

# Study on the Phenomena of Potential Well of Infinite Depth on the View Point of Natural Statistical Physics

By

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## Abstract

In this paper, we study the phenomena of potential well of infinite depth on the view point of natural statistical physics. The mathematical model of this physical system is the system of micro-particles moving periodically with constant velocity in the interval  $D = [-a, a]$ , ( $a > 0$ ).

Thereby we obtain the structure of this physical system at the stationary state.

Thus we clarify that this physical system is the composed state of the proper physical systems at the stationary state and the ratio of their composition is given by the sequence

$$\{|a_n|^2\}_{n=0} \cup \{|b_n|^2\}_{n=1}^\infty,$$

where  $a_n$ , ( $n \geq 0$ ) and  $b_n$ , ( $n \geq 1$ ) are the Fourier type coefficients of the initial state  $\psi \in L^2$ .

Then we obtain the energy expectation of the total physical system

$$\bar{E} = \sum_{n=0}^{\infty} \frac{n^2 \pi^2 \hbar^2}{2ma^2} (|a_n|^2 + |b_n|^2),$$

where  $m$  denotes the mass of one micro-particle.

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## Introduction

In this paper, we study the phenomena of potential well of infinite depth on the view point of natural statistical physics.

The mathematical model of this physical phenomenon is the system of micro-particles moving periodically with constant velocity in the interval  $D = [-a, a]$ , ( $a > 0$ ).

We study the derivation of the Schrödinger equation of this physical system and solve the initial and boundary value problem of this Schrödinger equation.

Thereby we obtain the structure of the physical system at the stationary state. Thus we obtain the information of the natural statistical distributions of the position variable and the momentum variable of this physical system. Thus we obtain the energy expectation

$$\bar{E} = \sum_{n=0}^{\infty} \frac{n^2 \pi^2 \hbar^2}{2ma^2} ( |a_n|^2 + |b_n|^2 )$$

of the total physical system. At last, we obtain the fact that this physical system is the composed state of the proper physical systems at the stationary states and the ratio of their composition is given by the sequence

$$\{|a_n|^2\}_{n=0}^{\infty} \cup \{|b_n|^2\}_{n=1}^{\infty}.$$

This sequence is determined by the  $L^2$ -density  $\psi$  for the initial states. Namely,  $a_n$ , ( $n = 0, 1, 2, \dots$ ), and  $b_n$ , ( $n = 1, 2, \dots$ ) are the Fourier type coefficients of  $\psi$ .

As for these symbols, we refer to section 4 in this paper. As for the related works, we refer to Ito[8], Ito[16], Ito[20], chapter 11, Ito[26], chapitre 9, Ito[27], chapter 3, Ito[40], [44], [55], and others in the Reference.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

## 1 Setting of the problem

We study the natural statistical phenomena for the physical system of infinitely many micro-particles moving under the potential well of infinite depth in one dimensional space  $\mathbf{R}$ . We neglect the interaction among micro-particles.

We assume that this physical system  $\Omega$  is the set of micro-particles moving on the interval of finite length whose both end points are the perfectly reflecting walls.

This is a certain approximating model.

In other words, this physical system  $\Omega$  is the set of micro-particles moving under the action of the potential  $V(x)$  in  $\mathbf{R}$ .

This potential  $V(x)$  is approximately equal to

$$V(x) = \begin{cases} 0, & (|x| \leq a), \\ \infty, & (|x| > a). \end{cases}$$

Here  $a > 0$  is a constant.

Assume that each micro-particle is a point mass which has the mass  $m > 0$  and some electric charge. This electric charge is equal to either one of a positive or negative value and 0.

Then, by virtue of the potential  $V(x)$ , the force  $F(x)$  acts on each micro-particle, where  $F(x)$  is equal to

$$F(x) = -\frac{dV(x)}{dx} = \infty\delta_{(-a)} - \infty\delta_a.$$

Here the derivative  $\frac{dV(x)}{dx}$  is defined in the sense of weak convergence, and  $\delta_{(-a)}$  and  $\delta_a$  are the Dirac masses.

