Development of L^p-calculus

By

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Abstract

In this paper, we define the derivative or the partial derivative of a L^p-function in the sense of L^p-convergence. We also define the derivative and the partial derivative of a $L^p_{\rm loc}$-function in the sense of $L^p_{\rm loc}$-convergence. Then we study their fundamental properties. Here assume that $1 \leq p \leq \infty$ holds.

We say that the branch of analysis on the bases of the concepts of L^p-convergence and $L^p_{\rm loc}$-convergence is the L^p-calculus.

As the results, we have the following conclusions for the differential calculus of classical functions.

Assume that $1 \leq p \leq \infty$. Then we have the inclusion relations $L^p \subset L^p_{\rm loc} \subset L^1_{\rm loc}$. In the L^p-calculus, the derivative or the partial derivatives of a L^p-function are the derivative or the partial derivatives of the function calculated in the sense of $L^1_{\rm loc}$-topology which are the L^p-functions for each p, $(1 < p \leq \infty)$ respectively.

For $L^p_{\rm loc}$-functions, we have the similar results.

Especially, the L^1-derivative or the partial L^1-derivatives of a L^1-function are the $L^1_{\rm loc}$-derivative or the partial $L^1_{\rm loc}$-derivatives in the above sense, respectively. But the inverse facts are not necessarily true.

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Introduction

In this paper, we define the derivative and the partial derivatives of a L^p-function in the sense of L^p-convergence. We also define the derivative and the partial derivatives of a L^p_{loc}-function in the sense of L^p_{loc}-convergence. Then we study their fundamental properties. Here we assume $1 \leq p \leq \infty$. For the calculation of such derivatives and partial derivatives, we need not use the concept of derivatives in the sense of distribution.

We say that the branch of analysis on the bases of the concepts of L^p-convergence and L^p_{loc}-convergence is the L^p-calculus.

In general, for L^1-functions, the differentiable functions in the sense of distributions exist more than the L^1-differentiable functions. Nevertheless, we study the L^p-differentiable functions principally in the L^p-calculus for $1 \leq p \leq \infty$.

It affects only the case of L^1-functions. Nevertheless, it is possible to study only the L^1-differentiable functions in the case of L^1-functions.

Further, because we have the inclusion relation $L^1 \subset L^1_{\mathrm{loc}}$, the L^1-derivative or the partial L^1-derivatives of a L^1-function in the sense of distribution are the L^1_{loc}-derivative or the partial L^1_{loc}-derivatives respectively which are the L^1-functions.

Also, in the cases of L^2-functions and L^2_{loc}-functions, these results carry out the fundamental roles for the study of solutions of Schrödinger equations.

Especially, we need really the concept of distributions when we study the distribution solutions of differential equations.

It is enough to use the L^p-calculus for studing the L^p-function solutions or the L^p_{loc}-function solutions of differential equations.

Until now, we study the weak derivative and the weak partial derivatives of a L^p-function or a L^p_{loc}-function by using their derivatives or their partial derivatives in the distributional sense. In this paper, we define their weak derivatives or their weak partial derivatives in the sense of the weak topology of L^p or in the sense of the weak topology of L^p_{loc}.

In this paper, we distinguish these weak derivatives or these weak partial derivatives and those in the sense of distribution. Further, under the certain condition, we prove the coincidence of three types of the derivatives or the partial derivatives of L^p-functions for the three types of calculations with respect to the strong topology of L^p, the weak topology of L^p or the topology in the sense of distribution. For L^p_{loc}-functions, we have the similar results. Therefore, for the derivation of L^p-functions or L^p_{loc}-functions, we only use the L^p-topology or the L^p_{loc}-topology respectively. Thus, in the study of analysis of classical functions, we need not use the theory of distributions.

As the results of this paper, we have the following conclusion in the derivation of classical functions.

We have the relations $L^p \subset L^p_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$ for $1 \leq p \leq \infty$. Thus, if we have $L^p \neq L^1$ in the L^p-calculus, the derivative or the partial derivatives of a L^p-

function are the derivative or the partial derivatives of this function calculated in the L^1_{loc}-topology which become the L^p-functions for each p, $(1 \leq p \leq \infty)$ respectively. For L^p_{loc}-functions, we have the similar results.

Especially, in the case of L^1-functions, the L^1-derivatives and the partial L^1-derivatives in the above sense are the L^1_{loc}-derivatives or the partial L^1_{loc}-derivatives respectively. Nevertheless we remark that the inverse does not neccessarily hold.

By virtue of the necessity for the study of Schrödinger equations, we assume that the functions considered in the sequel are the complex-valued functions of real variables.

In the study of mathematics, the problem is seen clear if we consider the problem by setting the theoretical framework of the considered problem.

When we meet the mathematical phenomena which do not fit the situation of the theoretical framework, we might consider the new theoretical establishment of the theoretical foundation of those mathematical phenomena. Those cases are found in many times in the history of mathematics.

1 Function spaces L^p

In this section, we assume that $1 \leq p \leq \infty$ and $d \geq 1$. Further we assume that $\boldsymbol{R}^d$ is the d-dimensional Euclidean space.

Let E be a Lebesgue measurable set in $\boldsymbol{R}^d$. Let $(E,\ \mathcal{M}_E,\ \mu)$ be the Lebesgue measure space.

Then we define the function space $L^p = L^p(E)$ in the following.

We define $L^p = L^p(E)$ to be the set of all complex-valued measurable functions $f(x)$ on E which satisfy the condition

$$\int_E |f(x)|^p dx < \infty.$$

We denote $L^p(\boldsymbol{R}^d)$ as L^p for simplification.

For $1 \leq p < \infty$, we define the norm of $f \in L^p(E)$ by the relation

$$\|f\|_p = \Big\{ \int_E |f(x)|^p dx \Big\}^{1/p}.$$

We call this the L^p-norm of f.

We denote the L^p-norm as $\|f\|$ for the simplification of $\|f\|_p$.

