac{1}{2}\langle \mu^\#, \mu^\# \rangle_t' = -\langle \phi(\mu^\#), \mu^\# \rangle_t = -\mu(\phi(\mu^\#)) \\ = \; & -\phi^*\mu(\mu^\#) = -\langle \phi^*\mu, \mu\rangle_t^* \end{aligned}$$

$$= (ZH_t)([g], \mu).$$

Suppose Z is a complete vector field on $M \times U$, and let χ_t is the one-parameter group of transformations generated by Z. It follows from Lemma 3.3 that χ_t satisfies (1.1). As seen in §1 χ_t is a homogeneous symplectic diffeomorphism if and only if Z satisfies

$$\mathcal{L}_Z \omega = 0, \tag{3.3}$$

and

$$(j_c)_* Z = Z \quad (c > 0), \tag{3.4}$$

j_c being the map $(g, \mu) \mapsto (g, c\mu)$.

The main result of this note is the following.

Theorem 3.4. *Let $\mathcal{H}_t = (M \times \mathrm{g}^*, \omega, H_t)$ $(t \in \boldsymbol{R})$ be a one-parameter family of Hamiltonian systems corresponding to the deformation of a metric which is induced from an almost-inner derivation ϕ. Suppose that there exist a conic and open subset U of $\dot{\mathrm{g}}^*$ and a smooth map Y of U into g such that*
(c.1) $M \times U$ is invariant under the flow of $\mathcal{H}_t$ for each t,
(c.2) the vector field Z given by (3.2) is complete in $M \times U$,
(c.3) $-\phi^*(\mu) = ad^*(Y(\mu))\mu$ holds for every $\mu \in U$,
(c.4) $Y(c\mu) = Y(\mu)$ holds for every positive real number c and $\mu \in U$, and
(c.5) $\nu\{\tau(Y(\mu))\} - \tau\{\nu(Y(\mu))\} = 0$ holds for every constant vector fields ν, $\tau : U \to \mathrm{g}^*$; $\nu(\mu) = \mu$, $\tau(\mu) = \tau$.
Then, $(M \times U, \omega, H_t) \cong (M \times U, \omega, H_0)$ holds for every $t \in \boldsymbol{R}$.

Proof. It suffices to check the conditions (3.3) and (3.4) for the vector field Z. The condition (3.4) follows directly from (c.4). For $\rho = (L_{g*}V, \nu), \sigma = (L_{g*}W, \tau) \in T_{[g]}M \times U$ $(V, W \in \mathrm{g})$ we have

$$\begin{aligned} &(\mathcal{L}_Z\omega)([g], \mu)(\rho, \sigma) \\ &= d(i_Z\omega)([g], \mu)(\rho, \sigma) \\ &= \rho\{\omega([g], \mu)(Z, \sigma)\} - \sigma\{\omega([g], \mu)(Z, \rho)\} - \omega([g], \mu)(Z, [\rho, \sigma]), \end{aligned}$$

where we regard in the last line ρ and σ as the vector fields $\rho([h], \zeta) = (L_{h*}V, \nu)$ and $\sigma([h], \zeta) = (L_{h*}W, \tau)$, respectively, on $M \times U$. Note that $[\rho, \sigma]_{([g], \mu)} = (L_{g*}[V, W], 0)$. By means of the conditions (c.3) and (c.5) the above turns out to be

$$\begin{aligned} \rho\{\phi^*\mu(W) + \tau(Y(\mu)) + \mu([Y(\mu), W])\} \quad &- \quad \sigma\{\phi^*\mu(V) + \nu(Y(\mu)) + \mu([Y(\mu), V])\} \\ &- \quad (\phi^*\mu)([V, W]) - \mu([Y(\mu), [V, W]]) = 0. \end{aligned}$$

Thus we get (3.3). ∎

Expressions by local coordinates. Let $\{V_1, \cdots, V_n\}$ be a basis of g and $\{V_1^*, \cdots, V_n^*\}$ its dual basis of g^*. Associated with these bases we take coordinates $(x^1, \cdots, x^n)$ of g and

$(\mu_1, \cdots, \mu_n)$ of g^*. Put $\phi^*(\mu) = \sum_{j=1}^n \phi_j^*(\mu)V_j^*$, $Y(\mu) = \sum_{j=1}^n Y^j(\mu)V_j$. Then, the condition (c.3) is expressed as

$$(c.3') \qquad -\begin{pmatrix} \phi_1^*(\mu) \\ \phi_2^*(\mu) \\ \vdots \\ \phi_n^*(\mu) \end{pmatrix} = \begin{pmatrix} 0 & \sum_j C_{21}^j\mu_j & \cdots & \sum_j C_{n1}^j\mu_j \\ \sum_j C_{12}^j\mu_j & 0 & \cdots & \sum_j C_{n2}^j\mu_j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_j C_{1n}^j\mu_j & \sum_j C_{2n}^j\mu_j & \cdots & 0 \end{pmatrix} \begin{pmatrix} Y^1(\mu) \\ Y^2(\mu) \\ \vdots \\ Y^n(\mu) \end{pmatrix},$$