Thus each micro-particle moves under the action of the force  $F(x)$  following the Newtonian equation of motion .

Therefore the equation of motion of each micro-particle is equal to

$$m\frac{dv}{dt} = -\frac{dV(x)}{dx} = \infty\delta_{(-a)} - \infty\delta_a.$$

Here  $v$  denotes the velocity of the micro-particle.

Then  $\infty\delta_{(-a)}$  and  $-\infty\delta_a$  denote the perfectly reflected walls at points  $x = -a$  and  $x = a$  respectively in the physical sense.

Every micro-particle is reflected perfectly to the positive direction at  $x = -a$ , and it is reflected perfectly to negative direction at  $x = a$ .

Therefore every micro-particle changes its velocity to the inverse direction with the same magnitude of velocity at the two points  $x = -a$  and  $x = a$ , and it moves with constant speed in the interior of the interval.

Then the law of conservation of mechanical energy holds for this system. Namely we have the equality

$$\frac{1}{2}mv^2 + V(x) = \text{constant}.$$

In the practical physical phenomena, the infinite energy does not appear. Really, every micro-particle in the physical system considered here moves periodically in the interval  $|x| \leq a$ .

By virtue of the law of conservation of mechanical energy, the total energy of one micro-particle

$$\frac{1}{2}mv^2 = \frac{1}{2m}p^2$$

does not depend on the time  $t$  in the interval  $|x| \leq a$ . Here we put  $p = mv$ .

Namely, it is proved that the law of conservation of total energy of one micro-particle holds.

Then the natural statistical state of the total physical system is determined by virtue of the law II in Ito [44], and it evolves with time  $t$  by virtue of the law III in Ito [44] starting from the certain initial state.

Practically, the total physical system is the composed state of the proper physical systems in the natural statistical states, determined by the law II in Ito[44].

These natural statistical states are determined by the solutions of the Schrödinger equation for the stationary states determined by the variational principle.

Then the  $L^2$ -density describing the natural statistical state of the total physical system is determined by the eigenfunction expansion by the solutions of the Schrödinger equation.

Even if each micro-particle moves with constant velocity, the natural statistical phenomenon for the probability distribution states of position variable and momentum variable of micro-particles appears when each micro-particle has its initial position and its initial velocity in the various manner.

## 2 Setting of the mathematical model

In this section, we propose the mathematical model of the natural statistical phenomena of the physical system composed of micro-particles moving periodically under the action of the potential well of infinite depth considered in section 1.

Here we consider this physical system  $\Omega = \Omega(\mathcal{B}, P)$  is a probability space. The cardinal number of  $\Omega$  is at least equal to the continuous cardinal number.  $\mathcal{B}$  denotes the  $\sigma$ -algebra of probability events in  $\Omega$  and  $P$  denotes the probability measure.

The elementary event  $\rho$  of  $\Omega$  is composed of one micro-particle in  $\Omega$ . Each micro-particle is moving periodically with constant velocity in the interval  $D = [-a, a]$  in  $\mathbf{R}$ .

Then we denote the position variable of one micro-particle  $\rho$  as  $x = x(\rho)$  and denote its momentum variable as  $p = p(\rho)$ . The variable  $x$  moves in the interval  $D$  and the variable  $p$  moves in its dual space  $\mathbf{P}_1$ . Here, by virtue of the law II, the law of natural probability distribution of  $x$  is determined by a  $L^2$ -density  $\psi$  and the law of natural probability distribution of  $p$  is determined by its Fourier type coefficients  $\hat{\psi}(p)$ .

Now, by virtue of Newtonian dynamics, the total energy of one micro-particle  $\rho$  is equal to

$$\frac{1}{2m}p(\rho)^2.$$

This energy variable is considered as a natural random variable defined on the probability space  $\Omega$ .

Here we assume that the time when the stationary state is realized is equal to 0.

