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Some Remarks on the Conformal Transformations of the Unit Disk

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Abstract

We consider a condition that arises in the theory of H-surface and the Plateau problem. This condition is related to the fact that the Plateau problem is invariant under the conformal transformations of the unit disk. Therefore some normalization is needed for the conformal transformations. Concerning this condition, we state preliminary results in this paper.

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§1. Notations

In some problems of H-surface parametrized with isothermal coordinates in the unit disk, the invariance under conformal transformations of the unit disk makes it necessary to choose some parameters. Three points condition is well-known one to normalize conformal transformations of the unit disk. But, in certain cases, three points condition does not work well and another condition is needed (cf. Struwe [4]).

To state this condition, let G be the Lie group of conformal transformations of the unit disk B, and let identify two-dimensional Euclidean plane $\boldsymbol{R}^2$ with complex plane $\boldsymbol{C}$. Then a member of G is of the form

$$(1) \quad g(z) = e^{i\phi}\frac{z-a}{1-\bar{a}z} \qquad \text{for } \phi \in \boldsymbol{R}, \quad a = \rho e^{i\alpha} \in B.$$

We identify G with $\tilde{G}$ which is the group of mappings from the unit circle onto itself. More precisely, any $g(z) \in G$ corresponds to

$$\tilde{g}(z) = g(z)|_{\partial B} \in \tilde{G}.$$

Furthermore, in polar coordinates, we can identify $\tilde{g}(z)$ with $\tilde{g}(\theta)$ that satisfies

$$\tilde{g}(z) = \exp i\, \tilde{g}(\theta) \qquad \text{for} \quad z = e^{i\theta}.$$

Then $\tilde{g}(\theta)$ is a function on $\boldsymbol{R}$ with the property $\tilde{g}(\theta + 2\pi) = \tilde{g}(\theta) + 2\pi$. Explicit form of $\tilde{g}(\theta)$ corresponding to $g(z)$ of the form (1) is given by

$$(2) \quad \tilde{g}(\theta) = \theta + \phi + 2\tan^{-1}\left(\frac{\rho \sin(\theta - \alpha)}{1 - \rho\cos(\theta - \alpha)}\right).$$

Hereafter we do not distinguish $\tilde{g}(z)$ from $\tilde{g}(\theta)$, and denote like $\tilde{g}(\theta) \in \tilde{G}$. Let $T_{id}\tilde{G}$ be a tangent space of $\tilde{G}$ at $id \in \tilde{G}$, then it is easy to see that

$$(3) \quad T_{id}\tilde{G} = span\ \{1, \sin\theta, \cos\theta\}.$$

Now denote

$$\mathcal{M} = \{\, x : x \in C(\boldsymbol{R}), x(\theta + 2\pi) = x(\theta) + 2\pi, x \text{ is non-decreasing}\,\},$$

and

$$(4) \quad \mathcal{M}^{\dagger} = \{\, x \in \mathcal{M} : \int_0^{2\pi} (x - id)\, \eta\, d\theta = 0, \qquad \text{for any } \eta \in T_{id}\tilde{G}\,\}.$$

Here, we do not explain the meaning of $x \in \mathcal{M}$ (see Struwe [4]). Then the condition is stated as follows. For any $x \in \mathcal{M}$, choose $\tilde{g}(\theta) \in \tilde{G}$ such that $x \circ \tilde{g}^{-1}(\theta) \in \mathcal{M}^{\dagger}$.

In the following sections, we consider some basic results relating to this condition. And in the last section, we will state a result (without proof) which is equivalent to this condition as an application.

§2. Basic Results

We show, in this section, preliminary results concerning above functions. First we see the fundamental formula concerning the equation (2).

Proposition 1. *For* $0 \leq r < 1$, *we have*

$$\tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) = \sum_{n=1}^{\infty} \frac{r^n}{n} \sin n\theta.$$

PROOF: The left-hand side of the above equation is an odd function, so it is expressed only by the terms of $\sin n\theta$. Then we must show

$$\frac{1}{\pi}\int_0^{2\pi} \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\sin n\theta\, d\theta = \frac{r^n}{n}.$$

This can be shown, for example, by the following computation. Integrating by parts, we see

$$\int_0^{2\pi} \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\sin n\theta\, d\theta = \frac{1}{n}\int_0^{2\pi} \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2}\cos n\theta\, d\theta.$$

So it is enough to compute the Fourier coefficient of $\cos n\theta$ of the function

$$\frac{r\cos\theta - r^2}{n(1-2r\cos\theta+r^2)}.$$

Let $z = e^{i\theta}$, then the above function is reduced to the form

$$\frac{r(1-rz-z(z-r))}{2n(1-rz)(r-z)} = \frac{1}{n}\sum_{k=1}^{\infty} r^k\left(\frac{z^k+z^{-k}}{2}\right).$$

Now it is easy to see that the Fourier coefficient is r^n/n. ∎

For an elegant proof of Proposition 1, see Zygmund [5, p. 2]. From the above Proposition 1, we obtain the following Fourier expansions of $\tilde{g}(\theta)$ and of $\tilde{g}'(\theta)$.