where $\{C_{ik}^j\}$ are the structure constants, i.e., $[V_i, V_k] = \sum_j C_{ik}^j V_j$. The matrix in the formula above is singular, hence, the vector $Y(\mu)$ is not uniquely determined. The condition (c.5) is equivalent to

$$(c.5') \qquad \frac{\partial Y^j}{\partial \mu_k} - \frac{\partial Y^k}{\partial \mu_j} = 0, \qquad 1 \le j, k \le n,$$

that means the one-form $\alpha = \sum_{j=1}^n Y^j d\mu_j$ on $\dot{g}^*$ to be closed.

Now we see in some examples that for an almost-inner derivation of the nilpotent algebra g there exist a conic, open and dense subset U of $\dot{g}^*$ and a smooth map Y of U into g which satisfy the conditions (c.1)-(c.5).

Example 1(see [4], for details). Let g be the six-dimensional Lie algebra with basis $\mathcal{B} = \{U_1, U_2, V_1, V_2, W_1, W_2\}$ and

$$[U_1, V_1] = [U_2, V_2] = W_1, \quad [U_1, V_2] = W_2,$$

all other brackets being zero. This is a two-step nilpotent Lie algebra with the center z being generated by $\{W_1, W_2\}$. One way to realize it as a matrix algebra is to let $\sum_{i=1}^2(x_iU_i + y_iV_i + z_iW_i)$ correspond to the 7×7 matrix

$$\left(\begin{array}{cccc|ccc} 0 & x_1 & x_2 & z_1 & & & \\ 0 & 0 & 0 & y_1 & & 0 & \\ 0 & 0 & 0 & y_2 & & & \\ 0 & 0 & 0 & 0 & & & \\ \hline & & & & 0 & x_1 & z_2 \\ & 0 & & & 0 & 0 & y_2 \\ & & & & 0 & 0 & 0 \end{array}\right).$$

Let $\phi : g \to g$ be the derivation defined by

$$\phi(U_1) = W_2,$$

with zero on the remaining elements of $\mathcal{B}$. Then ϕ belongs to AID(g). In fact, using the basis $\mathcal{B}^* = \{U_1^*, U_2^*, V_1^*, V_2^*, W_1^*, W_2^*\}$ of g^* dual to $\mathcal{B}$, we put $\mu = \kappa_1 W_1^* + \kappa_2 W_2^* + \mu_0$ with $\mu_0 = \sum_{i=1}^2(\mu_i U_i^* + \nu_i V_i^*) \in z^\perp := \{\mu_0 \in g^*; \mu_0(W) = 0 \text{ for all } W \in z\}$. Then,

$$\phi^*(\mu) = \begin{cases} ad^*(-V_2)\mu & (\kappa_1 = 0) \\ ad^*\left(-\dfrac{\kappa_2}{\kappa_1}V_1 + \dfrac{\nu_1\kappa_2}{\kappa_1^2}W_1 - \dfrac{\nu_1}{\kappa_1}W_2\right)\mu & (\kappa_1 \neq 0) \end{cases}.$$

Take the conic, open and dense subset $U = \{\mu = \kappa_1 W_1^* + \kappa_2 W_2^* + \mu_0;\ \kappa_1 \neq 0, \mu_0 \in \mathfrak{z}^\perp\}$ of $\mathfrak{g}^*$, and the map Y of U into $\mathfrak{g}$:

$$Y(\mu) = \frac{\kappa_2}{\kappa_1}V_1 - \frac{\nu_1\kappa_2}{\kappa_1^2}W_1 + \frac{\nu_1}{\kappa_1}W_2,$$

satisfies (c.3)-(c.5). Moreover, Z given by (3.2) is a Hamiltonian vector field associated with the function on $M \times U$:

$$F([g], \mu) = \frac{\nu_1\kappa_2}{\kappa_1}.$$

2. Let $\mathfrak{g}$ be the $(n+3)$-dimensional $(n \geq 2)$ Lie algebra with the basis $\mathcal{B} = \{U_1, U_2, V_j, W;\ 1 \leq j \leq n\}$ satisfying

$$[U_1, V_j] = V_{j+1}\ (1 \leq j \leq n-1), \quad [U_1, V_n] = W, \quad [U_1, W] = [U_1, U_2] = 0,$$
$$[U_2, V_j] = [U_1, [U_1, V_j]] = V_{j+2}\ (1 \leq j \leq n-2),$$
$$[U_2, V_{n-1}] = W, \quad [U_2, V_n] = [U_2, W] = 0,$$
$$[V_j, V_k] = 0\ (1 \leq j, k \leq n).$$