Then we calculate the expectation value of the energy variable defined in the above. Namely, we calculate the energy expectation value by virtue of the law II. When  $A$  is a Lebesgue measurable set in  $D$  and  $B$  is a subset of  $\mathbf{P}_1$ , we have the fundamental statistical formulas:

$$P(\{\rho \in \Omega; x(\rho) \in A\}) = \int_A |\psi(x)|^2 dx,$$

$$P(\{\rho \in \Omega; p(\rho) \in B\}) = \sum_{p \in B} |\hat{\psi}(p)|^2.$$

Then we have the energy expectation value in the following:

$$E\left[\frac{1}{2m}p(\rho)\right]^2 = \int_{-a}^a \frac{\hbar^2}{2m} \left| \frac{d\psi(x)}{dx} \right|^2 dx.$$

Here we used the Parseval equality for the Fourier series.

Now we denote **this energy expectation value** as

$$J[\psi] = \int_{-a}^a \frac{\hbar^2}{2m} \left| \frac{d\psi(x)}{dx} \right|^2 dx.$$

We say that  $J[\psi]$  is the **energy functional**.

In order to determine the natural probability distribution realized practically among the admissible natural probability distributions, we use the next principle I.

**Principle I(Variational principle)** The stationary state of the physical system considered in the above is realized so that it is the state where the energy expectation value of this physical system has its stationary value.

Following the law II and the principle I, we consider the following variational problem.

**Problem I(Variational problem)** Determine the stationary function of the energy functional  $J[\psi]$ . Here  $\psi(x)$  is a  $L^2$ -density satisfying the periodic boundary conditions  $\psi(-a) = \psi(a)$ .

### 3 Mathematical analysis

In this section, we study the Schrödinger equation which is the Euler equation obtained by solving the variational problem.

**Theorem 3.1** *For the considered physical system in section 2, we have the Schrödinger equation:*

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = \mathcal{E}\psi(x), \quad (|x| \leq a).$$

Here  $\hbar = \frac{h}{2\pi}$  and  $h$  is Planck's constant and the eigenvalue  $\mathcal{E}$  must satisfy the condition

$$\mathcal{E} \geq 0.$$

The Schrödinger equation in Theorem 3.1 is the Euler equation and the eigenvalue  $\mathcal{E}$  is the Lagrange's undetermined multiplier.

Here we solve the Schrödinger equation and obtain the solution  $\psi(x)$  concretely.

This is the solution of the eigenvalue problem in  $L^2 = L^2([-a, a])$ .

We have the following theorem.

**Theorem 3.2** *The solutions of the Schrödinger equation in Theorem 3.1 are given by the following formula:*

$$\begin{aligned} \psi_{\pm}^{(\mathcal{E})}(x) &= c(\mathcal{E}) \exp\left(\pm \frac{i}{\hbar} x \sqrt{2m\mathcal{E}}\right) \\ &= c(p) \exp\left(\pm ipx/\hbar\right), \quad (|x| \leq a). \end{aligned}$$

Here we put

$$\mathcal{E} = \frac{p^2}{2m}, \quad (-\infty < p < \infty)$$

and the signs  $\pm$  correspond to two linearly independent solutions,  $c(\mathcal{E})$  and  $c(p)$  denote the normalized constants.

Because the Schrödinger equation is symmetry with respect to the origin, two linearly independent eigenfunctions are obtained as the even function  $\psi_1^{(p)}(x)$  and the odd function  $\psi_2^{(p)}(x)$ . Namely, the eigenfunctions  $\psi_1^{(p)}(x)$  and  $\psi_2^{(p)}(x)$  satisfy the following relations:

$$\psi_1^{(p)}(-x) = \psi_1^{(p)}(x), \quad \psi_2^{(p)}(-x) = -\psi_2^{(p)}(x), \quad (|x| \leq a).$$

Now we determine the functions  $\psi_1^{(p)}(x)$  and  $\psi_2^{(p)}(x)$  concretely. Here we denote the solutions of the Schrödinger equation

$$\psi_{\pm}^{(p)}(x) = c(p)\exp(\pm ipx/\hbar), (|x| \leq a, -\infty < p < \infty).$$

Using these solutions, we determine the solutions  $\psi_1^{(p)}(x)$  and  $\psi_2^{(p)}(x)$ .