We define the norm of $L^\infty = L^\infty(E)$ by the relation

$$\|f\|_\infty = \operatorname*{ess.sup}_{x \in E} |f(x)| = \inf\{\alpha;\ |f(x)| \leq \alpha,\ (\text{a.e.}\, x \in E)\}.$$

For $f,\ g \in L^2(E)$, we define

$$(f,\ g) = \int_E f(x)\overline{g(x)}dx$$

and we say that $(f,\ g)$ is the inner product of f and g.

Especially, we have

$$\|f\|_2 = \sqrt{(f,\ f)}$$

for the L^2-norm. Then we have the following theorem.

Theorem 1.1 *For $1 \le p \le \infty$, $L^p(E)$ is a Banach space. Especially, $L^2(E)$ is a Hilbert space.*

Theorem 1.2 *For $1 \le p \le \infty$, if, for $f_n \in L^p(E)$, $(n = 0,\ 1,\ 2,\ \cdots)$, we have*

$$\lim_{n\to\infty} \|f_n - f_0\| = 0,$$

there exists a certain subsequence $\{f_{n(k)};\ 1,\ 2,\ \cdots\}$ of $\{f_n\}$ such that we have

$$\lim_{k\to\infty} f_{n(k)}(x) = f_0(x),\ (\text{a.e. } x \in E).$$

Theorem 1.3 *We assume that the conditions*

$$1 < p,\ q < \infty, \frac{1}{p} + \frac{1}{q} = 1$$

hold. Assume that E is a Lebesgue measurable set in $\boldsymbol{R}^d$. We put $L^p = L^p(E)$. Then we have the following isomorphisms

$$L^p \cong (L^q)' \cong (L^p)''.$$

Especially, we have the isomorphism

$$L^\infty \cong (L^1)'.$$

Theorem 1.4 *We assume that $d \ge 1$ and $1 \le p < \infty$ hold. We put $L^p = L^p(\boldsymbol{R}^d)$. We define that $\mathcal{D} = \mathcal{D}(\boldsymbol{R}^d)$ is the* TVS *of all C^∞-functions with compact support in $\boldsymbol{R}^d$. Then, $\mathcal{D}$ is dense in L^p.*

Theorem 1.5 *We assume that $d \ge 1$ and $1 < p \le \infty$ hold. We put $L^p = L^p(\boldsymbol{R}^d)$. We consider a sequence of functions $\{f_n\}$ in L^p and a function $f \in L^p$. Then the following three conditions* (1), (2) *and* (3) *are equivalent:*

(1) *We have $f_n \to f$ in the norm of L^p.*

(2) *We have* $f_n \to f$ *in the weak topology of* L^p.

(3) *We have* $f_n \to f$ *in the topology of* L^p *which is the induced topology of* $\mathcal{D}'$.

Theorem 1.6 *We assume that* $d \geq 1$ *and* $1 < p \leq \infty$ *hold. Then, for a sequence* $\{f_n\}$ *of* L^p*-functions, the following* (1) *and* (2) *are equivalent*:

(1) *The sequence of functions* $\{f_n\}$ *converges with respect to the strong topology of* L^p.

(2) *There exists* $f \in L^p$ *such that we have* $f_n \to f$ *with respect to the topology of* L^1_{loc}.

2 Function spaces L^p_{loc} and L^p_c

In this section, we study the function spaces L^p_{loc} and L^p_c. Here, we assume $1 \leq p \leq \infty$.

Assume that we have $d \geq 1$ and $\boldsymbol{R}^d$ is the d-dimensional Euclidean space.

For $1 \leq p < \infty$, we define that a complex-valued measurable function f is a locally p-th integrable if it satisfies the condition

$$\int_K |f(x)|^p dx < \infty$$

for an arbitrary compact set K in R^d.

Let $L^p_{\text{loc}} = L^p_{\text{loc}}(\boldsymbol{R}^d)$ be the complex TVS of all locally p-th integrable functions.

For $1 \leq p < \infty,\ f \in L^p_{\text{loc}}$ if and only if the condition

$$\int_{|x| \leq R} |f(x)|^p dx < \infty$$

is satisfied for any $R > 0$.

Especially, we say that an element of L^1_{loc} is a locally integrable function.

For $1 \leq p < \infty$, we define that a sequence of functions $\{f_n\}$ of L^p_{loc} converges to $f \in L^p_{\text{loc}}$ if we have the condition

$$\int_K |f_n(x) - f(x)|^p dx \to 0,\ (n \to \infty)$$

for an arbitary compact set K in $\boldsymbol{R}^d$. Namely, the topology of L^p_{loc} is the topology of L^p-convergence on each compact set of $\boldsymbol{R}^d$. Thereby L^p_{loc} becomes a TVS.

Especially, $L^\infty_{\rm loc} = L^\infty_{\rm loc}(\boldsymbol{R}^d)$ is a TVS of all complex-valued measurable functions which satisfy the condition

$$\|f\|_{\infty,\,K} = \operatorname*{ess.\,sup}_{x\in K} |f(x)|$$

$$= \inf \left\{ \ \alpha;\ |f(x)| \le \alpha,\ (\text{a.e.}\, x \in K)\ \right\} < \infty$$

for an arbitrary compact set K in $\boldsymbol{R}^d$. We define the semi-norm $\|\cdot\|_{\infty,\,K}$ by the relation

$$\|f\|_{\infty,\,K} = \operatorname*{ess.sup}_{x\in K} |f(x)|.$$

We define the topology of $L^\infty_{\rm loc}$ by using the system of semi-norms

$$\left\{ \ \|\cdot\|_{\infty,\,K};\ K \text{ is a compact set in } \boldsymbol{R}^d \ \right\}.$$

We define that a sequence of functions $\{f_n\}$ of $L^\infty_{\rm loc}$ converges to $f \in L^\infty_{\rm loc}$ if we have the condition

$$\|f_n - f\|_{\infty,\,K} \to 0,\ (n \to \infty)$$

for an arbitrary compact set K in $\boldsymbol{R}^d$. Namely, this topology of $L^\infty_{\rm loc}$ is the topology of L^∞-convergence on each compact set. Thereby, $L^\infty_{\rm loc}$ becomes a TVS.

