PAPER *Special Section on Nonlinear Theory and its Applications*

Bifurcation Phenomena in the Josephson Junction Circuit Coupled by a Resistor

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SUMMARY Bifurcation phenomena observed in a circuit containing two Josephson junctions coupled by a resistor are investigated. This circuit model has a mechanical analogue: Two damped pendula linked by a clutch exchanging kinetic energy of each pendulum. In this paper, firstly we study equilibria of the system. Bifurcations and topological properties of the equilibria are clarified. Secondly we analyze periodic solutions in the system by using suitable Poincaré mapping and obtain a bifurcation diagram. There are two types of limit cycles distinguished by whether the motion is in $S^1 \times \boldsymbol{R}^3$ or $T^2 \times \boldsymbol{R}^2$, since at most two cyclic coordinates are included in the state space. There is a typical structure of tangent bifurcation for 2-periodic solutions with a cusp point. We found chaotic orbits via the period-doubling cascade, and a long-period stepwise orbit.

key words: *Josephson junction, pendulum, frictional clutch, bifurcation, chaos*

1. Introduction

A circuit containing a Josephson junction (abbr. JJ) element composes a singular state space since its differential equation includes a sinusoidal function of the state variable. From this strong nonlinearity the circuit containing JJ elements exhibit a rich variety of nonlinear phenomena. We have been investigated the phenomena observed in JJ circuits; many heteroclinic orbits in a circuit containing a JJ element and an linear inductor with a d.c. source [3], a strange periodic orbit called caterpillar solution in an inductively coupled JJ circuit [4]. It is also noteworthy that the equation of a JJ element can be interpreted as a pendulum. Thus it might be possible that a mechanical system with rotational variables is realized as a circuit containing JJ elements.

In this paper, we treat the resistively coupled circuit with two JJ elements. The mechanical analog of this system is represented by two pendula connected by a friction clutch. Each of the angular velocities affects interactively to the other pendulum through the clutch.

To analyze the bifurcation phenomena of the system, firstly we consider an invariant relationship and classify qualitative properties of equilibria in the system Secondly we investigate bifurcation of periodic solutions by using the Poincaré mapping and obtain bifurcation diagrams. We clarify that there are two types of limit cycles which distinguished by whether the motion is in $S^1 \times \boldsymbol{R}^3$ or $T^2 \times \boldsymbol{R}^2$, since at most two cyclic coordinates are included in the state space. The former is a periodic solution and the latter is a quasi-periodic solution. We also clarify a typical structure of tangent bifurcation for 2-periodic solutions with a cusp point, chaotic orbits via the period-doubling cascade, and a long-period stepwise orbit.

Since this model can be considered as a simplest model which demonstrates the interactions between a rotational-energy supply and a load in mechanical systems, it is important to clarify the principal bifurcation structures of the periodic solutions observed in the model. It is interesting that the chaotic vibration is observed when the pendula are not perfectly in the clutch.

Manuscript received February 2, 1996.
Manuscript revised April 11, 1996.

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2. The JJ Circuit Coupled by a Resistor

Figure 1 shows the JJ circuit coupled by a resistor. It is obtained from the circuit discussed in Ref. [4] by using R instead of L as a coupling device. Assume that the current-voltage characteristic of a JJ element is described as:

$$i_J = I_C \sin\phi_i, \quad \frac{d\phi_i}{dt} = \frac{2e}{\hbar}v_i \quad i = 1, 2. \qquad (1)$$

where, I_C, e, $\hbar$ are a threshold current associated with the tunnelling current, the electron charge and Dirac's constant, respectively. ϕ is the phase difference of the wave function at the center of the junction plane. If ϕ is considered as magnetic flux, then the JJ element behaves as an nonlinear inductor which is controlled by the magnetic flux. Usually a nonlinear conductance is assumed for G. We treat it as a linear one for the sake of mechanical interpretation, global behavior of the system is, however, not changed by this assumption.