Because every micro-particle moves with constant velocity, we determine the solutions using the periodic boundary conditions. Namely we assume the following boundary conditions:

$$(B.C.) \begin{cases} \psi_1^{(p)}(-a) = \psi_1^{(p)}(a), \\ \psi_2^{(p)}(-a) = -\psi_2^{(p)}(a) = \psi_2^{(p)}(a) = 0. \end{cases}$$

Then we have the following theorem.

**Theorem 3.3** *In the interval  $|x| \leq a$ ,  $\psi_1^{(p)}(x)$  and  $\psi_2^{(p)}(x)$  are given as follows:*

$$(1) \quad \psi_1^{(0)}(x) = \frac{1}{\sqrt{2a}}.$$

$$(2) \quad \psi_1^{(n)}(x) = \frac{1}{\sqrt{a}} \cos \frac{n\pi x}{a}, (n = 1, 2, \dots).$$

$$(3) \quad \psi_2^{(n)}(x) = \frac{1}{\sqrt{a}} \sin \frac{n\pi x}{a}, (n = 1, 2, \dots).$$

Here we put  $p = \frac{n\pi\hbar}{a} = \frac{nh}{2a}$ .

Then the space of momentum variables is discrete. We denote this as

$$\mathbf{P}_1 = \left\{ p = \frac{n\pi\hbar}{a}; n = 0, 1, 2, \dots \right\}.$$

Now we prove that the solutions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$  in Theorem 3.3 are the stationary functions.

**Theorem 3.4** *For  $p = \frac{n\pi\hbar}{a} = \frac{nh}{2a}$ , ( $n = 0, 1, 2, \dots$ ), the solutions  $\psi_1^{(p)}(x) = \psi_1^{(n)}(x)$  and  $\psi_2^{(p)}(x) = \psi_2^{(n)}(x)$  satisfy the following equalities:*

$$J[\psi_j^{(p)}] = \mathcal{E}_n, (|x| \leq a),$$

for  $j = 1, 2, ; n = 0, 1, 2, \dots$ . Here we put

$$\mathcal{E}_n = \frac{p^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \geq 0, (n = 0, 1, 2, \dots).$$

Proof The functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$  are the solutions of the Schrödinger equation.

Namely we have the equality

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi_j^{(n)}(x)}{dx^2} = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \psi_j^{(n)}(x) = \mathcal{E}_n \psi_j^{(n)}(x), (|x| \leq a)$$

for  $j = 1, 2, ; n = 0, 1, 2, \dots$ . Therefore the functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$  are the stationary functions of the energy functionals as follows:

$$J[\psi_j^{(n)}] = \mathcal{E}_n, (j = 1, 2, ; n = 0, 1, 2, \dots),$$

$$\mathcal{E}_n = \frac{p^2}{2m}, p = \frac{nh}{2a}, (n = 0, 1, 2, \dots). //$$

Next, by using the properties of the trigonometrical functions, we prove the orthonormality condition of functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$ . Then we have the following theorem.

**Theorem 3.5** *The functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$ , ( $|x| \leq a$ ) are defined in Theorem 3.3. Then we have the orthonormality conditions as follows:*

- (1)  $\int_{-a}^a \psi_1^{(n')}(x) \psi_1^{(n)}(x) dx = \delta_{n, n'}, (n, n' = 0, 1, 2, \dots).$
- (2)  $\int_{-a}^a \psi_2^{(n')}(x) \psi_2^{(n)}(x) dx = \delta_{n, n'}, (n, n' = 0, 1, 2, \dots).$
- (3)  $\int_{-a}^a \psi_1^{(n')}(x) \psi_2^{(n)}(x) dx = 0, (n = 1, 2, \dots, n' = 0, 1, 2, \dots).$

Next we prove the completeness condition of the functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$ . Namely we have the following theorem.