Then, for $1 \le p \le \infty$, we have the inclusion relation

$$\left(\bigcup_{m=0}^{\infty} C^m \right) \cup L^p \subset L^p_{\rm loc} \subset L^1_{\rm loc}.$$

Here, for $0 \le m \le \infty$, C^m is the TVS of all C^m-functions on $\boldsymbol{R}^d$.

For $1 \le p \le \infty$, L^p_c denotes the TVS of all L^p-functions on $\boldsymbol{R}^d$ with compact support.

Then we have the following theorem.

Theorem 2.1 *Assume* $1 \le p \le \infty$. *Assume that a sequence of compact sets* $\{K_j\}$ *of* $\boldsymbol{R}^d$ *satisfies the following conditions* (i) *and* (ii) :

(i) $K \subset K_2 \subset \cdots \subset \boldsymbol{R}^d,\ \boldsymbol{R}^d = \bigcup_{j=1}^{\infty} K_j.$

(ii) $K_j = \text{cl}(\text{int}(K_j)),\ K_j \subset \text{int}(K_{j+1}),\ (j \ge 1).$

Then we have the following ismorphisms (1) *and* (2):

(1) $L^p_{\rm loc} \cong \varprojlim L^p(K_j).$

(2) $L^p_c \cong \varinjlim L^p(K_j).$

Then L^p_{loc} is a FS*-space and L^p_c is a DFS*-space. Thus L^p_{loc} and L^p_c are reflexive.

Therefore we have the following theorem.

Theorem 2.2 *Assume that two real numbers p and q satisfy the conditions*

$$1 \leq p,\ q \leq \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Then we have the following isomorphisms (1) *and* (2):

(1) $L^p_{\mathrm{loc}} \cong (L^q_c)' \cong (L^p_{\mathrm{loc}})''$.

(2) $L^q_c \cong (L^p_{\mathrm{loc}})' \cong (L^q_c)''$.

Theorem 2.3 *For $1 \leq p \leq \infty$, the function space $\mathcal{D}$ is dense in L^p_c.*

Theorem 2.4 *Assume $1 \leq p \leq \infty$. For a sequence of functions $\{f_n\}$ of L^p_{loc} and a function f* of L^p_{loc}, *the following* (1) $\sim$ (3) *are equivalent*:

(1) *We have $f_n \to f$ with respect to the strong topology of L^p_{loc}.*

(2) *We have $f_n \to f$ with respect to the weak topology of L^p_{loc}.*

(3) *We have $f_n \to f$ with respect to the topology of L^p_{loc} induced from the topology of $\mathcal{D}'$.*

Theorem 2.5 *Assume that $1 \leq p \leq \infty$ and we have $\{f_n\} \subset L^p_{\mathrm{loc}}$. Then the following* (1) *and* (2) *are equivalent*:

(1) *The sequence of functions $\{f_n\}$ converges with respect to the topology of L^p_{loc}.*

(2) *There exists $f \in L^p_{\mathrm{loc}}$ such that we have $f_n \to f$ with respect to the topology of L^1_{loc}.*

3 Differential calculus of L^p-functions

3.1 L^p-differentiability

In this section, we study the concept of L^p-differentiability.

We define that the function space $L^p = L^p(-\infty, \infty)$ is the space of all p-th integrable functions on the open interval $(-\infty, \infty)$. Here we assume $1 \leq p \leq \infty$.

Then we define the concept of L^p-differentiability. Namely we define the concept of differential calculus of L^p-functions in the sense of convergence of L^p-norm.

Then we give the following definition 3.1.

Definition 3.1 (L^p-differentiability) Assume $1 \leq p \leq \infty$. Assume that a function $y = f(x)$ is a L^p-function defined on the open interval $(-\infty,\ \infty)$. Then we denote the increment Δy of the function $y = f(x)$ corresponding to the increment Δx of the independent variable x as follows:

$$\Delta y = f(x + \Delta x) - f(x) = A(x)\Delta x + \varepsilon(x,\ \Delta x)\Delta x.$$

Here $A(x)$ is a function of x which does not depend on Δx. $\varepsilon = \varepsilon(x,\ \Delta x)$ is a function of x and Δx.

Then we define that the function $y = f(x)$ is differentiable in the sense of L^p-convergence on the open interval $(-\infty,\ \infty)$ if we have the condition $\varepsilon(x,\ \Delta x) \to 0$ in the sense of L^p-convergence on the open interval $(-\infty,\ \infty)$ when $\Delta x \to 0$.

Namely this is equivalent to the condition

$$\lim_{\Delta x \to 0} \|\varepsilon(x,\ \Delta x)\|_p = 0.$$

Then we extend the definition as $\varepsilon(x,\ 0) = 0,\ (x \in (-\infty,\ \infty))$.

Here, if a function is differentiable in the sense of L^p-convergence, we say that it is L^p-differentiable for simplification.

Now, we denote the function space of all p-th integrable functions on a general open interval $(a,\ b)$ as $L^p = L^p(a,\ b)$.

Then, if put

$$\widetilde{f}(x) = \begin{cases} f(x), & (x \in (a,\ b), \\ 0, & (x \notin (a,\ b) \end{cases}$$

for an arbitrary $f \in L^p(a,\ b)$, we have $\widetilde{f}(x) \in L^p(-\infty,\ \infty)$.

Then the correspondence of $f(x) \in L^p(a,\ b)$ to $\widetilde{f}(x) \in L^p(-\infty,\ \infty)$ is one to one. Thus we may consider that $L^p(a,\ b)$ is a subspace of $L^p(-\infty,\ \infty)$.

Therefore we say that $f \in L^p(a,\ b)$ is L^p-differentiable if it is L^p-differentiable as the function in $L^p(-\infty,\ \infty)$.

Now we assume that a function $y = f(x)$ is L^p-differentiable on the open interval $(a,\ b)$.

Then, by virtue of the condition of definition 3.1, we have the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

in the sense of L^p-convergence. We define that this limit $f'(x)$ is a L^p-derivative of $y = f(x)$.