This is an $(n+1)$-step nilpotent Lie algebra with the one dimensional center $\mathfrak{z}$ generated by W. One realization as a matrix algebra is obtained by letting $\sum_{i=1}^2 x_i U_i + \sum_{j=1}^n y_j V_j + zW$ correspond to the $(n+2) \times (n+2)$ matrix

$$\begin{pmatrix} 0 & x_1 & x_2 & 0 & \cdots & 0 & z \\ & 0 & x_1 & x_2 & \cdots & 0 & y_n \\ & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & x_2 & y_3 \\ & & & & \ddots & x_1 & y_2 \\ & 0 & & & & 0 & y_1 \\ & & & & & & 0 \end{pmatrix}.$$

Let ϕ be the derivation of $\mathfrak{g}$ defined by

$$\phi(U_2) = W$$

with zero on the remaining elements of $\mathcal{B}$. Let $\mathcal{B}^* = \{U_1^*, U_2^*, V_j^*, W^*\}$ be the basis of $\mathfrak{g}^*$ dual to $\mathcal{B}$. For $\mu = \kappa W^* + \mu_1 U_1^* + \mu_2 U_2^* + \sum_{j=1}^n \nu_j V_j^*$, we have

$$\phi^*\mu = \begin{cases} ad^*(cV_n)\mu & (\kappa = 0) \\ ad^*\left(-V_{n-1} + \dfrac{\nu_n}{\kappa}V_n - \dfrac{\nu_n^2}{2\kappa^2}W\right)\mu & (\kappa \neq 0) \end{cases},$$

c being constant. Thus, ϕ belongs to AID(g). The map

$$Y(\mu)=V_{n-1}-\frac{\nu_n}{\kappa}V_n+\frac{\nu_n^2}{2\kappa^2}W$$

of the conic, open and dense subset $U=\{\mu=\kappa W^*+\mu_0;\ \kappa\neq 0, \mu_0\in \mathrm{z}^\perp\}$ of $\dot{\mathrm{g}}^*$ satisfies (c.3)-(c.5), and is the Hamiltonian vector field associated with

$$F([g],\mu)=\nu_{n-1}-\frac{\nu_n^2}{2\kappa}.$$

3(cf. [5]). Let g be the six-dimensional Lie algebra with the basis $\mathcal{B}=\{U_1,\cdots,U_5,W\}$ which satisfies that

$$[U_1,U_2]=U_3,\quad [U_1,U_3]=U_4,$$
$$[U_1,U_4]=U_5,\quad [U_2,U_3]=U_5,$$
$$[U_5,U_2]=W,\quad [U_3,U_4]=W,$$

and all other brackets are zero. This is a five-step nilpotent Lie algebra with the one dimensional center z generated by W. Let ϕ be the derivation of g defined by

$$\phi(U_2)=U_5$$

with zero on the remaining elements of $\mathcal{B}$. Then ϕ is almost-inner. Let $Y(\mu)$ be a vector in g satisfying $-\phi^*\mu=ad^*(Y(\mu))\mu$. Put $\mu=\kappa W^*+\sum_{i=1}^5\mu_iU_i^*$ with respect to the basis $\mathcal{B}^*=\{U_1^*,\cdots,U_5^*,W^*\}$ of g^* dual to $\mathcal{B}$, and take the conic, open and dense subset $U=\{\mu=\kappa W^*+\mu_0;\ \kappa\neq 0,\mu_0\in\mathrm{z}^\perp\}$ of $\dot{\mathrm{g}}^*$. Then, $Y(\mu)$ is given as

$$Y(\mu)=-\frac{\mu_5}{\kappa}U_5+\frac{\mu_5^2}{2\kappa^2}W,$$

which satisfies (c.4) and (c.5), and Z given by (3.2) is the Hamiltonian vector field associated with

$$F([g],\mu)=-\frac{\mu_5^2}{2\kappa}.$$

It remains for us to check the conditions (c.1) and (c.2) in the examples above. Note that $\mathrm{z}^*\cong \mathrm{g}^*/\mathrm{z}^\perp$, and we can show (c.1) and (c.2) in every example from the fact that the vectors X_{H_t} and Z are tangent to $M\times \mathrm{z}^\perp$.

We conclude this note with the following conjecture.

Conjecture. Let $M=\Gamma\backslash G$ be a compact nilmanifold. Let $\langle,\rangle_t$ be the isospectral deformation of a metric on M which is induced from an almost-inner derivation of g. Then there exists a conic, open and dense subset U of $\dot{\mathrm{g}}^*$ such that $(M\times U,\omega,H_t)\cong(M\times U,\omega,H_0)$.

Remark. It is proved by an elementary consideration that if the almost-inner derivation ϕ is not inner, then the map $Y(\mu)$ satisfying (3.1) cannot be smooth (continuous) in the whole space $\dot{T}^*M$.

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