**Theorem 3.6** *The functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$ , ( $|x| \leq a$ ) are given in Theorem 3.3. Then the functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$  satisfy the completeness condition in the following:*

$$\sum_{n=0}^{\infty} \psi_1^{(n)}(x') \psi_1^{(n)}(x) + \sum_{n=1}^{\infty} \psi_2^{(n)}(x') \psi_2^{(n)}(x) = \delta(x' - x), (|x|, |x'| \leq a).$$

We can prove Theorem 3.6 by virtue of the completeness of the system of trigonometric functions.



Then we can prove that the Theorem 3.6 is equivalent to Corollary 3.1 in the following.

**Corollary 3.1** For  $\psi(x) \in L^2$ , we have the following equality

$$\int_{-a}^a |\psi(x)|^2 dx = \sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |b_n|^2.$$

Here we put

$$a_n = \int_{-a}^a \psi_1^{(n)}(x) \psi(x) dx, \quad (n = 0, 1, 2, \dots),$$

$$b_n = \int_{-a}^a \psi_2^{(n)}(x) \psi(x) dx, \quad (n = 1, 2, \dots).$$

If, in the initial state, the state of the natural statistical distribution of the total physical system  $\Omega$  is determined by the  $L^2$ -density  $\psi(x)$ , we have the following by virtue of the theory of Fourier series in  $L^2$ .

By the study until now, the considered physical system of micro-particles are moving periodically with constant velocity in the interval  $[-a, a]$ .

Therefore we have to consider that the  $L^2$ -density  $\psi(x)$  which determines the natural statistical state is always a periodic function which satisfies the periodic condition

$$\psi(-a) = \psi(a).$$

Then  $\psi(x)$  is represented as

$$\psi(x) = \sum_{n=0}^{\infty} a_n \psi_1^{(n)}(x) + \sum_{n=1}^{\infty} b_n \psi_2^{(n)}(x),$$

$$a_n = \int_{-a}^a \psi_1^{(n)}(x) \psi(x) dx, \quad (n = 0, 1, 2, \dots),$$

$$b_n = \int_{-a}^a \psi_2^{(n)}(x) \psi(x) dx, \quad (n = 1, 2, \dots).$$

Here we define the Schrödinger operator for the stationary state as to be

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}.$$

Then the energy expectation value  $\bar{E}$  is calculated as follows:

$$\bar{E} = J[\psi] = \int_{-a}^a \psi^* H \psi(x) dx$$

$$= \sum_{n=0}^{\infty} \frac{p^2}{2m} |a_n|^2 + \sum_{n=1}^{\infty} \frac{p^2}{2m} |b_n|^2.$$

In the above calculation, we use the orthonormality condition of the functions  $\psi_1^{(n)}(x)$  and  $\psi_2^{(n)}(x)$ .

Further, because the function  $\psi(x)$  satisfies the normalization condition

$$\int_{-a}^a |\psi(x)|^2 dx = 1,$$

we have the equality

$$\sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |b_n|^2 = 1.$$

Then we have the equality

$$\begin{aligned} \bar{E} &= \sum_{n=0}^{\infty} \frac{n^2 \pi^2 \hbar^2}{2ma^2} |a_n|^2 + \sum_{n=1}^{\infty} \frac{n^2 \pi^2 \hbar^2}{2ma^2} |b_n|^2 \\ &= \sum_{n=0}^{\infty} \frac{n^2 \pi^2 \hbar^2}{2ma^2} (|a_n|^2 + |b_n|^2) = \sum_{n=1}^{\infty} \frac{n^2 \hbar^2}{8ma^2} (|a_n|^2 + |b_n|^2). \end{aligned}$$

Therefore we have the following theorem.