By virtue of the completeness of L^p, $f'(x)$ is an element of $L^p(a,\ b)$. By the property of L^p-convergence, $f'(x)$ has the determined complex values almost everywhere on $(a,\ b)$.

3.2 Fundamental properties of L^p-derivatives

We put $L^p = L^p(\boldsymbol{R})$. Then we define the concept of weak derivatives of L^p-functions.

Definition 3.2 Assume $f(x) \in L^p$ for $1 < p \leq \infty$. We use the same notation as in definition 3.1. Then we define that a function $y = f(x)$ is differentiable in the sense of the weak convergence of L^p if we have the condition $\varepsilon(x,\ \Delta x) \to 0$ in the sense of the weak topology of L^p on $\boldsymbol{R}$ when $\Delta x \to 0$.

Namely this is equivalent to the condition

$$\lim_{\Delta x \to 0} \big(\ \varepsilon(x,\ \Delta x\ \big),\ \varphi) = 0$$

for $\varphi \in L^q$. Here we have the relations

$$1 \leq q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Then we extend the definition as $\varepsilon(x,\ 0) = 0,\ (x \in \boldsymbol{R})$.

Here, if a function is differentiable in the sense of the weak topology of L^p, we say that it is weakly L^p-differentiable for simplification.

Then, by virtue of the condition of definition 3.2, we have the weak limit

$$w\text{-}\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = w\text{-}\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = w\text{-}f'(x)$$

in the sense of the weak topology of L^p. We define that this weak limit w-$f'(x)$ is a weak L^p-derivative of $f(x)$.

By virtue of the weak completeness of L^p, w-$f'(x)$ is an element of L^p.

Then we have the following theorem.

Theorem 3.1 *Assume that $1 < p \leq \infty$ and $f(x) \in L^p$ hold. If $f(x)$ is L^p-differentiable, $f(x)$ is weakly differentiable and its derivative $f'(x)$ in the sense of L^p-convergence coincides with the weak derivative w-$f'(x)$. Namely we have the equality*

$$f'(x) = w\text{-}f'(x)$$

or the equality

$$(f',\ \varphi) = (w\text{-}f',\ \varphi),\ (\varphi \in L^q).$$

Here we assume the relations

$$1 < p \leq \infty,\ 1 \leq q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Then the weak derivative w-$f'(x)$ of $f(x) \in L^p$ is a L^p-function.

Theorem 3.2 *Assume that $1 < p \leq \infty$ and $f(x) \in L^p$ hold. If we have the weak derivative w-$f'(x)$ of $f(x)$ and w-$f'(x) \in L^p$, $f(x)$ is L^p-differentiable and we have w-$f'(x) = f'(x)$ for the derivative $f'(x)$ of $f(x)$ in the sense of L^p-convergence.*

Theorem 3.3 *Assume $1 \leq p \leq \infty$. If, for a sequence of functions $f_n(x) \in L^p,\ (n = 1,\ 2,\ 3,\ \cdots)$, there exist $f,\ g \in L^p$ such that we have*

$$f_n \to f,\ (n \to \infty),\ f_n' \to g,\ (n \to \infty),$$

we have $f' \in L^p$ such that we have the equality

$$f' = g.$$

Namely, the differential operator $\dfrac{d}{dx}$ is a closed linear operator.

By virtue of theorem 1.5, for $1 < p \leq \infty$, the L^p-differentiability, the weak L^p-differentiability and the differentiability in the sense of distributions coincide.

Further, for L^p-functions, the L^p-derivative, the weak L^p-derivative and the derivative in the sense of distribution coincide.

3.3 L^p-differentiability

Let $L^p = L^p(\boldsymbol{R}^d)$ be the function space of all p-th integrable functions on $\boldsymbol{R}^d$. Here we assume that $d \geq 2$ and $1 \leq p \leq \infty$ hold.

Then we define the concept of L^p-differentiability. Namely we study the concept of differential calculus of L^p-functions in the sense of L^p-convergence.

Then we give the following definition 3.3.

Definition 3.3(L^p-differentiability) Assume $1 \leq p \leq \infty$. We assume that a functions $f(x)$ is a L^p-function defined on $\boldsymbol{R}^d$. Then we denote the increment Δy of a function $y = f(x)$ corresponding to the increment Δx of the independent variables x as

$$\Delta y = f(x + \Delta x) - f(x) = \sum_{i=1}^{d} A_i(x)\Delta x_i + \varepsilon(x,\ \Delta x)\rho.$$

Here $\rho = \|\Delta x\|$ and $A_i(x),\ (i = 1,\ 2,\ \cdots,\ d)$ are the functions of x which do not depend on Δx. $\varepsilon(x,\ \Delta x)$ is the function of x and Δx.

Then we define that the function $y = f(x)$ is differentiable in the sense of L^p-convergence on $\boldsymbol{R}^d$ if we have the condition

$$\varepsilon(x,\ \Delta x) \to 0,\ (\Delta x \to 0)$$

in the sense of L^p-convergence on $\boldsymbol{R}^d$.

Namely this is equivalent to the condition

$$\lim_{\Delta x \to 0} \|\varepsilon(x,\ \Delta x)\|_p = 0.$$

Then we extend the definition as $\varepsilon(x,\ 0) = 0,\ (x \in \boldsymbol{R}^d)$.

Here we say that a function is L^p-differentiable for simplification if it is differentiable in the sense of L^p-convergence.

Now we denote the function space of all p-th integrable functions in a generel domain D in $\boldsymbol{R}^d$ as $L^p = L^p(D)$. In a similar way as in the case of functions of one variable, we may consider that $L^p(D)$ is a subspace of $L^p(\boldsymbol{R}^d)$.

Therefore we define that a function in $L^p(D)$ is L^p-differentiable if it is L^p-differentiable considering that the function f belongs to $L^p(\boldsymbol{R}^d)$.

3.4 Fundamental properties of partial L^p-derivatives

Assume that $d \geq 2$ and $1 \leq p \leq \infty$ hold.