To normalize the equation, we define the following variables and parameters:

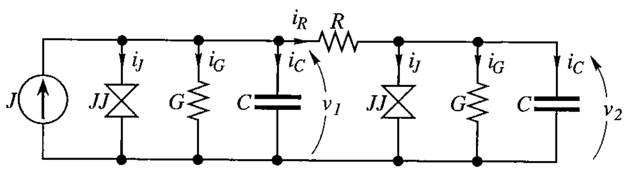

Fig. 1 The JJ circuit coupled by a resistor.

$$k = \sqrt{\frac{\hbar}{2eI_C C}} G$$
$$c = \sqrt{\frac{\hbar}{2eI_C C}} \frac{1}{R}$$
$$B_0 = \frac{J}{I_C}$$
$$\tau = \sqrt{\frac{2eI_C}{\hbar C}} t$$
$$x_i = \phi_i$$
$$y_i = \sqrt{\frac{2eC}{\hbar I_C}} v_i \quad (2)$$

where $i = 1, 2$. By rewriting τ as t, we have a four-dimensional autonomous differential equation:

$$\frac{dx_1}{dt} = y_1$$
$$\frac{dy_1}{dt} = -ky_1 - \sin x_1 - c(y_1 - y_2) + B_0$$
$$\frac{dx_2}{dt} = y_2$$
$$\frac{dy_2}{dt} = -ky_2 - \sin x_2 - c(y_2 - y_1). \quad (3)$$

System (3) can be regarded as parallel pendula connected by a friction clutch. k is interpreted as the coefficient of damping, c as frictional transmission constant, and B_0 as an input torque, see Fig. 2. The input is directly applied to the a primary pendulum P_1. By the clutch not only a part of kinetic energy of P_1 is transmitted to a secondary pendulum P_2, but also both kinetic energy of P_1 and P_2 affect each other. Thus $c \to 0$ (let out the clutch) means that two pendula are independent, i.e., the input drives only P_1, and then P_2 behaves as a simple pendulum without an input, at last rests in the origin. Otherwise, $c \to \infty$ (let in the clutch) means that the P_1 and P_2 can be regarded as a single pendulum because both pendula are connected equivalently by a rigid rod.

There exists an invariant transformation in system (3):

$$\mathcal{T} : (x_1, y_1, x_2, y_2) \mapsto (x_1 \pm 2n\pi, y_1, x_2 \pm 2m\pi, y_2) \quad (4)$$

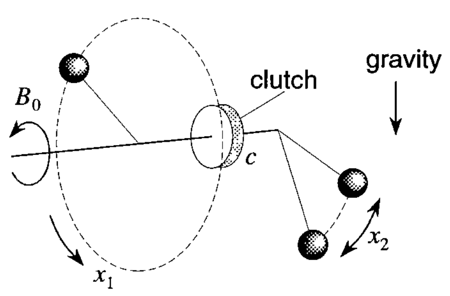

Fig. 2 Frictionally coupled pendula model.

where, n, m are integers. Equation (4) suggests that x_1 and x_2 can be added integral multiples of 2π independently, i.e., there exist two cyclic coordinates $S^1 \times S^1 = T^2$ in the system. It should be noted that the system is very different from the case of Ref. [4] upon two points: 1) the state space can include T^2, say, a coupling inductor constructs $S^1 \times \boldsymbol{R}^3$, and a coupling resistor can construct not only $S^1 \times \boldsymbol{R}^3$ but also $T^2 \times \boldsymbol{R}^2$. 2) there are very few equilibria in this system at all values of the parameters. We consider the latter point in the following section.

3. Equilibria in the System

Equilibria of the system are obtained from Eq. (3):

$$x_1 = \begin{cases} \mathrm{Sin}^{-1} B_0 \\ \pi - \mathrm{Sin}^{-1} B_0, \end{cases} \quad y_1 = 0$$
$$x_2 = \begin{cases} 0 \\ \pi, \end{cases} \quad y_2 = 0 \quad (5)$$

where, x_1, x_2 are in $[0, 2\pi)$. So we obtain a schematic diagram of the torus phase space for equilibria in x_1-x_2. See Fig. 3.

In the system, there exist only three types of equilibria and at most four equilibria in the fixed parameters. Figure 3 also shows directions of solution flow with respect to each the equilibria on x_1-x_2 plane and Table 1 indicate the classification for stability of the equilibria. In Fig. 3, it is noted that ${}_2O$ is regarded that each of the

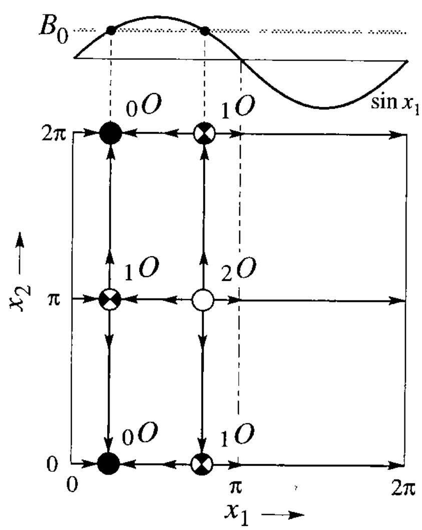

Fig. 3 Phase portrait of the system (3) in torus.