**Theorem 3.7** *For the physical system considered here, we assume that the initial state is the stationary state. Then we assume that the initial distribution of the total physical system is determined by a  $L^2$ -density  $\psi(x)$ . Further we assume that  $\psi(x)$  satisfies the periodic condition*

$$\psi(-a) = \psi(a).$$

Then we have

$$\begin{aligned} \psi(x) &= \sum_{n=0}^{\infty} a_n \psi_1^{(n)}(x) + \sum_{n=1}^{\infty} b_n \psi_2^{(n)}(x), \\ a_n &= \int_{-a}^a \psi_1^{(n)}(x)^* \psi(x) dx, \quad (n = 0, 1, 2, \dots), \\ b_n &= \int_{-a}^a \psi_2^{(n)}(x)^* \psi(x) dx, \quad (n = 1, 2, \dots). \end{aligned}$$

Then we have the energy expectation value  $\bar{E}$  of the total physical system as follows:

$$\bar{E} = \sum_{n=1}^{\infty} \frac{n^2 \hbar^2}{8ma^2} (|a_n|^2 + |b_n|^2).$$

**Theorem 3.8** Let  $J[\psi]$  be the energy functional defined in Problem I. Assume that  $\{\psi_1^{(n)}(x)\}$  and  $\{\psi_2^{(n)}(x)\}$  are as same as in Theorem 3.3 and  $\{\mathcal{E}_n\}$  are as same as in Theorem 3.4.

Now we denote the closed subspace  $\mathcal{L}(\psi_j^{(m)}(x); j = 1, 2, m = n, n + 1, n + 2, \dots)$  of  $L^2 = L^2([-a, a])$  as  $\mathcal{H}_n$ . Then we have the following:

$$\begin{aligned} \min_{\psi \in \mathcal{H}_n, \|\psi\|=1} J[\psi] &= \mathcal{E}_n, \\ J[\psi_j^{(n)}] &= \mathcal{E}_n, \quad (j = 1, 2, ; n = 0, 1, 2, \dots), \\ 0 \leq \mathcal{E}_0 &< \mathcal{E}_1 < \mathcal{E}_2 < \dots < \mathcal{E}_n < \dots, \\ \lim_{n \rightarrow \infty} \mathcal{E}_n &= \infty. \end{aligned}$$

Here the eigenvalue  $\mathcal{E}_0 = 0$  has the multiplicity 1 and the eigenvalue  $\mathcal{E}_n$ , ( $n \geq 1$ ) has the multiplicity 2.

By virtue of this Theorem 3.8, this solutions of  $\{\psi_1^{(n)}(x)\}$  and  $\{\psi_2^{(n)}(x)\}$  of the eigenvalue problem in Theorem 3.1 are the complete solution of the variational problem of the energy functional  $J[\psi]$ .

Here we derive the time-evolving Schrödinger equation by the inverse process of the separation of variables.

At first, we put

$$\begin{aligned} \psi_1^{(n)}(x, t) &= \psi_1^{(n)}(x) \exp\left(-i \frac{\mathcal{E}_n}{\hbar} t\right), \quad (n = 0, 1, 2, \dots), \\ \psi_2^{(n)}(x, t) &= \psi_2^{(n)}(x) \exp\left(-i \frac{\mathcal{E}_n}{\hbar} t\right), \quad (n = 1, 2, \dots). \end{aligned}$$

By differentiating partially these functions with respect to the time variable  $t$ , we have the equalities

$$\begin{aligned} i\hbar \frac{\partial \psi_1^{(n)}(x, t)}{\partial t} &= \mathcal{E}_n \psi_1^{(n)}(x) \exp\left(-i \frac{\mathcal{E}_n}{\hbar} t\right), \quad (n = 0, 1, 2, \dots), \\ i\hbar \frac{\partial \psi_2^{(n)}(x, t)}{\partial t} &= \mathcal{E}_n \psi_2^{(n)}(x) \exp\left(-i \frac{\mathcal{E}_n}{\hbar} t\right), \quad (n = 1, 2, \dots). \end{aligned}$$

Then we have the equalities

$$\begin{aligned} H\psi_1^{(n)}(x) &= \mathcal{E}_n \psi_1^{(n)}(x), \quad (n = 0, 1, 2, \dots), \\ H\psi_2^{(n)}(x) &= \mathcal{E}_n \psi_2^{(n)}(x), \quad (n = 1, 2, \dots). \end{aligned}$$