Now, if $f(x) \in L^p = L^p(\boldsymbol{R}^d)$ is L^p-differentiable, we have

$$\frac{\partial y}{\partial x_j} = \lim_{h \to 0} \frac{(\tau_{-he_j} f)(x) - f(x)}{h}$$

for $1 \leq j \leq d$ in the sense of L^p-topology. Here, let $\{e_1,\ e_2,\ \cdots e_d\}$ be the standard basis of $l^2(d)$ and $\tau_y,\ (y \in \boldsymbol{R}^d)$ be the translation operator.

Then the partial derivatives $\dfrac{\partial y}{\partial x_j},\ (1 \leq j \leq d)$ are L^p-functions.

We say that they are the partial L^p-derivatives. Then $\dfrac{\partial y}{\partial x_j},\ (1 \leq j \leq d)$ have the determined complex values almost everywhere.

If there exist the partial L^p-derivatives of a L^p-function $y = f(x)$, we say that $y = f(x)$ is partially L^p-differentiable.

Therefore, if $f(x) \in L^p$ is L^p-differentiable, we may consider that its partial derivatives in the sense of L^p-convergence are the weak partial L^p-derivatives. Nevertheless, it is hard to prove the inverse statement.

Here we give the definition of weak partial derivatives in the following.

Definition 3.4 Assume $f(x) \in L^p$ for $1 < p \leq \infty$. We use the same notation as in definition 3.3. Then we define that a function $y = f(x)$ is differentiable in the sense of the weak convergence of L^p if we have the condition

$$\varepsilon(x,\ \Delta x) \to 0,\ (\Delta x \to 0)$$

in the sense of the weak topology of L^p on $\boldsymbol{R}^d$. Namely this is equivalent to the condition

$$\lim_{\Delta x \to 0} \big(\ \varepsilon(x,\ \Delta x),\ \varphi\ \big) = 0$$

for $\varphi \in L^q$. Here we assume the relations

$$1 \leq q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Then we extend the definition as $\varepsilon(x,\ \Delta x) = 0,\ (x \in \boldsymbol{R}^d)$.

Here, if a function is differentiable in the sense of the weak topology of L^p, we say that it is weakly L^p-differentiable for smiplification.

Then, by virtue of the condition of definition 3.4, we have the weak limit

$$w\text{-}\frac{\partial y}{\partial x_j} = w\text{-}\lim_{h \to 0} \frac{(\tau_{-he_j} f)(x) - f(x)}{h}$$

in the sense of the weak topology of L^p. We define that this weak limit $w\text{-}\dfrac{\partial y}{\partial x_j}$ is a weak partial L^p-derivative for $1 \leq j \leq d$. By virtue of the weak completeness of L^p, $w\text{-}\dfrac{\partial y}{\partial x_j}$, $(1 \leq j \leq d)$ are the elements of L^p.

Theorem 3.4 *We assume that* $1 < p \leq \infty,\ 1 \leq j \leq d$ *and* $f(x) \in L^p$ *hold. If* $f(x)$ *is partially* L^p*-differentiable, then* $f(x)$ *is weakly partial differentiable and its partial derivative* $\dfrac{\partial f}{\partial x_j}$ *in the sense of* L^p*-convergence coincides with the weak partial* L^p*-derivative* $w\text{-}\dfrac{\partial f}{\partial x_j}$*. Namely, we have the equalities*

$$\frac{\partial f}{\partial x_j} = w\text{-}\frac{\partial f}{\partial x_j},\ (1 \leq j \leq d).$$

Namely we have the equalities

$$\big(\ \frac{\partial f}{\partial x_j},\ \varphi\ \big) = \big(\ w\text{-}\frac{\partial f}{\partial x_j},\ \varphi\ \big),\ (\varphi \in L^q,\ 1 \leq j \leq d).$$

Here we assume the relations

$$1 \leq q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 3.5 *Assume that* $1 < p \leq \infty$, $1 \leq j \leq d$ *and* $f(x) \in L^p$ *hold. Then, if we have the weak partial* L^p*-derivative* $w\text{-}\dfrac{\partial f}{\partial x_j}$ *of* $f(x)$ *and we have* $w\text{-}\dfrac{\partial f}{\partial x_j} \in L^p$, *then* $f(x)$ *is partially* L^p*-differentiable, and we have the equality*

$$w\text{-}\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_j},\ (1 \leq j \leq d)$$

for the partial L^p*-derivative* $\dfrac{\partial f}{\partial x_j}$ *of* $f(x)$ *in the sense of* L^p*-convergence.*

Next we prove the commutativity of the order of partial differentiation.

Theorem 3.6 *Assume* $1 \leq p \leq \infty$ *and* $f(x) \in L^p$ *hold.*

If, for $1 \leq i,\ j \leq d,\ (i \neq j)$, *we have* $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ *and* $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ *in the sense of* L^p*-convergence, we have the equality*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Theorem 3.7 *Assume* $1 \leq p \leq \infty$. *If, for a sequence of functions* $f_n(x),\ (n = 1,\ 2,\ 3,\ \cdots)$, *we have* $f,\ g \in L^p$ *such that we have*

$$f_n \to f,\ (n \to \infty),\ \frac{\partial f}{\partial x_j} \to g,\ (n \to \infty),$$

we have $\dfrac{\partial f}{\partial x_j} \in L^p$ *such that we have*

$$\frac{\partial f}{\partial x_j} = g.$$

Here we assume $1 \leq j \leq d$. *Namely the partial differential operator* $\dfrac{\partial}{\partial x_j}$ *is a closed linear operator.*

By virtue of theorem 1.5, the partial L^p-differentiability, the weak partial L^p-differentiability and the partial differentiability in the sense of distribution coincide for each $p,\ (1 < p \leq \infty)$. This facts hold for the L^p-differentiability, the weak L^p-differentiability and the differentiability in the sense of distribution.

Further, for $1 < p \leq \infty$, the partial L^p-derivatives, the weak partial L^p-derivatives and the partial derivatives in the sense of distribution for a L^p-function coincide. For L^p-functions on $\boldsymbol{R}$, we have the similar results.