Corollary 2. *For $\tilde{g}(\theta)\in \tilde{G}$ and $\tilde{g}'(\theta)$, we have*

$$(5)\quad \tilde{g}(\theta) = \theta + \phi + 2\sum_{n=1}^{\infty}\frac{\rho^n}{n}\sin n(\theta-\alpha),$$

$$(6)\quad \tilde{g}'(\theta) = 1 + 2\sum_{n=1}^{\infty}\rho^n\cos n(\theta-\alpha).$$

PROOF: The equation (5) is a direct consequence of Proposition 1. Now it is easy to see the equation (6). ∎

Note that the above expression of $\tilde{g}'(\theta)$ is exactly the well-known Poisson kernel. Incidentally, in order to obtain the above relation (6) briefly, we may differentiate the equation (2) and see that

$$\tilde{g}'(\theta) = \frac{1-\rho^2}{1-2\rho\cos(\theta-\alpha)+\rho^2}.$$

Then the right-hand side of the above equation is the Poisson kernel, so we obtain the formula (6).

Next we show the elementary formulas that will be used in the next section to obtain the formulas for $\sin\tilde{g}(\theta)$ and $\cos\tilde{g}(\theta)$.

Proposition 3. *For $0 \le r < 1$, we have*

$$\begin{aligned}\frac{(1-r^2)\sin\theta}{1-2r\cos\theta+r^2} &= (1-r^2)\sum_{n=0}^{\infty} r^n \sin(n+1)\theta,\\ \frac{(1+r^2)\cos\theta-2r}{1-2r\cos\theta+r^2} &= (1-r^2)\sum_{n=0}^{\infty} r^n \cos(n+1)\theta - r.\end{aligned}$$

PROOF: Let $z = e^{i\theta}$, then the left-hand side of the first equation is reduced to the form

$$\frac{(1-r^2)(z^2-1)}{2i(1-rz)(z-r)} = (1-r^2)\sum_{n=0}^{\infty} r^n \left(\frac{z^{n+1}-z^{-(n+1)}}{2i}\right).$$

So we have the first equation.

By the same way, let $z = e^{i\theta}$ in the left-hand side of the second equation, then we have

$$\frac{1}{2}\left(\frac{1-rz}{z-r}+\frac{r-z}{1-rz}\right) = (1-r^2)\sum_{n=0}^{\infty} r^n \left(\frac{z^{n+1}+z^{-(n+1)}}{2}\right) - r.$$

This shows that the second formula is also valid. ∎

Finally, we show elementary formulas that will be used in the next section. By the definition, we have

$$\begin{aligned}\exp i\,\tilde{g}(\theta) &= \tilde{g}(e^{i\theta}) = e^{i\phi}\frac{e^{i\theta}-\rho e^{i\alpha}}{1-\rho e^{i(\theta-\alpha)}}\\ &= e^{i(\alpha+\phi)}\frac{e^{i(\theta-\alpha)}-2\rho+\rho^2 e^{-i(\theta-\alpha)}}{1-2\rho\cos(\theta-\alpha)+\rho^2}.\end{aligned}$$

Then comparing the real and imaginary parts of the above equation, we have

$$\begin{aligned}(7)\quad \cos(\tilde{g}(\theta)) &= \frac{(1+\rho^2)\cos(\theta-\alpha)-2\rho}{1-2\rho\cos(\theta-\alpha)+\rho^2}\cos(\alpha+\phi)\\ &\quad -\frac{(1-\rho^2)\sin(\theta-\alpha)}{1-2\rho\cos(\theta-\alpha)+\rho^2}\sin(\alpha+\phi),\end{aligned}$$

$$\begin{aligned}(8)\quad \sin(\tilde{g}(\theta)) &= \frac{(1-\rho^2)\sin(\theta-\alpha)}{1-2\rho\cos(\theta-\alpha)+\rho^2}\cos(\alpha+\phi)\\ &\quad +\frac{(1+\rho^2)\cos(\theta-\alpha)-2\rho}{1-2\rho\cos(\theta-\alpha)+\rho^2}\sin(\alpha+\phi).\end{aligned}$$

§3. Further Results

Using the results in the former section, we can show the next formulas.