Here we have the equalities

$$i\hbar \frac{\partial \psi_1^{(n)}(x, t)}{\partial t} = \{H\psi_1^{(n)}(x)\} \exp\left(-i \frac{\mathcal{E}_n}{\hbar} t\right) = H\psi_1^{(n)}(x, t), \quad (n = 0, 1, 2, \dots),$$

$$i\hbar \frac{\partial \psi_2^{(n)}(x, t)}{\partial t} = \{H\psi_2^{(n)}(x)\} \exp\left(-i\frac{\mathcal{E}_n}{\hbar}t\right) = H\psi_2^{(n)}(x, t), \quad (n = 1, 2, \dots).$$

Therefore, if we put

$$\psi(x, t) = \sum_{n=0}^{\infty} a_n \psi_1^{(n)}(x, t) + \sum_{n=1}^{\infty} b_n \psi_2^{(n)}(x, t),$$

we have

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2}.$$

This is the time-evolving Schrödinger equation of the total physical system. Here we have the following theorem.

**Theorem 3.9** *Assume that the function  $\psi(x)$  and  $\psi(x, t)$  are given as in the above. Then  $\psi(x, t)$  is the solution of the initial and boundary value problem of the time-evolving Schrödinger equation*

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2}, \quad (|x| \leq a, 0 < t < \infty),$$

$$\psi(x, 0) = \psi(x), \quad (|x| \leq a), \quad (\text{Initial condition}),$$

$$\psi(-a) = \psi(a), \quad \psi(-a, t) = \psi(a, t), \quad (0 < t < \infty), \quad (\text{Boundary condition}).$$

## 4 Motion of micro-particles under the action of potential well of infinite depth

Assume a micro-particle moves under the action of potential  $V(x)$  in  $\mathbf{R}$ .

The action of this force are given approximately by the potential well  $V(x)$  of infinite depth:

$$V(x) = \begin{cases} 0, & (|x| \leq a), \\ \infty, & (|x| > a). \end{cases}$$

Then each micro-particle is reflected completely to the positive direction at  $x = -a$  and it is reflected completely to the negative direction at  $x = a$ .

Therefore each particle changes the velocity to the opposite direction only at two points  $x = -a$  and  $x = a$ , and moves with constant velocity between the two points  $x = -a$  and  $x = a$ .

Here it is considered that each particle moves periodically with constant velocity in the interval  $[-a, a]$ .

The time-evolving of the natural statistical state of the total physical system is described by the solution  $\psi(x, t)$  of the Schrödinger equation in Theorem 3.8.

If the initial distribution  $\psi(x)$  is given as the natural probability distribution localized in the interval  $|x| \leq a$ , the natural probability distribution varying in the future by virtue of the motion of micro-particles is determined by  $\psi(x, t)$ .

Even if each one of micro-particles moves with constant velocity, the natural statistical phenomena appear for the state of the natural probability distribution of position variable and momentum variable of micro-particles when the initial position and the initial velocity of each micro-particle are given in the various manner. The natural statistical variation is reflected on the natural probability distribution defined by  $\psi(x, t)$ .

Thus it is known that the motion of each micro-particle is controlled by the potential well of infinite depth.

Then the physical system  $\Omega$  considered in the stationary state  $\Omega$  is divided into the direct sum

$$\Omega = \sum_{n=0}^{\infty} \Omega_1^{(n)} + \sum_{n=1}^{\infty} \Omega_2^{(n)}.$$

Then, for every event  $A \in \mathcal{B}$ , we have the equality

$$P(A) = \sum_{n=0}^{\infty} P(\Omega_1^{(n)})P_{\Omega_1^{(n)}}(A) + \sum_{n=1}^{\infty} P(\Omega_2^{(n)})P_{\Omega_2^{(n)}}(A).$$

Here  $P_{\Omega_j^{(n)}}(A)$ , ( $j = 1, 2$ ) denotes the conditional probability.

Then, for the cases (1)  $j = 1$ ,  $n = 0, 1, 2, \dots$ , and (2)  $j = 2$ ,  $n = 1, 2, \dots$ , we say that the probability space  $(\Omega_j^{(n)}, \mathcal{B} \cap \Omega_j^{(n)}, P_{\Omega_j^{(n)}})$  is  $(j, n)$ -th **proper physical system**.