4 Differential calculus of L^p_{loc}-functions

4.1 L^p_{loc}-differentiability

In this section, we study the concept of L^p_{loc}-differentiability. We define that $L^p_{\mathrm{loc}} = L^p_{\mathrm{loc}}(a,\ b)$ is the function space of all locally p-th integrable functions defined on an open interval $(a,\ b)$. Here we assume $1 \le p \le \infty$. Then, for a locally p-th integrable function defined on the open interval $(a,\ b)$, we study the concept of differentiability in the sense of L^p_{loc}-convergence.

Then we have the following definition 4.1.

Definition 4.1(L^p_{loc}-differetiability) Assume that a function $y = f(x)$ is a locally p-th integrable function defined on an open interval $(a,\ b)$. Here we assume $1 \le p \le \infty$.

Then we define that the increment Δy of a function $y = f(x)$ corresponding to the increment Δx of the independent variable x is

$$\Delta y = f(x + \Delta x) - f(x) = A(x)\Delta x + \varepsilon(x,\ \Delta x)\Delta x.$$

Here $A(x)$ is a function of x which does not depend on Δx. $\varepsilon(x,\ \Delta x)$ is a function of x and Δx.

Then we define that the function $f(x)$ is differentiable in the sense of L^p_{loc}-convergence on the open interval $(a,\ b)$ if we have the condition

$$\varepsilon(x,\ \Delta x) \to 0,\ (\Delta x \to 0)$$

in the sense of L^p_{loc}-convergence on the open interval $(a,\ b)$.

Namely, this is equivalent to the condition that, for an arbitrary pair $c,\ d$ of real numbers such as $a < c < d < b$, we have

$$\lim_{\Delta x \to 0} q_{[c,\ d]}\ \big(\ \varepsilon(x,\ \Delta x)\ \big) = \lim_{\Delta x \to 0} \Big(\ \int_c^d |\varepsilon(x,\ \Delta x)^p| dx\ \Big)^{1/p} = 0.$$

Then we extend the definition as $\varepsilon(x,\ 0) = 0,\ (x \in (a,\ b))$.

Here we say that a function $f(x)$ is L^p_{loc}-differentiable for simplification if it is differentiable in the sense of L^p_{loc}-convergence.

Now we assume that a function $y = f(x)$ is L^p_{loc}-differentiable in the open interval $(a,\ b)$. Then, by virtue of the condition of definition 4.1, we have the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

in the sense of L^p_{loc}-convergence on the open interval $(a,\ b)$.

Then we say that this limit $f'(x)$ is a $L^p_{\rm loc}$-derivative of $y = f(x)$. By virutue of the completeness of $L^p_{\rm loc}$, $f'(x)$ belongs to $L^p_{\rm loc}(a,\ b)$.

By virtue of the property of $L^p_{\rm loc}$-convergence, $f'(x)$ has the determined complex values almost everywhere on $(a,\ b)$.

4.2 Properties of $L^p_{\rm loc}$-derivatives

Assume that $\mathcal{D} = \mathcal{D}(\boldsymbol{R})$ is the function space of all C^∞-functions with compact support on $\boldsymbol{R}$. Here we define the concept of weak derivatives.

Definition 4.2 Assume that $1 \le p \le \infty$ and $f(x) \in L^p_{\rm loc}$ hold.

We use the same notation as in definition 4.1. Then we define that a function $y = f(x)$ is differentiable in the sense of the weak convergence of $L^p_{\rm loc}$ if we have the condition $\varepsilon(x,\ \Delta x) \to 0$ in the sense of the weak topology of $L^p_{\rm loc}$ on $\boldsymbol{R}$ when $\Delta x \to 0$.

Namely this is equivalent to the condition

$$\lim_{\Delta x \to 0} \big(\, \varepsilon(x,\ \Delta x),\ \varphi \,\big) = 0$$

for $\varphi \in L^q_c$. Here we assume that the relations

$$1 \le q \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1$$

hold.

Then we extend the definition as $\varepsilon(x,\ 0) = 0,\ (x \in \boldsymbol{R})$.

Here, if a function is differentiable in the sense of the weak topology of $L^p_{\rm loc}$, we say that it is weakly $L^p_{\rm loc}$-differentiable for simplification.

Then, by virtue of the condition of definition 4.2, we have the weak limit

$$w\text{-}\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = w\text{-}\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = w\text{-}f'(x)$$

in the sense of the weak topology of $L^p_{\rm loc}$. We define that this weak limit w-$f'(x)$ is a weak $L^p_{\rm loc}$-derivative. By virtue of the weak completenes of $L^p_{\rm loc}$, w-$f'(x)$ is an element of $L^p_{\rm loc}$.

Theorem 4.1 *Assume that $1 \le p \le \infty$ and $f(x) \in L^p_{\rm loc}$ hold. If $f(x)$ is $L^p_{\rm loc}$-differentiable, $f(x)$ is weakly $L^p_{\rm loc}$-differentiable and the derivative $f'(x)$ in the sense of $L^p_{\rm loc}$-convergence coincides with the weak derivative w-$f'(x)$. Namely we have the equality*

$$f'(x) = w\text{-}f'(x).$$

Namely we have the equality

$$(f',\ \varphi) = (w\text{-}f',\ \varphi),\ (\varphi \in L^q_c).$$

Here p, q satisfy the relations

$$1 \leq q \leq \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Then the weak $L^p_{\rm loc}$-derivative w-$f'(x)$ of $f(x) \in L^p_{\rm loc}$ is a $L^p_{\rm loc}$-function.