Table 1 Classification of equilibria in Eq. (3).

symbol	notation	topological property
●	${}_0O$	completely stable
⊗	${}_1O$	1-dimensionally unstable
⊗	${}_1O$	1-dimensionally unstable
○	${}_2O$	2-dimensionally unstable

pendula is on an unstable equilibrium (saddle point) simultaneously. Note also that ${}_3O$ does not exist in this system.

The condition of the parameter region in which there exist equilibria is $|B_0| < 1$. $B_0 = 1$ is tangent bifurcation value of equilibria. In the other value of B_0, there exists no equilibrium and at least one limit cycle since the system is dissipative.

4. Bifurcation of Periodic Solutions

Equations (3) are rewritten as:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \tag{6}$$

where, $\boldsymbol{x} = (x_1, y_1, x_2, y_2)$. Suppose that the solution of Eq. (6) is denoted by:

$$\boldsymbol{x}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}_0), \quad \boldsymbol{x}_0 = \boldsymbol{x}(0). \tag{7}$$

We should notice that there is no periodic solution in $\boldsymbol{R}^4$ if $k \neq 0$. The periodic solution we called in this paper is a modified periodic solution defined in Ref. [5]:

$$\begin{aligned} x_1(t_0) &= x_1(0) + 2\pi\ell \\ y_1(t_0) &= y_1(0) \\ x_2(t_0) &= x_2(0) + 2\pi m \\ y_2(t_0) &= y_2(0) \end{aligned} \tag{8}$$

where $t_0 > 0$ is a period and $\ell \geqq 1, m \geqq 0$ are integers. This definition means that P_1 must rotate to generate a periodic solution. We classify periodic solutions with respect to their behavior of the motions as follows:

1. **(Type-I)**: P_1 is winding around the circle $S^1 = \{0 \leqq x_1 < 2\pi\}$, but P_2 is rolling about a constant angular. Thus the motion is in $S^1 \times \boldsymbol{R}^3$.
2. **(Type-II)**: Each of the pendula is winding around the circle $S^1 \times S^1 = T^2$. The whole system behaves as quasi-periodic motion in $T^2 \times \boldsymbol{R}^2$.

Type-I solution evolves to chaos by changing the system parameters. In this paper, we mainly treat the bifurcation of this type of solutions. In the following we fix the system parameter k as 0.2.

To calculate bifurcation parameter we define the Poincaré mapping T and its section Π as:

$$\begin{aligned} T : \Pi &\rightarrow \Pi \\ \boldsymbol{x} &\mapsto \{\boldsymbol{\varphi}(t_0, \boldsymbol{x}) \mid x_1 - 2\pi, \\ &\qquad \left.\begin{array}{l} \text{if } x_2 \geqq 2\pi \text{ then } x_2 - 2\pi \\ \text{if } x_2 \leqq 2\pi \text{ then } x_2 \end{array}\right\} \end{aligned} \tag{9}$$

where t_0 is the time of first return of the initial state $\boldsymbol{x}$ to Π. Because all types of periodic solutions have at least an S^1 for x_1, we choose the Poincaré section as:

$$\Pi = \{\boldsymbol{x} \in S^1 \times \boldsymbol{R}^3 \subset \boldsymbol{R}^4 : x_1 = 0\} \tag{10}$$

Thereby we can calculate the bifurcation values of the parameters by Newton's method using T and its derivatives.