Then, as the results from the above consideration, we have the following:

(1) For  $j = 1$ ,  $n = 0, 1, 2, \dots$ , we have

$$P(\Omega_1^{(n)}) = |a_n|^2.$$

(2) For  $j = 2$ ,  $n = 1, 2, \dots$ , we have

$$P(\Omega_2^{(n)}) = |b_n|^2.$$

Here we have

$$\sum_{n=0}^{\infty} P(\Omega_1^{(n)}) + \sum_{n=1}^{\infty} P(\Omega_2^{(n)}) = \sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} |b_n|^2 = 1.$$

Then, for a Lebesgue measurable set  $A$  in  $[-a, a]$  and a subset  $B$  of

$$P_1 = \left\{ p = \frac{n\pi\hbar}{a}; n = 0, 1, 2, \dots \right\},$$

we have the following:

(1) For  $j = 1$ ,  $n = 0, 1, 2, \dots$ , we have

$$P_{\Omega_1^{(n)}} (\{ \rho \in \Omega_1^{(n)}; x(\rho) \in A \}) = \int_A |\psi_1^{(n)}(x)|^2 dx,$$

$$P_{\Omega_1^{(n)}} (\{ \rho \in \Omega_1^{(n)}; p(\rho) \in B \}) = \sum_{p \in B} |\hat{\psi}_1^{(n)}(p)|^2.$$

(2) For  $j = 2$ ,  $n = 1, 2, \dots$ , we have

$$P_{\Omega_2^{(n)}} (\{ \rho \in \Omega_2^{(n)}; x(\rho) \in A \}) = \int_A |\psi_2^{(n)}(x)|^2 dx,$$

$$P_{\Omega_2^{(n)}} (\{ \rho \in \Omega_2^{(n)}; p(\rho) \in B \}) = \sum_{p \in B} |\hat{\psi}_2^{(n)}(p)|^2.$$

Therefore the energy expectation of the proper physical system  $\Omega_j^{(n)}$  is equal to

$$E_{\Omega_j^{(n)}} \left[ \frac{1}{2m} p(\rho)^2 \right] = \int_{-a}^a \frac{\hbar^2}{2m} \left| \frac{d\psi_j^{(n)}(x)}{dx} \right|^2 dx$$

$$= J[\psi_j^{(n)}] = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad (j = 1, 2; n = 0, 1, 2, \dots).$$

Then, by virtue of the relation between the total physical system and the proper physical systems, the energy expectation  $\bar{E}$  of the total physical system is equal to

$$\bar{E} = E \left[ \frac{1}{2m} P(\rho)^2 \right] = \sum_{j=1}^2 \sum_{n=0}^{\infty} P(\Omega_j^{(n)}) E_{\Omega_j^{(n)}} \left[ \frac{1}{2m} P(\rho)^2 \right]$$

$$= \sum_{n=0}^{\infty} \frac{n^2 \pi^2 \hbar^2}{2ma^2} (|a_n|^2 + |b_n|^2).$$

Thus we know that, on the stationary state, the considered physical system is realized as the composed state of the proper physical systems. The ratio of this composition is determined by the sequence

$$\{|a_n|^2\}_{n=0}^{\infty} \cup \{|b_n|^2\}_{n=1}^{\infty}.$$

This sequence is determined by the method of setting the initial state. As the result, it is known that, for  $j = 1, 2$ ,  $(j, n)$ -th eigenvalue  $\mathcal{E}_n$  is equal to the energy expectation of  $(j, n)$ -th proper physical system.

By observing the energy expectation or the spectrum of the physical system under a certain initial condition, we can test this theory by comparing the theoretical energy expectation  $\bar{E}$  and the observed data.

## 5 Unsolved problems

In this section, we give the two unsolved problems for the natural statistical phenomena of potential well of infinite depth in the following (1) and (2):

- (1) By virtue of the numerical experiment, study the various phases of the phenomena of potential well of infinite depth concretely.
- (2) By virtue of the physical experiment, carry out the demonstrative experiment and the application of the phenomena of potential well of infinite depth.

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