Theorem 4.2 *Assume that $1 \leq p \leq \infty$ and $f(x) \in L^p_{\rm loc}$ hold. If there exists the weak $L^p_{\rm loc}$-derivative w-$f'(x)$ of $f(x)$ and we have w-$f'(x) \in L^p_{\rm loc}$, $f(x)$ is $L^p_{\rm loc}$-differentiable and we have the equality w-$f'(x) = f'(x)$ for the derivative $f'(x)$ of $f(x)$ in the sense of $L^p_{\rm loc}$-convergence.*

Theorem 4.3 *Assume $1 \leq p \leq \infty$. Then f is $L^p_{\rm loc}$-differentiable if and only if f is $L^1_{\rm loc}$-differentiable and the $L^1_{\rm loc}$-derivative f' belongs to $L^p_{\rm loc}$. Then f' is the $L^p_{\rm loc}$-derivative of f.*

Theorem 4.4 *Assume $1 \leq p \leq \infty$. If, for a sequence of functions $f_n(x) \in L^p_{\rm loc}$, $(n = 1,\ 2,\ 3,\ \cdots)$, we have $f,\ g \in L^p_{\rm loc}$ such that we have*

$$f_n \to f,\ (n \to \infty),\ f'_n \to g,\ (n \to \infty),$$

we have $f' \in L^p_{\rm loc}$ such that we have the equality

$$f' = g.$$

Namely, the differential operator $\dfrac{d}{dx}$ is a closed linear operator.

Assume $1 \leq p \leq \infty$. By virtue of theorem 2.4, the $L^p_{\rm loc}$-differentiability, the weak $L^p_{\rm loc}$-differetiability and the differentiability in the sense of distribution coincide. Therefore, the $L^p_{\rm loc}$-derivative, the weak $L^p_{\rm loc}$-derivative and the derivative in the sense of distribution of $f(x) \in L^p_{\rm loc}$ are identical.

For $1 \leq p \leq \infty$, we have the inclusion relation $L^p_{\rm loc} \subset L^1_{\rm loc}$.

By virtue of theorem 4.3, the $L^p_{\rm loc}$-derivative of $f \in L^p_{\rm loc}$ is the $L^1_{\rm loc}$-derivative f' of f which is a $L^p_{\rm loc}$-function.

Therefore, for $1 \leq p \leq \infty$, the derivative of $f \in L^p_{\rm loc}$ is calculated by using the topology of $L^1_{\rm loc}$-convergence.

The differentiability of a function and the calculation of derivative are the local properties. Especially, because we have the inclusion relation $L^1 \subset L^1_{\rm loc}$, we may calculate the derivative of a L^1-function considering it as a $L^1_{\rm loc}$-function.

4.3 $L^p_{\rm loc}$-differentiability

Let D be a general domain in $\boldsymbol{R}^d$. Here assume $d \geq 2$.

Let $L^p_{\rm loc} = L^p_{\rm loc}(D)$ be the function space of all locally p-th integrable functions defined on the domain D.

Here assume $1 \leq p \leq \infty$.

Then we study the differentiability in the sense of $L^p_{\rm loc}$-convergence for a locally p-th integrable function defined on the domain D. Here we give the following definition 4.3.

Definition 4.3($L^p_{\rm loc}$-differentiability) Assume that a function $y = f(x)$ is a locally p-th integrable function defined on a domain D. Here assume that $1 \leq p \leq \infty$.

Then, the increment Δy of the function $y = f(x)$ corresponding to the increment Δx of the independent variables x is

$$\Delta y = f(x + \Delta x) - f(x) = \sum_{i=1}^{d} A_i(x)\Delta x_i + \varepsilon(x,\ \Delta x)\rho.$$

Here $\rho = \|\Delta x\|$ and $A_i(x),\ (i = 1,\ 2,\ \cdots,\ d)$ are the functions of x which do not depend on Δx. $\varepsilon = \varepsilon(x,\ \Delta x)$ is the function of x and Δx.

Then we define that the function $y = f(x)$ is differentiable in the sense of $L^p_{\rm loc}$-convergence on the domain D if we have the condition

$$\varepsilon(x,\ \Delta x) \to 0,\ (\Delta x \to 0)$$

in the sense of $L^p_{\rm loc}$-convergence on the domain D.

Namely, for $1 \leq p < \infty$, this is equivalent to the condition

$$\lim_{\Delta \to 0} q_K(\varepsilon(x,\ \Delta x)) = \lim_{\Delta x \to 0} \Big(\int_K |\varepsilon(x,\ \Delta x)|^p dx \Big)^{1/p} = 0$$

for an arbitrary compact subset K of the domain D.

For $p = \infty$, we have the similar condition with respect to the system of semi-norms of $L^\infty_{\rm loc}(D)$.

Then we extend the definion as $\varepsilon(x,\ 0) = 0,\ (x \in D)$.

Here we say that a function is $L^p_{\rm loc}$-differentiable for simplification if it is differentiable in the sense of $L^p_{\rm loc}$-convergence.

4.4 Properties of partial $L^p_{\rm loc}$-derivatives

Let $L^p_{\rm loc} = L^p_{\rm loc}(D)$ for a general domain D in $\boldsymbol{R}^d$. Here assume that $d \geq 2$ and $1 \leq p \leq \infty$ hold.

Now, if $f(x) \in L^p_{\rm loc}$ is $L^p_{\rm loc}$-differentiable, we have the limit

$$\frac{\partial y}{\partial x_j} = \lim_{h \to 0} \frac{(\tau_{-he_j} f)(x) - f(x)}{h}$$

in the sense of $L^p_{\rm loc}$-convergence for $1 \leq j \leq d$. Here assume that $\{e_1,\ e_2,\ \cdots,\ e_d\}$ is the standard basis of $l^2(d)$ and $\tau_y,\ (y \in \boldsymbol{R}^d)$ denotes the translation operator.

Then the partial derivatives $\dfrac{\partial y}{\partial x_j},\ (1 \leq j \leq d)$ are the $L^p_{\rm loc}$-functions.

We say that these are the partial $L^p_{\rm loc}$-derivatives.

Here $\dfrac{\partial y}{\partial x_j},\ (1 \leq j \leq d)$ have the determined complex values almost everywhere in D.

Now we give the definition of the weak partial derivatives.

Definition 4.4 Assume that $1 \leq p \leq \infty,\ 1 \leq j \leq d$ and $f(x) \in L^p_{\rm loc}$ hold. We use the same notation as in definition 4.1.