Proposition 4. *For* $\sin(\tilde{g}(\theta))$ *and* $\cos(\tilde{g}(\theta))$, *we have*

$$(9)\quad \sin(\tilde{g}(\theta)) = (1-\rho^2)\cos(\alpha+\phi)\sum_{n=0}^{\infty}\rho^n \sin(n+1)(\theta-\alpha) + (1-\rho^2)\sin(\alpha+\phi)\sum_{n=0}^{\infty}\rho^n\cos(n+1)(\theta-\alpha) - \rho\sin(\alpha+\phi),$$

$$(10)\quad \cos(\tilde{g}(\theta)) = (1-\rho^2)\cos(\alpha+\phi)\sum_{n=0}^{\infty}\rho^n \cos(n+1)(\theta-\alpha) - (1-\rho^2)\sin(\alpha+\phi)\sum_{n=0}^{\infty}\rho^n\sin(n+1)(\theta-\alpha) - \rho\cos(\alpha+\phi).$$

PROOF: By (8), we have

$$\begin{aligned}\sin(\tilde{g}(\theta)) &= \frac{(1-\rho^2)\sin(\theta-\alpha)}{1-2\rho\cos(\theta-\alpha)+\rho^2}\cos(\alpha+\phi) \\ &\quad + \frac{(1+\rho^2)\cos(\theta-\alpha)-2\rho}{1-2\rho\cos(\theta-\alpha)+\rho^2}\sin(\alpha+\phi).\end{aligned}$$

Then, using Proposition 3, we easily obtain the first formula. The second formula is proved by the same method. ∎

The condition that $x \circ \tilde{g}^{-1}(\theta) \in \mathcal{M}^{\dagger}$ is eqivalent to

$$\int_0^{2\pi} (x\circ\tilde{g}^{-1}(\theta)-\theta)\,\eta(\theta)\,d\theta = 0 \qquad \text{for any } \eta\in T_{id}\tilde{G},$$

and this can be written as

$$\int_0^{2\pi} (x(\theta)-\tilde{g}(\theta))\,\eta(\tilde{g}(\theta))\,\tilde{g}'(\theta)\,d\theta = 0.$$

It is more convenient to express the above equation as

$$(11)\quad \int_0^{2\pi}(x(\theta)-\theta)\eta(\tilde{g}(\theta))\tilde{g}'(\theta)\,d\theta = \int_0^{2\pi}(\tilde{g}(\theta)-\theta)\eta(\tilde{g}(\theta))\tilde{g}'(\theta)\,d\theta.$$

Because of (3), we need to take as η only 1, $\sin\theta$, and $\cos\theta$. From the above consideration, we may see that it is useful to obtain the expressions of

$\sin(\tilde{g}(\theta))\tilde{g}'(\theta)$ and $\cos(\tilde{g}(\theta))\tilde{g}'(\theta)$.

Proposition 5. *For* $\sin(\tilde{g}(\theta))\tilde{g}'(\theta)$ *and* $\cos(\tilde{g}(\theta))\tilde{g}'(\theta)$ *where* $\tilde{g}(\theta) \in \tilde{G}$, *we have*

$$\sin(\tilde{g}(\theta))\tilde{g}'(\theta) = (1-\rho^2)\cos(\alpha+\phi)\sum_{n=0}^{\infty}(n+1)\rho^n \sin(n+1)(\theta-\alpha) + (1-\rho^2)\sin(\alpha+\phi)\sum_{n=0}^{\infty}(n+1)\rho^n \cos(n+1)(\theta-\alpha),$$

$$\cos(\tilde{g}(\theta))\tilde{g}'(\theta) = (1-\rho^2)\cos(\alpha+\phi)\sum_{n=0}^{\infty}(n+1)\rho^n \cos(n+1)(\theta-\alpha) - (1-\rho^2)\sin(\alpha+\phi)\sum_{n=0}^{\infty}(n+1)\rho^n \sin(n+1)(\theta-\alpha).$$

PROOF: By Proposition 4, the proof of Proposition 5 is straightforward. ∎

Finally, we make some remarks about further development briefly. For any $x \in \mathcal{M}$, we set $x - id = f$ (note that f is a 2π-periodic), and denote by F the solution of

$$(12)\quad \triangle F = 0 \quad \text{in } B, \qquad F|_{\partial B} = f.$$

Expanding f in Fourier series as

$$(13)\quad f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta),$$

we have the expression of the above solution F in polar coordinates

$$(14)\quad F(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta).$$

With these notations, we will be able to show the following statement.

For any $x \in \mathcal{M}$, *the condition that* $\tilde{g}(\theta) \in \tilde{G}$ *satisfies*

$$(15)\quad x \circ \tilde{g}^{-1}(\theta) \in \mathcal{M}^{\dagger}$$

is equivalent to the condition that there exist a point $\rho e^{i\alpha} \in B$ *and* $\phi \in \boldsymbol{R}$ *such that*

$$(16)\quad F(\rho,\alpha) = \phi, \quad \frac{\partial F}{\partial r}(\rho,\alpha) = 0, \quad \frac{1}{\rho}\frac{\partial F}{\partial \theta}(\rho,\alpha) = \frac{2\rho}{1-\rho^2}.$$

The proof of the above statement will be given in [3]. The condition (16) determines ϕ and $a = \rho e^{i\alpha}$, so it determines the explicit form of $\tilde{g}(\theta)$ which normalize $x \in \mathcal{M}$ and also of corresponding $g(z)$.

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