Figure 4 shows bifurcation diagram of the system (3) in B_0-c plane. The following properties are found:

1. I^i, G^i, $i = 1, 2$ indicate period-doubling and tangent bifurcation of the Type-I i-periodic solution, respectively. In the shaded region edged by I^1 and G^1, there exists stable periodic solution with the fixed point, x_1 winds around S^1 and x_2 tumbles around a constant angle, see Figs. 5 (a-1), (a-2). One of 2-periodic orbits bifurcated by I^1 is shown in Figs. 5 (b-1), (b-2).
2. Every solution bifurcated by I^2 evolves to chaos via period-doubling cascade. Figures 5 (c-1), (c-2) show the a chaotic orbit and the Poincaré mapping of the chaotic attractor.
3. In whole parameters, there exist quasi-periodic (Type-II) solutions. Figures 5 (d-1) and (d-2) show the orbit and an invariant closed curve of the Poincaré mapping on the cylindrical phase space. As c increases, the solution bind to a synchronized orbit.
4. I^2 curves are overhung each other by G^2 with a cusp point. In this region, we can observe coexistent 2-periodic and higher-periodic solutions. This is a typical structure of the higher-periodic bifurcation set.
5. The end-points of all bifurcation curves are caused by touching the homoclinic orbits. Also the chaotic orbits are frequently disappeared by meeting this global bifurcation and attracted into Type-II orbits.
6. The equilibrium is disappeared in $B_0 = 1.0$. Just

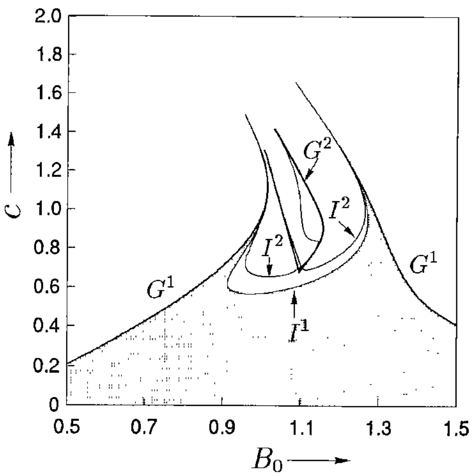

Fig. 4 Bifurcation diagram for periodic solutions.

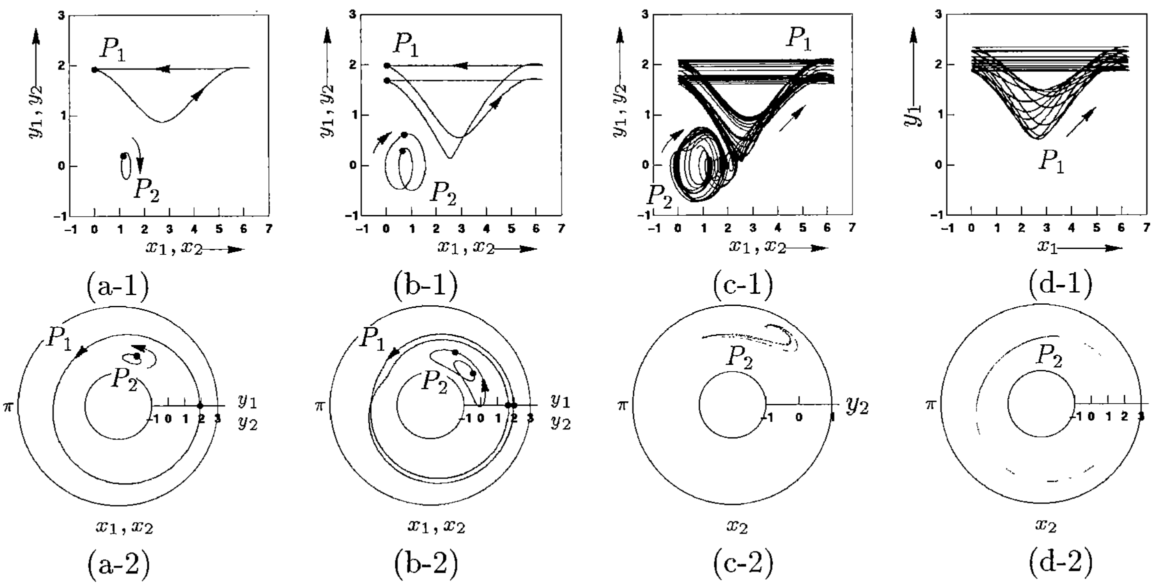

Fig. 5 Phase portraits. (a-1), (a-2): periodic orbit (Type I) with a fixed point. $B_0 = 1.3$, $c = 0.7$. (b-1), (b-2): 2-periodic orbit bifurcated by I^1, $B_0 = 0.97$, $c = 0.66$. (c-1), (c-2): Chaotic orbit via a period-doubling cascade and its Poincaré mapping $B_0 = 1.12$, $c = 0.73$. (d-1), (d-2): Quasi-periodic (Type II) orbit, (d-2) shows the Poincare mapping of x_2, $B_0 = 0.7$, $c = 0.7$.