Then we define that the function is weakly differentiable in $L^p_{\rm loc}$ if we have $\varepsilon(x,\ \Delta x) \to 0,\ (\Delta x \to 0)$ in the sense of weak topology of $L^p_{\rm loc}$.

If the function $f(x)$ is weakly $L^p_{\rm loc}$-differentiable, we have the weak limit

$$w\text{-}\frac{\partial y}{\partial x_j} = w\text{-}\lim_{h \to 0} \frac{(\tau_{-he_j} f)(x) - f(x)}{h}.$$

We define that this weak limit $w\text{-}\dfrac{\partial y}{\partial x_j}$ is a weak partial $L^p_{\rm loc}$-derivative of $y = f(x),\ (1 \leq j \leq d)$.

By virtue of the weak completenes of $L^p_{\rm loc}$, we have $w\text{-}\dfrac{\partial y}{\partial x_j} \in L^p_{\rm loc},\ (1 \leq j \leq d)$.

Then we have the following theorem.

Theorem 4.5 *Assume that* $1 \leq p \leq \infty,\ 1 \leq j \leq d$ *and* $f(x) \in L^p_{\rm loc}$ *hold. If* $f(x)$ *is partially* $L^p_{\rm loc}$*-differentiable,* $f(x)$ *is weakly partial* $L^p_{\rm loc}$*-differentiable and its partial* $L^p_{\rm loc}$*-derivative* $\dfrac{\partial f}{\partial x_j}$ *in the sense of* $L^p_{\rm loc}$*-convergence coincides with the weak partial* $L^p_{\rm loc}$*-derivative* $w\text{-}\dfrac{\partial f}{\partial x_j}$.

Namely, we have the equalities

$$\frac{\partial f}{\partial x_j} = w\text{-}\frac{\partial f}{\partial x_j},\ (1 \le j \le d).$$

Also we have the equalities

$$\Big(\ \frac{\partial f}{\partial x_j},\ \varphi\ \Big)=\Big(\ w\text{-}\frac{\partial f}{\partial x_j},\ \varphi\ \Big),\ (\varphi \in L^q_c,\ 1 \le j \le d).$$

Here we assume the relations

$$1 \le q \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Then the weak partial $L^p_{\rm loc}$-derivative of $f(x) \in L^p_{\rm loc}$ is a $L^p_{\rm loc}$-function, $(1 \le j \le d)$.

Theorem 4.6 *Assume that $1 \le p \le \infty$ and $f(x) \in L^p_{\rm loc}$ hold. Then, if, for $1 \le j \le d$, we have the weak partial $L^p_{\rm loc}$-derivative $w\text{-}\dfrac{\partial f}{\partial x_j}$ of $f(x)$, $f(x)$ is partially $L^p_{\rm loc}$-differentiable and we have the equalities*

$$w\text{-}\frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_j},\ (1 \le j \le d)$$

for the partial derivatives $\dfrac{\partial f}{\partial x_j}$ of $f(x)$ in the sense of $L^p_{\rm loc}$-convergence, $(1 \le j \le d)$.

Further, we have the commutativity of the order of partial differentiation.

Theorem 4.7 *Assume that $1 \le p \le \infty$ and $f(x) \in L^p_{\rm loc}$ hold. If, for $1 \le i,\ j \le d,\ (i \ne j)$, we have $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ and $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ in the sense of $L^p_{\rm loc}$-convergence, we have the equality*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Theorem 4.8 *Assume that $1 \le p \le \infty$ and $f(x) \in L^p_{\rm loc}$ hold. Then f is $L^p_{\rm loc}$-differentiable if and only if f is $L^1_{\rm loc}$-differentiable and the partial $L^1_{\rm loc}$-derivatives $\dfrac{\partial f}{\partial x_j}$, $(1 \le j \le d)$ are the $L^p_{\rm loc}$-functions. Then the partial $L^1_{\rm loc}$-derivatives $\dfrac{\partial f}{\partial x_j}$, $(1 \le j \le d)$ are the partial $L^p_{\rm loc}$-derivatives of f.*

Theorem 4.9 *Assume* $1 \le p \le \infty$*. Then, if, for a sequence of functions* $f_n(x) \in L^p_{\mathrm{loc}}$*,* $(n = 1,\ 2,\ 3,\ \cdots)$*, we have* $f,\ g \in L^p_{\mathrm{loc}}$ *such that we have*

$$f_n \to f,\ (n \to \infty),\ \frac{\partial f_n}{\partial x_j} \to g,\ (n \to \infty),$$

we have $\dfrac{\partial f}{\partial x_j} \in L^p_{\mathrm{loc}}$ *such that*

$$\frac{\partial f}{\partial x_j} = g$$

holds. Here assume $1 \le j \le d$*. Thus the partial differential operator* $\dfrac{\partial}{\partial x_j}$ *is a closed linear operator.*

Assume $1 \le p \le \infty$. Then, by virtue of theorem 2.4, the partial L^p_{loc}-differentiability, the weakly partial L^p_{loc}-differentiability and the partial differentiability in the sense of distribution coincide. These facts are also true for the L^p_{loc}-differentiability, the weak L^p_{loc}-differentiability and the differentiability in the sense of distribution.

Further, the partial L^p_{loc}-derivatives, the weak partial L^p_{loc}-derivatives and the partial derivatives in the sense of distribution of L^p_{loc}-function coincide.

For $1 \le p \le \infty$, we have the inclusion relation $L^p_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$. Thus, by virtue of Theorem 4.8, the partial L^p_{loc}-derivatives of $f \in L^p_{\mathrm{loc}}$ are the partial L^1_{loc}-derivatives $\dfrac{\partial f}{\partial x_j}$, $(1 \le j \le d)$ which are the L^p_{loc}-functions.

Because we have the inclusion relation $L^1 \subset L^1_{\mathrm{loc}}$, the weak partial derivative of a L^1-function is the partial L^1_{loc}-derivative which is a L^1-function.

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