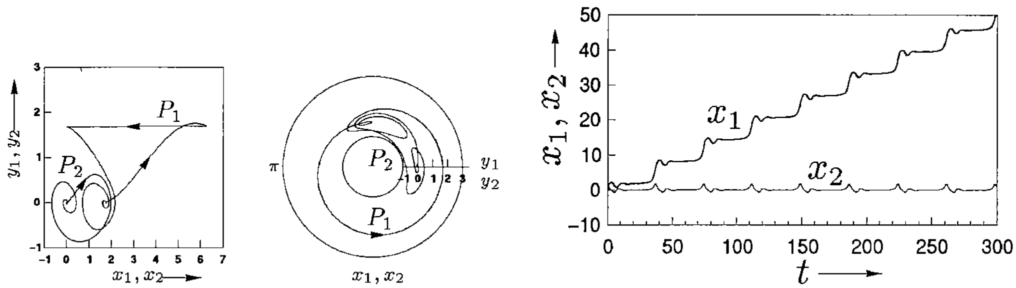

Fig. 6 Periodic orbit with a fixed point near the disappearance of the equilibria, $B_0 = c = 1.0$.

at this parameter, the orbit stopped at a stable equilibrium begins to start and approaches the neighborhood of the point in which the equilibrium exists originally. This orbit shown in Fig. 6 resembles Shil'nikov type homoclinic orbit, but this orbit is only a periodic solution because the manifolds of saddle-type equilibrium is also disappeared. It is interesting that the solution has a long period and a stepwise response.

7. Figure 7 illustrates the schematic bifurcation diagram. There exist chaotic orbits in the dark shaded region, however all of them are disappeared by homoclinic orbits by changing the parameter toward the center of the island edged by I^2. It is emphasized that a simple mechanical transmission can behave chaotically by controlling the clutch.

5. Conclusions

Some properties of system (3) are investigated. The results of this analysis compared with Ref. [4] suggests that the coupling element governs the structure of phase space without changing its order of the circuit equations. In the mechanical analog, we observe the chaotic vibrations in $S^1 \times \boldsymbol{R}^3$ at typical value of the parameter, i.e., they are caused when the pendula are not perfectly in the clutch. Our future researches are as follows:

- to investigate the bifurcation diagram for higher-periodic solutions in detail. In case that k is small, it might have more complex phenomena.
- to obtain the homoclinic orbits in this system. Particularly the relationship with disappearance of chaotic attractor is interesting.

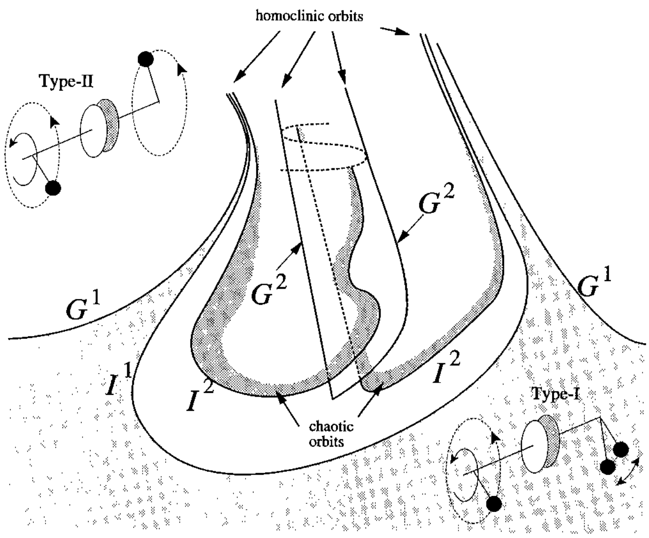

Fig. 7 Schematic diagram of bifurcations. The sheets shown in the area edged by G^2 indicate the manifolds of 2-periodic orbits.

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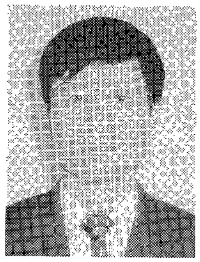

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